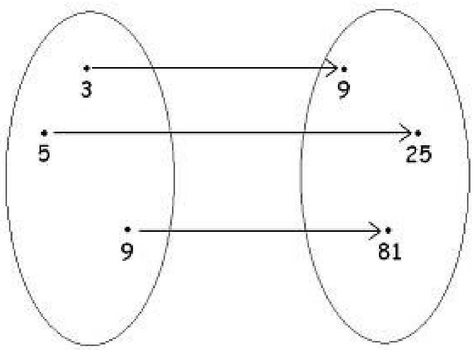
Computer Science 1FC3
Functions

Jessica Cao (caoff@mcmaster.ca)
Don Vo (vod2@mcmaster.ca)

Function

- a function assigns one element of one set to each element of another set
- let *a* be an element in set A and *b* an element in set B: if *b* is assigned to *a*, this is written as *f*(*a*) = *b*
- if *f* is a function from A to B, we write *f*: A --> B

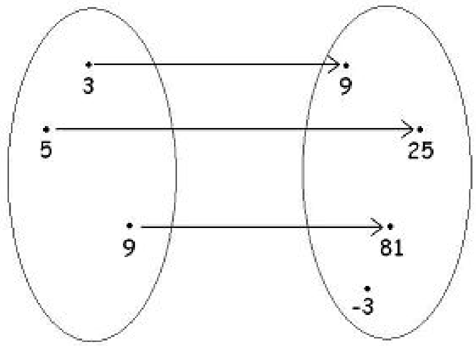
Eg. for the function $f(x) = x^2$, $f(3) = 9$, $f(5) = 25$, $f(9) = 81$ etc.



There is really nothing new about functions – we've been using them in Maple already. Recall the AND function:

[> AND := (*x*, *y*) → *x* and *y*;

The input to this function, or it's ***domain***, is the pair of truth values:
(true, true), (true, false), (false, true) and (false,false).
The output of this function, or it's ***codomain***, is the truth values:
true and false.
If *f(a)* = *b*, we say that *b* is the ***image*** of *a* and *a* is a ***pre-image*** of *b*.
What is the image of (true, false) for the AND function?
The ***range*** of *f* is the set of all images of elements of A.
What's the difference between range and codomain? (see graph below)



Real-valued functions with the same domain can be added and multiplied:

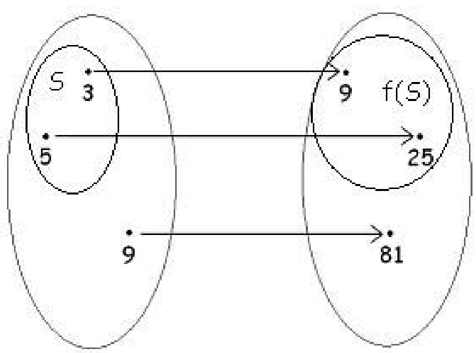
$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 f_2)(x) = f_1(x)f_2(x)$

What do you think is the domain and codomain of $(f_1 + f_2)$ and f_1f_2?

Subset

- if S is a subset of A, the *image* of S is the *subset* of B that consists of the images of the elements of S.
- $f(S)$ is the image of S, so that $f(S) = \{f(s) \mid s \in S\}$



One-to-one functions

- also called *injective* functions
- a function is an injection if and only if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f.
- what is an example of an one-to-one function?

Increasing/Decreasing functions

- a function f is strictly increasing if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f
- what do you think a strictly decreasing function is?

Onto functions

- also called *surjective* functions
- a function is a surjection if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- what is an example of an onto function?

One-to-one Correspondence
- a function is a one-to-one correspondence, or a *bijection*, if it is both one-to-one and onto.
- what is an example of an bijection?

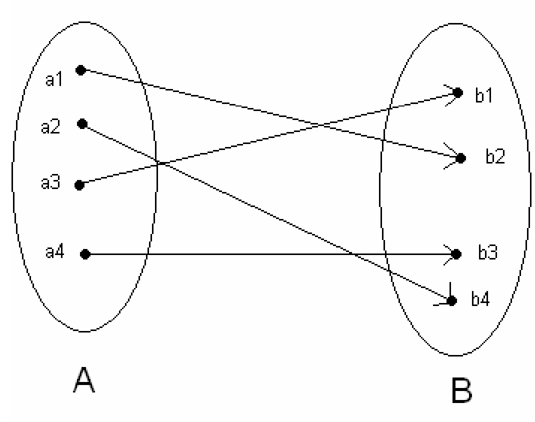
Inverse function
- assigns to an element b belonging to B the unique element a in A such that $f(a) = b$
- denoted by f^{-1}
- $f^{-1}(b) = a$ when $f(a) = b$
- what is the inverse function of $f(x) = x^2$?

Composite functions
- the *composition* of two functions f and g is denoted by $f \circ g$
- $(f \circ g)(a) = f(g(a))$

Exercises

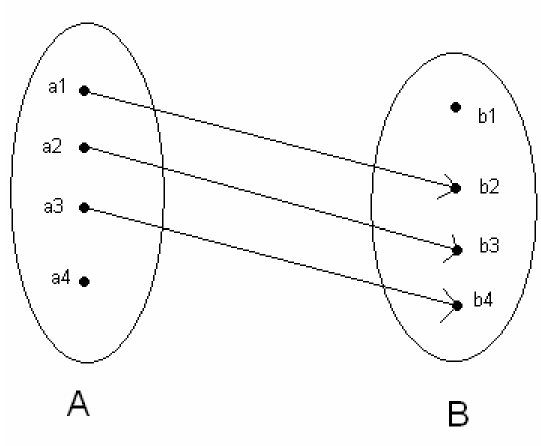
1) For the following functions, state the i) domain ii) codomain iii) image iv) range

a)



b) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$

c)



d) f: $\mathbb{N} \rightarrow \mathbb{N}$, $f(x) = \lfloor 100 / x \rfloor$

2) Describe the following functions as one-to-one, onto, bijective or none of the above.
Does the function have an inverse? If so, find it.

a) f: $\mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x - 1$

b) f: $\mathbb{N} \rightarrow \mathbb{N}$, $f(x) = 2x$

c) g: A→B where A = {1, 4, 7, 13, 42} and B = {2, 4, 108, 256} and g = {(1, 108), (4, 256), (7, 2), (13, 4)}

d) g: A→B where A = {10, 42, 64, 111} and B = {2, 4, 108, 256, 720} and g = {(10, 4), (42, 2), (64, 2), (111, 720)}

e) h: A→B where A = {a, b, x, y} and B = {$, #, %, !} and h = {(a, !), (b, %), (x, $), (y, #)}

f) f: $\mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 + 1$

g) f: $\mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$

3) Determine the function that results from the composition, $f \circ g$.

a) $f = x + 2$, $g = x^4$

b) $f = 2x + 2$, $g = x/2$

c) $f = x^2 + 4x + 3$, $g = (x^2 - 5)/(x + 13)$