Predicate Logic

March 29, 2005

What's wrong with Propositional logic?

- It cannot express some ideas!
- Examples:
  - All men are mortal
  - All students in this class are registered at McMaster
    - Every instance **can** be expressed!
  - Some students in this class are not in first year

What is missing ?

- For all $\forall$
- There Exists $\exists$
- Examples: $\forall$

$$\forall x : Man(x) \rightarrow Mortal(x)$$

$$\forall x : (Student(x) \wedge CS1MC3(x)) \rightarrow R(x)$$

$$\exists x : Student(x) \wedge \neg FirstYear(x)$$

A little more formal

- Predicate logic is made up of propositional logic plus:
- $\forall, \exists$ and : added to the **syntax**
- The following rules for formula formation:
  - If x is a symbol and t_0 is a term then $\forall x : t_0$ and $\exists x : t_0$ are terms

Scope and capture

- Predicate logic requires **scoping**, just like a programming language!
- Examples (on blackboard)

Semantics

- Need to also add evaluation rules!
- $\forall x : t$ true if for all possible substitutions of values for x in t; it is false otherwise.
  - It is like a generalized **and**
- $\exists x : t$ true if there is one possible substitution of values for x in t; it is false otherwise
  - It is like a generalized **or**

Relations

$\neg\exists X{:}p(X) = \forall X{:}\neg\, p(X)$

$\neg\, \forall X{:}p(X) = \exists X{:}\neg\, p(X)$

$\exists X{:}p(X) = \exists Y{:}p(Y)$

$\forall X{:}q(X) = \forall Y{:}q(Y)$

$\forall X{:}(p(X) \wedge q(X)) = \forall X{:}p(X) \wedge \forall Y{:}q(Y)$

$\exists X{:}(p(X) \vee q(X)) = \exists X{:}p(X) \vee \exists Y{:}q(Y)$

Quantification

- Quantification is the process of “going over” all the elements of a specific set
- $\forall$X:t and $\exists$ X:t quantify over all the **constants** of the current theory
- Such logical systems are called **first-order**

Additional rule

- Rule of Universal Instantiation: an individual may be substituted for a universal

  $$\frac{(\forall x)\ \text{Human}(x)}{\text{Human (Socrates)}}$$

Terminology

- **Defn:** a formula F is **satisfiable** if there is an evaluation of the variables of F for which the formula is true.
- **Defn**: a formula H is a **consequence** of a set of formulas $G=\{F_1,\ldots,F_n\}$ if for all evaluation of the variables which satisfy G then H is also true.
- Defn: a formula H is **valid** if it is a consequence of the empty set.

Truth and provability

- Informally, something is true (valid) if it is a tautology.
- Something is **provable** if it can be derived from some axioms and applying deduction rules
- **Theorem**: in first-order logic, something is true iff it is provable

Second order logic

- Allow to quantify over more than constants.
  - Example: over all formulas
  - Ex: Induction principle for the integers:

$$\forall P : (P(0) \wedge (\forall x : P(x) \rightarrow P(x+1))) \rightarrow \forall y P(y)$$

But…

- In second order logic, there are true statements that cannot be proved
- Not only that, but this is inescapable: any system of logic with “enough expressiveness” will display this behaviour
- Gödel incompleteness theorem