Deriving on Steroids - for proof assistants

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The Big Picture: MetaProgramming on Theories

(Co)Limits of diagrams, fibered functors

$\Downarrow$

Concepts and theory combinators

$\Downarrow$

(Generalized) Algebraic theories

$\Downarrow$

Biform theories

$\Downarrow$

Generic algorithm

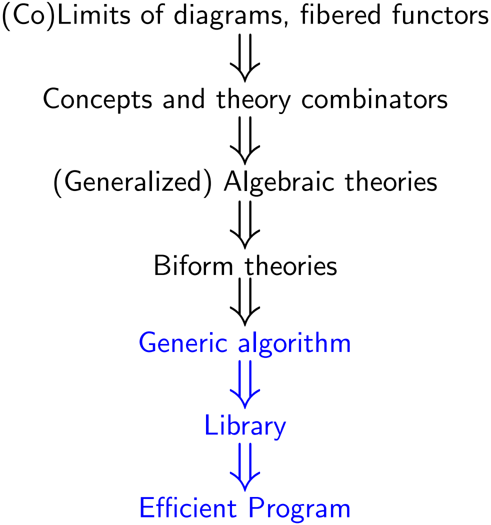
$\Downarrow$

Library

$\Downarrow$

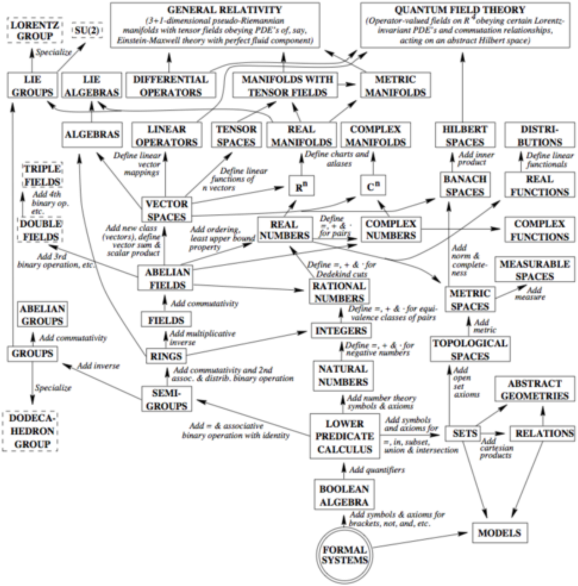
Efficient Program

The Big Picture: MetaProgramming on Theories

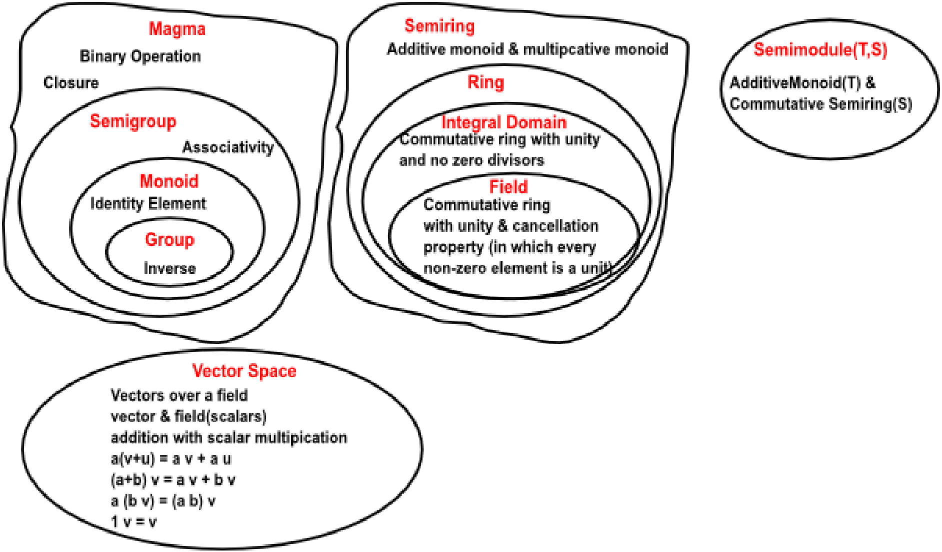


$$\begin{tikzcd}[row sep=small]
\text{(Co)Limits of diagrams, fibered functors} \arrow[d, Rightarrow] \\
\text{Concepts and theory combinators} \arrow[d, Rightarrow] \\
\text{(Generalized) Algebraic theories} \arrow[d, Rightarrow] \\
\text{Biform theories} \arrow[d, Rightarrow] \\
\text{Generic algorithm} \arrow[d, Rightarrow] \\
\text{Library} \arrow[d, Rightarrow] \\
\text{Efficient Program}
\end{tikzcd}$$

Theory graphs



Theory graphs



Magma

Binary Operation

Closure

Semigroup

Associativity

Monoid

Identity Element

Group

Inverse

Semiring

Additive monoid & multipcative monoid

Ring

Integral Domain

Commutative ring with unity and no zero divisors

Field

Commutative ring with unity & cancellation property (in which every non-zero element is a unit)

Semimodule(T,S)

AdditiveMonoid(T) & Commutative Semiring(S)

Vector Space

Vectors over a field

vector & field(scalars)

addition with scalar multipication

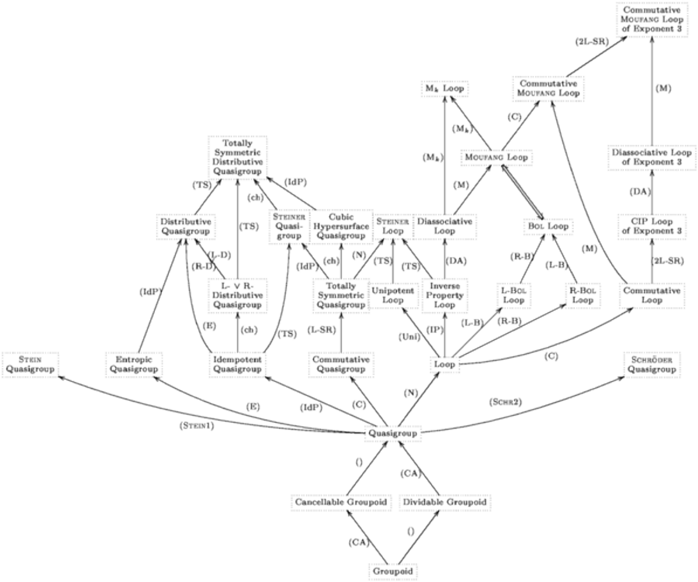
a(v+u) = a v + a u

(a+b) v = a v + b v

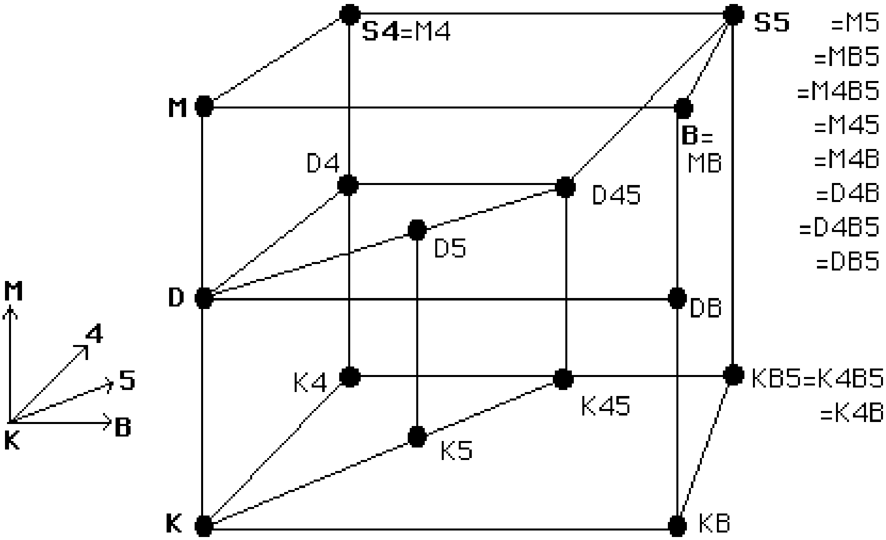
a (b v) = (a b) v

1 v = v

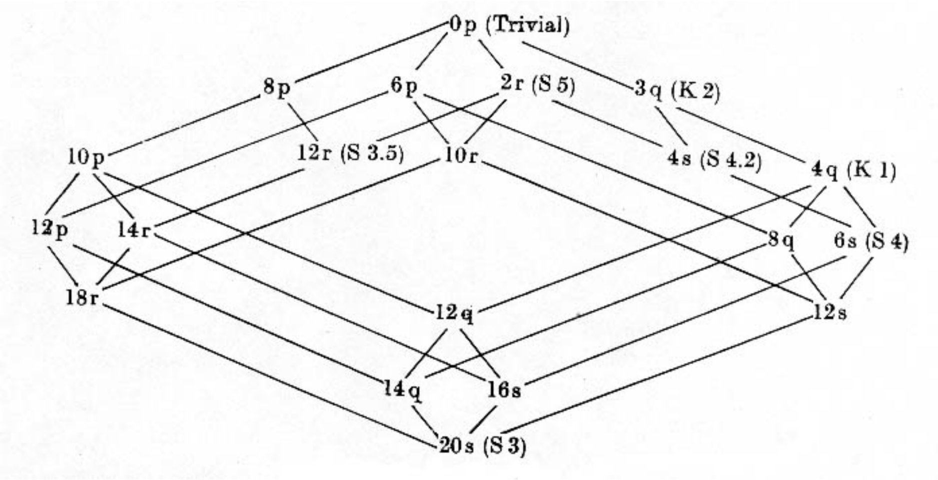
Theory graphs



Theory graphs



Theory graphs



(Presentations of) Algebraic Theories

```
Monoid := Theory {
  U : type;
  * : (U, U) -> U;
  e : U;
  axiom rightIdentity_*_e : forall x:U. x*e = x;
  axiom leftIdentity_*_e : forall x:U. e*x = x;
  axiom associative_* : forall x,y,z:U. (x*y)*z=x*(y*z)}
```

(Presentations of) Algebraic Theories

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Monoid := Theory {
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  axiom associative_* : forall x,y,z:U. (x*y)*z=x*(y*z)}

CommutativeMonoid := Theory {
  U : type;
  * : (U, U) −> U;
  e : U;
  axiom rightIdentity_*_e : forall x:U. x*e = x;
  axiom leftIdentity_*_e : forall x:U. e*x = x;
  axiom associative_* : forall x,y,z:U. (x*y)*z=x*(y*z)
  axiom commutative_* : forall x,y,z:U. x*y=y*x}
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(Presentations of) Algebraic Theories

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Monoid := Theory {
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AdditiveMonoid := Theory {
  U : type;
  + : (U, U) -> U;
  0 : U;
  axiom rightIdentity_+_0 : forall x:U. x+0 = x;
  axiom leftIdentity_+_0 : forall x:U. 0+x = x;
  axiom associative_+ : forall x,y,z:U. (x+y)+z=x+(y+z) }
```

(Presentations of) Algebraic Theories

```
Monoid := Theory {
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  e : U;
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  axiom leftIdentity_*_e : forall x:U. e*x = x;
  axiom associative_* : forall x,y,z:U. (x*y)*z=x*(y*z)}

AdditiveCommutativeMonoid := Theory {
  U : type;
  + : (U, U) -> U;
  0 : U;
  axiom rightIdentity_+_0 : forall x:U. x+0 = x;
  axiom leftIdentity_+_0 : forall x:U. 0+x = x;
  axiom associative_+ : forall x,y,z:U. (x+y)+z=x+(y+z)
  axiom commutative_+ : forall x,y,z:U. x+y=y+x}
```

Pseudo-Combinators for (presentations of) theories

Following Burstall & Goguen (OBJ); Kapur, Musser, Stepanov (Tecton)

Extension:

```
CommutativeMonoid := Monoid extended by {
  axiom commutative_∗ : forall x,y,z:U. x∗y=y∗x}
```

Pseudo-Combinators for (presentations of) theories

Following Burstall & Goguen (OBJ); Kapur, Musser, Stepanov (Tecton)

Extension:

```
CommutativeMonoid := Monoid extended by {
  axiom commutative_* : forall x,y,z:U. x*y=y*x}
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Renaming:

```
AdditiveMonoid := Monoid[ * |-> +, e |-> 0 ]
```

Pseudo-Combinators for (presentations of) theories

Following Burstall & Goguen (OBJ); Kapur, Musser, Stepanov (Tecton)

Extension:

```
CommutativeMonoid := Monoid extended by {
  axiom commutative_* : forall x,y,z:U. x*y=y*x}
```

Renaming:

```
AdditiveMonoid := Monoid[ * |-> +, e |-> 0 ]
```

Combination:

```
AdditiveCommutativeMonoid :=
  combine AdditiveMonoid, CommutativeMonoid over Monoid
```

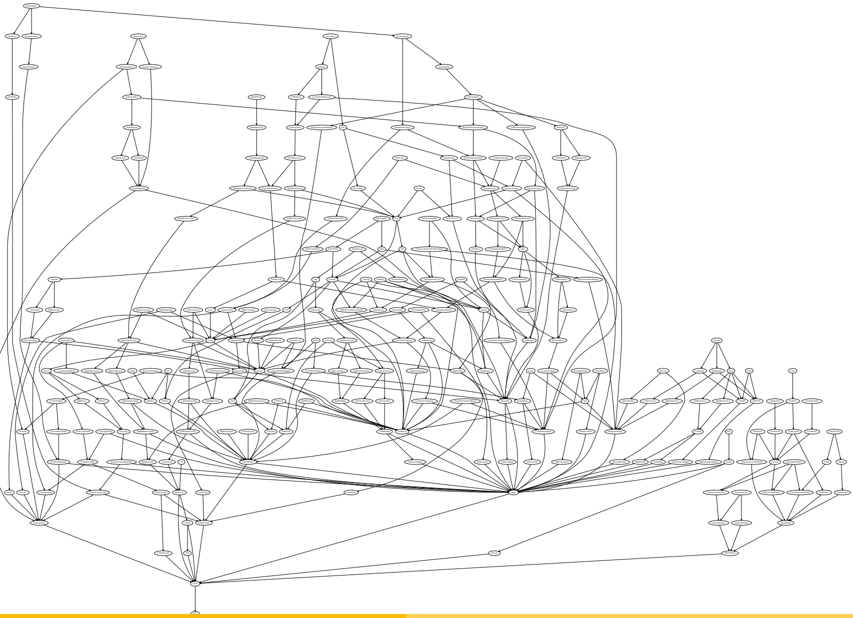
Library fragment 1

```
MoufangLoop := combine Loop, MoufangIdentity over Magma
LeftShelfSig := Magma[ * |-> |> ]
LeftShelf := LeftDistributiveMagma [ * |-> |> ]
RightShelfSig := Magma[ * |-> <| ]
RightShelf := RightDistributiveMagma[ * |-> <| ]
RackSig := combine LeftShelfSig, RightShelfSig over Carrier
Shelf := combine LeftShelf, RightShelf over RackSig
LeftBinaryInverse := RackSig extended by {
    axiom leftInverse_|>_<| : forall x,y:U. (x |> y) <| x = y }
RightBinaryInverse := RackSig extended by {
    axiom rightInverse_|>_<| : forall x,y:U. x |> (y <| x) = y }
Rack := combine RightShelf, LeftShelf, LeftBinaryInverse,
    RightBinaryInverse over RackSig
LeftIdempotence := IdempotentMagma[ * |-> |> ]
RightIdempotence := IdempotentMagma[ * |-> <| ]
LeftSpindle := combine LeftShelf, LeftIdempotence over LeftShelfSig
RightSpindle := combine RightShelf, RightIdempotence over RightShelfSig
Quandle := combine Rack, LeftSpindle, RightSpindle over Shelf
```

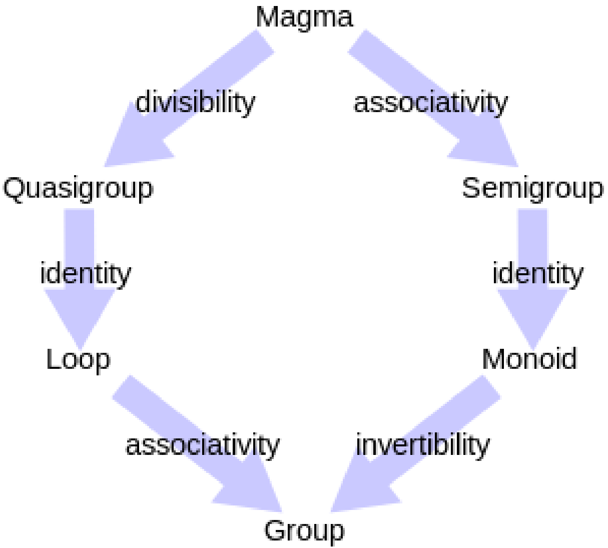
Library fragment 2

```
NearSemiring := combine AdditiveSemigroup, Semigroup, RightRingoid ove
NearSemifield := combine NearSemiring, Group over Semigroup
Semifield := combine NearSemifield, LeftRingoid over RingoidSig
NearRing := combine AdditiveGroup, Semigroup, RightRingoid over Ringoi
Rng := combine AbelianAdditiveGroup, Semigroup, Ringoid over RingoidSi
Semiring := combine AdditiveCommutativeMonoid, Monoid1, Ringoid, Left0
SemiRng := combine AdditiveCommutativeMonoid, Semigroup, Ringoid over
Dioid := combine Semiring, IdempotentAdditiveMagma over AdditiveMagma
Ring := combine Rng, Semiring over SemiRng
CommutativeRing := combine Ring, CommutativeMagma over Magma
BooleanRing := combine CommutativeRing, IdempotentMagma over Magma
NoZeroDivisors := Ringoid0Sig extended by {
  axiom onlyZeroDivisor_*_0: forall x:U.
    ((exists b:U. x*b = 0) and (exists b:U. b*x = 0)) implies (x = 0)
Domain := combine Ring, NoZeroDivisors over Ringoid0Sig
IntegralDomain := combine CommutativeRing, NoZeroDivisors over Ringoid
DivisionRing := Ring extended by {
  axiom divisible : forall x:U. not (x=0) implies
    ((exists! y:U. y*x = 1) and (exists! y:U. x*y = 1)) }
Field := combine DivisionRing, IntegralDomain over Ring
```

A fraction of the Algebraic Zoo



Breaks down


Magma
divisibility
associativity
Quasigroup
Semigroup
identity
identity
Loop
Monoid
associativity
invertibility
Group

Breaks down

```
Thy1 := Empty extended by { U : type }
Thy2 := Empty extended by { U : type }
Thy3 := combine Thy1, Thy2 over Empty
```

Breaks down

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Thy1 := Empty extended by { U : type }
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Thy3 := combine Thy1, Thy2 over Empty
```

Lesson from PL theory

Find a good denotational semantics, then come back to the syntax

A little theory

Given some dependent type theory, its category of contexts $\mathbb{C}$ has objects

$$\Gamma := \langle x_0 : \sigma_0; \ldots; x_{n-1} : \sigma_{n-1} \rangle\,,$$

such that for each $i < n$ the judgement

$$\langle x_0 : \sigma_0; \ldots; x_{i-1} : \sigma_{i-1} \rangle \vdash \sigma_i : \mathrm{Type}\ (\mathrm{or}\ \ : \mathrm{Prop})$$

holds. A morphism $\Gamma \to \Delta (= \langle y : \sigma \rangle_0^{m-1})$ is an assignment (substitution) $[y_0 \mapsto t_0, \ldots, y_m \mapsto t_{m-1}]$ such that

$$\Gamma \vdash t_0 : \tau_0 \qquad \ldots \qquad \Gamma \vdash t_{m-1} : \tau_{m-1}\,[y \mapsto t]_0^{m-2}$$

$$\begin{tikzcd}
\Gamma^+ \arrow[r, "f^+"] \arrow[d, "A"'] & \Delta^+ \arrow[d, "B"] \\
\Gamma \arrow[r, "f^-"'] & \Delta
\end{tikzcd}$$

Definition

The category of general extensions $\mathbb{E}$ has all general extensions from $\mathbb{B}$ as objects, and given two general extensions $A : \Gamma^+ \to \Gamma$ and $B : \Delta^+ \to \Delta$, an arrow $f : A \to B$ is a commutative square from $\mathbb{B}$.

and just a bit more theory

Theorem

The functor $\mathrm{cod} : \mathbb{E} \to \mathbb{B}$ *is a fibration.*

and just a bit more theory

Theorem

The functor cod : $\mathbb{E} \to \mathbb{B}$ *is a fibration.*

$$\begin{aligned} a, b, c &\in \text{labels} \\ A, B, C &\in \text{names} \\ l &\in \text{declarations}^* \\ r &\in (a_i \mapsto b_i)^* \end{aligned}$$

$$\begin{aligned} \text{tpc} ::= &\ \text{extend } A \text{ by } \{l\} \\ &\mid \text{combine } A\ r_1,\ B\ r_2 \\ &\mid A\ ;\ B \\ &\mid A\ r \\ &\mid \text{Empty} \\ &\mid \text{Theory } \{l\} \end{aligned}$$

and just a bit more theory

Theorem

The functor $\mathsf{cod} : \mathbb{E} \to \mathbb{B}$ *is a fibration.*

$$\begin{aligned}
\llbracket - \rrbracket_{\mathbb{B}} &: \mathsf{tpc} \rightharpoonup |\mathbb{B}| \\
\llbracket \mathsf{Empty} \rrbracket_{\mathbb{B}} &= \langle\, \rangle \\
\llbracket \mathsf{Theory}\ \{l\} \rrbracket_{\mathbb{B}} &\cong \langle l \rangle \\
\llbracket A\ r \rrbracket_{\mathbb{B}} &= \llbracket r \rrbracket_{\pi} \cdot \llbracket A \rrbracket_{\mathbb{B}} \\
\llbracket A; B \rrbracket_{\mathbb{B}} &= \llbracket B \rrbracket_{\mathbb{B}} \\
\llbracket \mathsf{extend}\ A\ \mathsf{by}\ \{l\} \rrbracket_{\mathbb{B}} &\cong \llbracket A \rrbracket_{\mathbb{B}} \fatsemi \langle l \rangle \\
\llbracket \mathsf{combine}\ A_1 r_1, A_2 r_2 \rrbracket_{\mathbb{B}} &\cong D
\end{aligned}$$

$$\begin{tikzcd}
D \arrow[r, "\llbracket r_1 \rrbracket_{\pi} \circ \delta_{A_1}"] \arrow[d, "\llbracket r_2 \rrbracket_{\pi} \circ \delta_{A_2}"] & A_1 \arrow[d, "\delta_A"] \\
A_2 \arrow[r, "\delta_A"'] & A
\end{tikzcd}$$

and just a bit more theory

Theorem

The functor cod : $\mathbb{E} \to \mathbb{B}$ *is a fibration.*

$$\begin{aligned}
\llbracket - \rrbracket_{\mathbb{E}} &: \ \mathsf{tpc} \rightharpoonup |\mathbb{E}| \\
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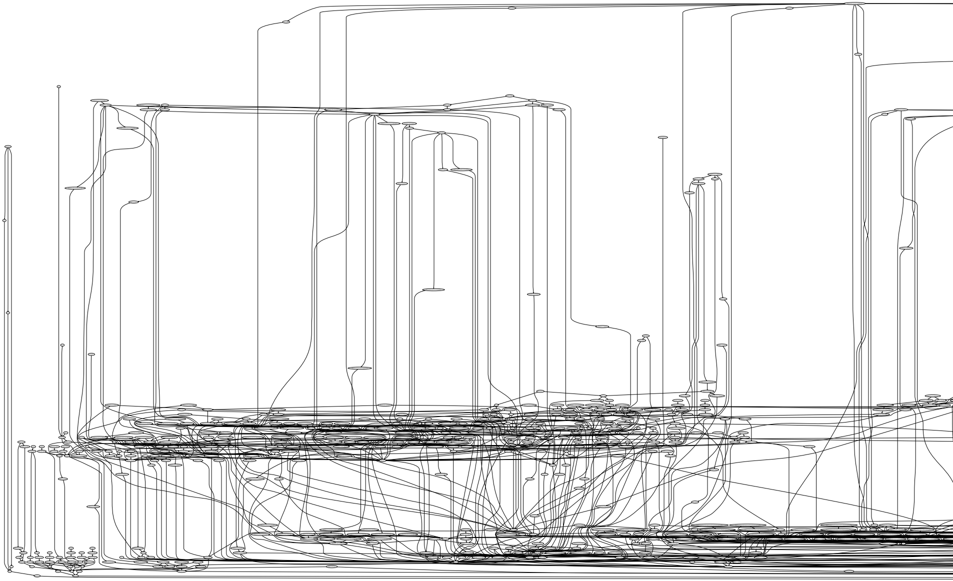
\llbracket \mathsf{extend}\ A\ \mathsf{by}\ \{l\} \rrbracket_{\mathbb{E}} &\cong \ \delta_{A} \\
\llbracket \mathsf{combine}\ A_1 r_1, A_2 r_2 \rrbracket_{\mathbb{E}} &\cong \ \llbracket r_1 \rrbracket_{\pi} \circ \delta_{T_1} \circ \llbracket A_1 \rrbracket_{\mathbb{E}} \\
&\cong \ \llbracket r_2 \rrbracket_{\pi} \circ \delta_{T_2} \circ \llbracket A_2 \rrbracket_{\mathbb{E}}
\end{aligned}$$

$$\begin{tikzcd}
D \arrow[r, "\llbracket r_1 \rrbracket_{\pi} \circ \delta_{T_1}"] \arrow[d, "\llbracket r_2 \rrbracket_{\pi} \circ \delta_{T_2}"] & T_1 \arrow[d, "A_1"] \\
T_2 \arrow[r, "A_2"'] & T
\end{tikzcd}$$

That does scale



That does scale

- 1046 theories
- 2429 lines of code (including comments and many theories defined over 3 lines for human readability)
- expanded theories: 13751 lines
- type checked by export to Matita

More structure

- Order
- Partial operations
- Equality
- Multivalued operations

More structure

- Order
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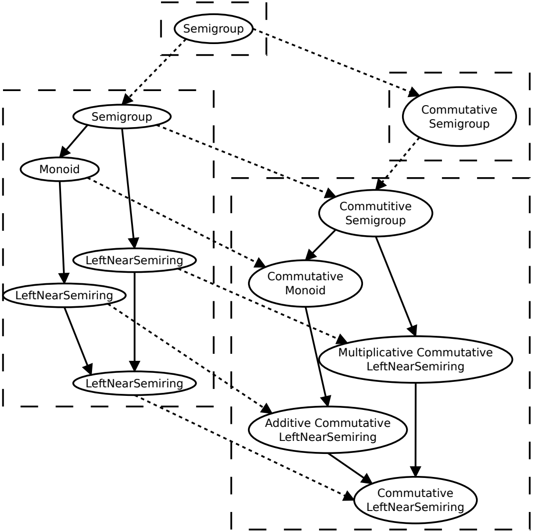
- Equality
- Multivalued operations



Biform monoids

```
Monoid := Theory {
  U : type;
  e : U;
  * : (U, U) -> U;
  ax: forall x:U. e*x = x;
  ax: forall x:U. x*e = x;
  ax: forall x,y,z:U.(x*y)*z=x*(y*z)}
```

Syntax (term language)

```
MonoidTerm := Theory {
  type MTerm = (data X .
    #e : X |
    #* : (X, X) -> X)
}
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Biform Theory: axiomatic + syntactic theory + transformers.

```
length :: MTerm -> Nat
length trm = gfold (+) 1 trm
```

Biform monoids

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length :: MTerm -> Nat
length trm = gfold (+) 1 trm

leftSimp  :: MTerm -> MTerm
leftSimp  = fun (#*(a,b)) when a = #e -> b
rightSimp :: MTerm -> MTerm
rightSimp = fun (#*(a,b)) when b = #e -> b
```

Biform monoids

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length :: MTerm -> Nat
length trm = gfold (+) 1 trm

simp :: MTerm -> MTerm
simp t = match t with
 | (#* (a,b)) when a = #e -> b
 | (#* (a,b)) when b = #e -> b
 | _                      -> t
```

Biform monoids

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Monoid := Theory {
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Biform Theory: axiomatic + syntactic theory + transformers.

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length :: MTerm -> Nat
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Generic

```
simp :: MTerm -> MTerm
simp t = match t with
  | (#* (a,b)) when a = #e -> b
  | (#* (a,b)) when b = #e -> b
  | _                      -> t
```

Derived from length reducing axioms

Different interpretations of theories [1]

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Monoid := Theory {
  U : type;
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  * : (U, U) -> U;
  ax: forall x:U. e*x = x;
  ax: forall x:U. x*e = x;
  ax: forall x,y,z:U.(x*y)*z=x*(y*z)}
```

Monoid type, as values

```
module type MONOID = sig
  type n
  val plus : n -> n -> n
  val zero : n
end
```

---

[1] *simplified metaocaml for clarity*

Different interpretations of theories [1]

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Monoid := Theory {
 U : type;
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 ax: forall x:U. x*e = x;
 ax: forall x,y,z:U.(x*y)*z=x*(y*z)}
```

Monoid type, as code

```
module type MONOIDCODE = sig
  type n
  type nc = n code
  val plus : nc -> nc -> nc
  val zero : nc
end
```

Monoid type, as values

```
module type MONOID = sig
  type n
  val plus : n -> n -> n
  val zero : n
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---

[1] *simplified metaocaml for clarity*

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Monoid type, as values

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module type MONOID = sig
  type n
  val plus : n -> n -> n
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end
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Monoid type, as code

```
module type MONOIDCODE = sig
  type n
  type nc = n code
  val plus : nc -> nc -> nc
  val zero : nc
end
```

Monoid type, staged

```
type x staged = Now of x
              | Later of x code
module type MONOIDSTAGED = sig
  type n
  type ns = n staged
  val plus : ns -> ns -> ns
  val zero : ns
end
```

---

[1] *simplified metaocaml for clarity*

From syntax to code

```
MonoidTerm := Theory {
  type MTerm = (data X .
    #e : X |
    #* : (X, X) -> X)
}
```

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module type MONOIDSTAGED = sig
  type n
  type ns = n staged
  val zero : ns
  val plus : ns -> ns -> ns
end
```

From syntax to code

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MonoidTerm := Theory {
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module type MONOIDSTAGED = sig
  type n
  type ns = n staged
  val zero : ns
  val plus : ns -> ns -> ns
end
```

Equality is “free”

```
simp :: MTerm -> MTerm
simp t = match t with
  | (#* (a,b)) when a = #e -> b
  | (#* (a,b)) when b = #e -> b
  | _                      -> t
```

Equality is “now”

```
let monoid zero plusN plusL x y =
  match x, y with
  | (Now a), b when a = zero -> b
  | a, (Now b) when b = one -> a
  | _ -> lift2 plusN plusL x y
```

Concrete Monoids

```
module IntM = struct
  type n = Int
  let plus = (+)
  let zero = 0
end
```

```
module IntMC = struct
  type n = Int
  type 'a nc = ('a,n) code
  let plus = .<fun x y -> .~x + .~y>.
  let zero = .< 0 >.
end
```

```
module IntMS = struct
  type n = Int
  type 'a ns = ('a,n) staged
  let plus = monoid IntM.zero IntM.plus IntMC.plus
  let zero = Now IntM.zero
end
```

Concrete Monoids

```
module IntM = struct
  type n = Int
  let plus = (+)
  let zero = 0
end
```

```
module IntMC = struct
  type n = Int
  type 'a nc = ('a,n) code
  let plus = .<fun x y -> .~x + .~y>.
  let zero = .< 0 >.
end
```

```
module IntMS = struct
  type n = Int
  type 'a ns = ('a,n) staged
  let plus = monoid IntM.zero IntM.plus IntMC.plus
  let zero = Now IntM.zero
end
```

Machinery for free

Given a structured graph of theories, one can get a (naïve) optimizing compiler.

MSL

```
Monoid := Theory {
 U : type;
 * : (U,U) -> U;
 e : U;
 axiom right_identity_*_e :
  forall x : U . (x * e) = x
 axiom left_identity_*_e :
  forall x : U . (e * x) = x;
 axiom associativity_* :
  forall x,y,z : U .
   ((x * y) * z) = (x * (y * z));
}
```

Coq

```
Class Monoid {A : type}
 (dot : A -> A -> A)
 (one : A) : Prop := {
 dot_assoc :
   forall x y z : A,
     (dot x (dot y z))
     = dot (dot x y) z
 unit_left :
  forall x, dot one x = x
 unit_right :
  forall x, dot x one = x
}
```

Alternative Definition:

```
Record monoid := {
 dom : Type;
 op : dom -> dom -> dom
  where "x * y" := (op x y);
 id : dom where "1" := id ;
 assoc : forall x y z, x * (y * z) = (x * y) * z;
 left_neutral : forall x, 1 * x = x;
 right_neutral : forall x, x * 1 = x
}.
```

Haskell

```
class Semigroup a => Monoid a where
 mempty :: a
 mappend :: a -> a -> a
 mappend = (<>)
 mconcat :: [a] -> a
 mconcat = foldr mappend mempty
```

Isabelle

```
class semigroup =
 fixes mult :: α => α => α
      (infixl ⊗ 70)
 assumes assoc :: (x ⊗ y) ⊗ z
          = x ⊗ (y ⊗ z)
class monoidl = semigroup +
 fixes neutral :: α (1)
 assumes neutl : 1 ⊗ x = x
class monoid = monoidl +
 assumes x ⊗ 1 = x
```

Lean

```
universe u
variables{ α : Type u }
class monoid (α : Type u) extends
  semigroup α, has_one α :=
  (one_mul : ∀ a : α, 1 * a = a)
  (mul_one : ∀ a : α, a * 1 = a)
```

Agda

```
data Monoid (A : Set)
  (Eq : Equivalence A) : Set
where
 monoid :
  (z : A)
  (_+_ : A -> A -> A)
  (left_Id : LeftIdentity Eq z _+_)
  (right_Id : RightIdentity Eq z _+_)
  (assoc : Associative Eq _+_) ->
  Monoid A Eq
```

Alternative Definition:

```
record Monoid c ℓ :
 Set (suc (c ⊔ ℓ)) where
 infixl 7 _•_
 infix 4 _≈_
 field
  Carrier : Set c
  _≈_ : Rel Carrier ℓ
  _•_ : Op₂ Carrier
  isMonoid :
  IsMonoid _≈_ _•_ ϵ
```

where

```
record IsMonoid (• : Op₂) (ε : A)
   : Set (a ⊔ ℓ) where
 field
  isSemigroup : IsSemigroup •
  identity : Identity ε •

  identityˡ : LeftIdentity ε •
  identityˡ = proj₁ identity

  identityʳ : RightIdentity ε •
  identityʳ = proj₂ identity
```

Universal Algebra...

Most of these work for Generalized Algebraic Theories (à la Cartmell):

- Signature
- Term Algebra
  - "generic functions" (à la *Scrap your Boilerplate*)
  - Structural induction
- Term Algebra parametrized by a "theory" of variables
  - predicate for ground terms
  - "simplifier" for open terms (correct but usually incomplete)
- Homomorphism; homomorphism composition; isomorphism
- kernel of homomorphism
- Theory of congruence relations over a theory
- Induced congruence of a homomorphism
- Interpreter from Term Algebra to any instance of a theory
- Partial evaluator
- Sub-theory, Product Theory, Co-product Theory
- Internalization (making a record that represents a theory)

B

ack down! Given:

- The theory presentation of 2-categories, can you specialize to category?
- Category to monoid?
- Monoidal Category to... monoid?
- Braided Category to... ?

- How do you (as a library builder) not repeat yourself,
- while giving end-users a huge, rich, as-they-expect it to look library?

Computer Science?

Axiomatic presentations of the theories of:

- Data Structures

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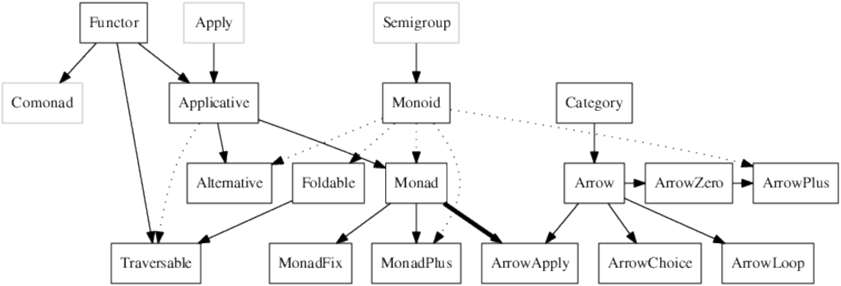
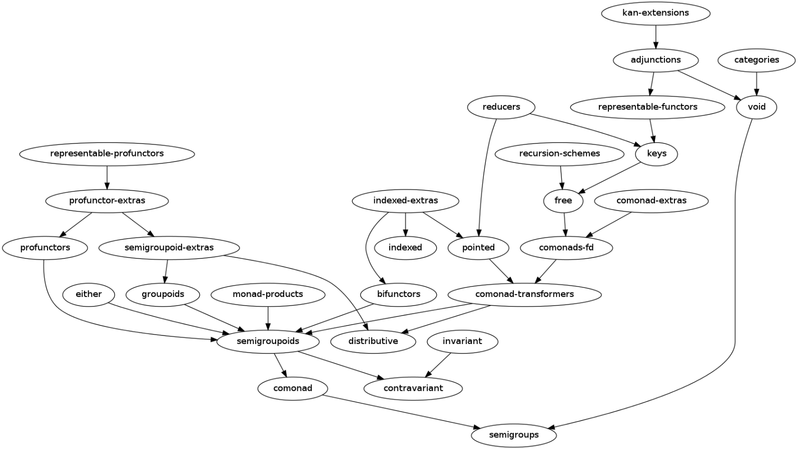


Computer Science?

Axiomatic presentations of the theories of:

- Data Structures



Computer Science?

Axiomatic presentations of the theories of:

- Data Structures
- Algorithms
  - Douglas Smith's *SpecWare*
  - Ralf Hinze's (et al.)'s *recursion schemes* extracted from categorical adjustion and/or Kan extensions.
- Models of Computation
  - FSM, DPDA, TM, 23 Registers Machines, SECD
  - ongoing work with Ph.D. student Lijun Zhu