Symbolic Evaluation of Hadamard-Toffoli Quantum Circuits

Jacques Carette, Amr Sabry, Gerardo Ortiz

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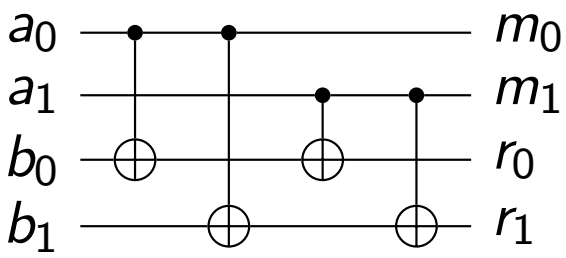
McMaster University

The Idea

Traditional partial evaluation and symbolic evaluation techniques apply to an interesting class of quantum circuits with surprising effectiveness!

Reversible gates



$$\begin{aligned} \mathrm{CX}(0, b) &= (0, b) \\ \mathrm{CX}(1, b) &= (1, \overline{b}) \end{aligned}$$

So, writing inputs/outputs horizontally:

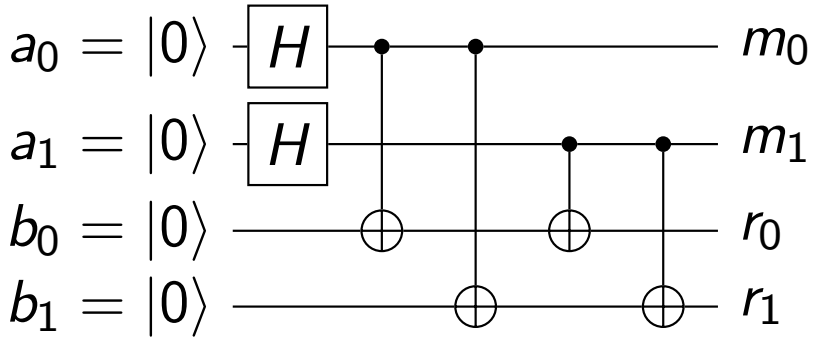
$$\begin{aligned} |0111\rangle &\Rightarrow |0111\rangle \\ &\Rightarrow |0111\rangle \\ &\Rightarrow |0101\rangle \\ &\Rightarrow |0100\rangle \end{aligned}$$

$$\begin{aligned} |1010\rangle &\Rightarrow |1000\rangle \\ &\Rightarrow |1001\rangle \\ &\Rightarrow |1001\rangle \\ &\Rightarrow |1001\rangle \end{aligned}$$

$$\begin{aligned} |1100\rangle &\Rightarrow |1110\rangle \\ &\Rightarrow |1111\rangle \\ &\Rightarrow |1101\rangle \\ &\Rightarrow |1100\rangle \end{aligned}$$

$$\begin{aligned} |1111\rangle &\Rightarrow |1101\rangle \\ &\Rightarrow |1100\rangle \\ &\Rightarrow |1110\rangle \\ &\Rightarrow |1111\rangle \end{aligned}$$

Superpositions

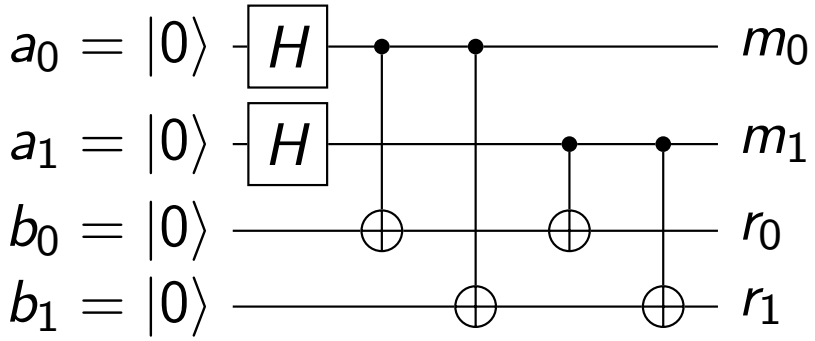


$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

So the top two wires are in the state

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle)$$

Superpositions



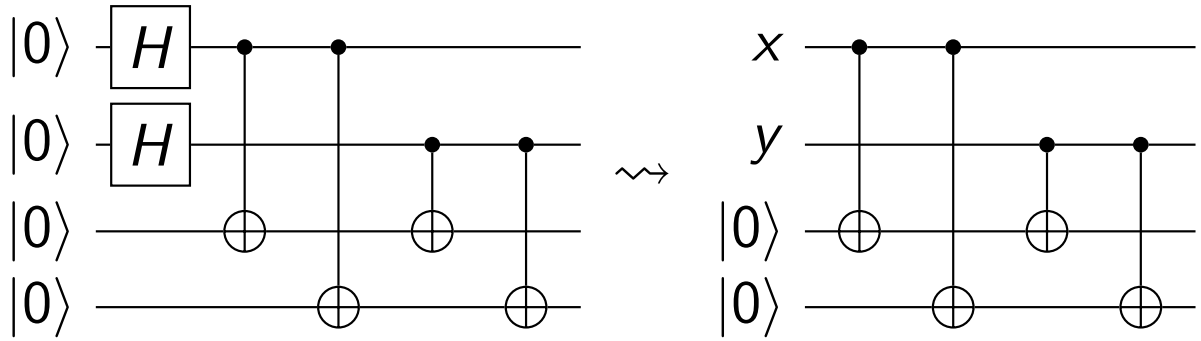
$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Thus for the whole circuit:

$$\begin{aligned}
|0000\rangle &\Rightarrow \frac{1}{2}(|0000\rangle + |0100\rangle + |1000\rangle + |1100\rangle) \\
&\Rightarrow \frac{1}{2}(|0000\rangle + |0100\rangle + |1010\rangle + |1110\rangle) \\
&\Rightarrow \frac{1}{2}(|0000\rangle + |0100\rangle + |1011\rangle + |1111\rangle) \\
&\Rightarrow \frac{1}{2}(|0000\rangle + |0110\rangle + |1011\rangle + |1101\rangle) \\
&\Rightarrow \frac{1}{2}(|0000\rangle + |0111\rangle + |1011\rangle + |1100\rangle)
\end{aligned}$$

Symbolic Execution (new)

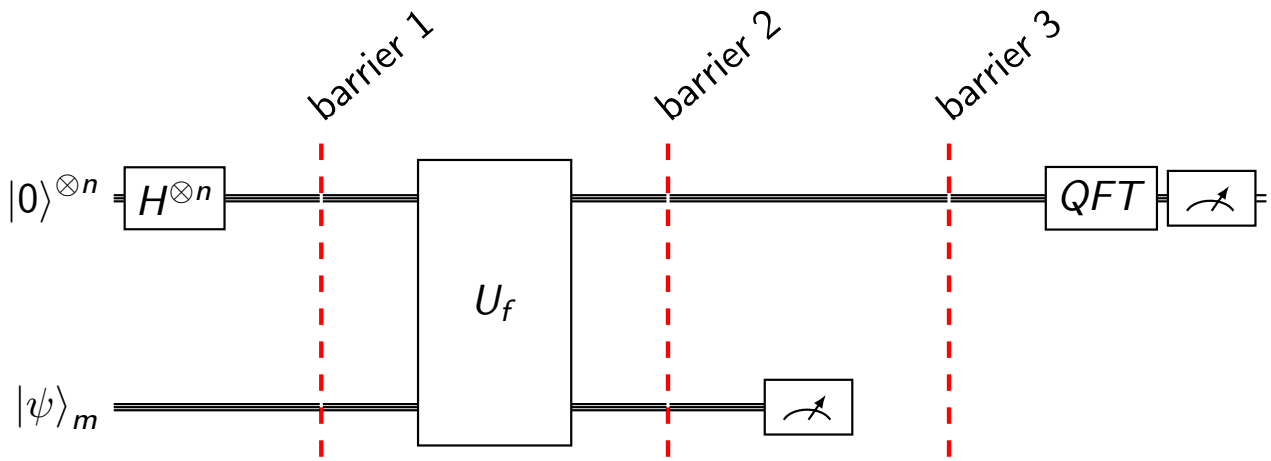
Replace $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ by a symbolic variable:



Maintain formula in algebraic normal form (ANF).
So

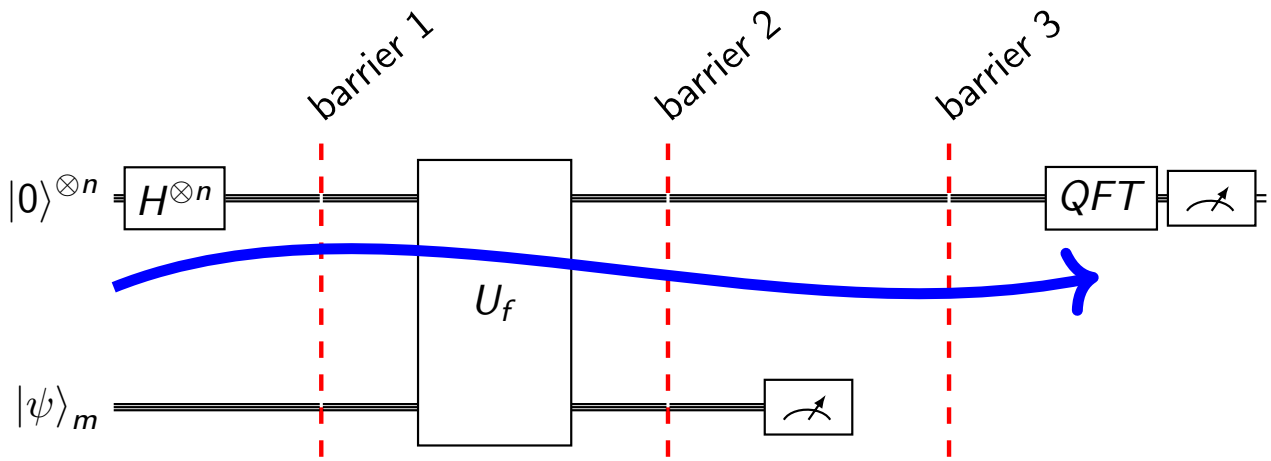
$$\begin{aligned} |xy00\rangle &\Rightarrow |xyx0\rangle \\ &\Rightarrow |xyxx\rangle \\ &\Rightarrow |xy(x \oplus y)x\rangle \\ &\Rightarrow |xy(x \oplus y)(x \oplus y)\rangle \end{aligned}$$

Quantum Circuits



- U_f is a *classical reversible* circuit representing f, i.e. $U_f(\mathbf{x}, \mathbf{y}) = |\mathbf{x} \quad (\mathbf{y} \oplus \mathbf{f}(\mathbf{x}))\rangle$.
- H is the Hadamard gate; introduces *quantum parallelism* to evaluate U_f for many inputs simultaneously.
- QFT is the Quantum Fourier Transform used to analyze the spectral properties of the output.

Quantum Circuits



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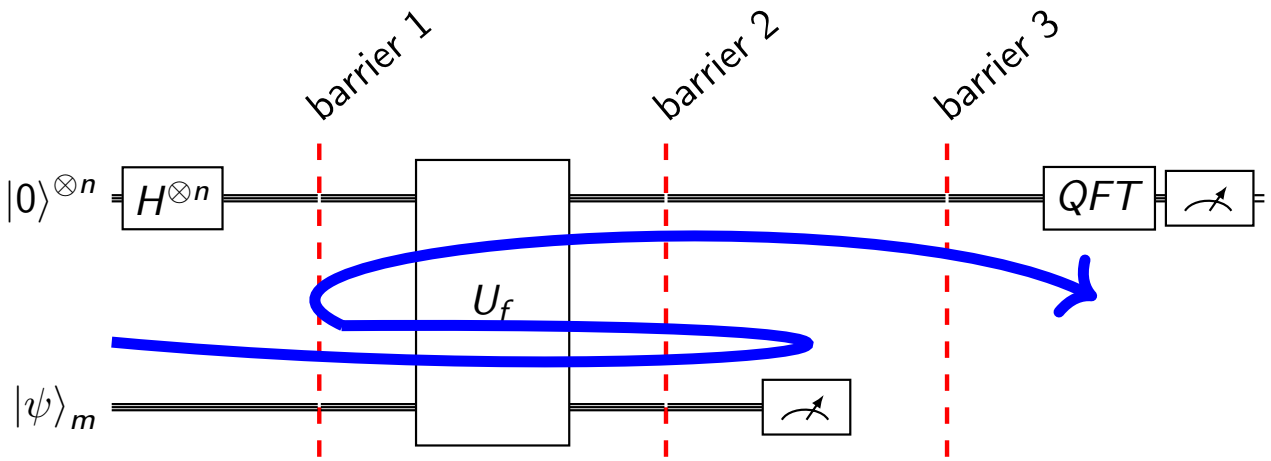
Quantum Circuits Retrodictive Evaluation



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Examples

1. Deutsch
2. Deutsch-Jozsa
3. Bernstein-Varizani
4. Simon
5. Grover
6. Shor

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Examples: Deutsch and Deutsch-Josza

Definition

A boolean function is **balanced** *if it outputs the same number of 0/1 outputs.*

Deutsch:

Problem

Given $f : \mathbb{B} \to \mathbb{B}$, *decide if* f *is constant or balanced.*

Deutsch-Josza:

Problem

Given $f : \mathbb{B}^n \to \mathbb{B}$, *where* f *is known to be constant or balanced, decide which one it is.*

Examples: Grover and Shor

Grover:

Problem

Given $f : \mathbb{B}^n \to \mathbb{B}$ *where there exists a unique* u *such that* $f(u) = 1$. *Find* u.

Shor:

Problem

Factor a given N. *Do this by using* $f(x) = a^x \mod N$ *for suitable* a *and* $f : \mathbb{B}^Q \to \mathbb{B}^n$ *with* $Q = \lceil \log_2 (N^2) \rceil$, $n = \lceil \log_2 N \rceil$.

Requirements

Variabilities:

1. multiple representations of boolean values,
2. multiple representations of boolean formulae,
3. different evaluation means (directly, symbolically, forwards, backwards, retrodictive).

Possible to implement:

4. a reusable representation of our circuits,
5. a reusable representation of the inputs, outputs and ancillae,
6. a synthesis algorithm for $\mathbb{B}^n \to \mathbb{B}$ functions
7. a reusable library of circuits

Also, non-functional characteristics to hold:

8. evaluation of reasonably-sized circuits should be reasonably efficient.

Design


```
class (Show v, Enum v) => Value v where
  zero :: v
  one  :: v
  snot :: v -> v
  sand :: v -> v -> v
  sxor :: v -> v -> v

  -- has a default implementation
  snand :: [v] -> v -- n-ary and
  snand = foldr sand one
```

Implemented four times.

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Implemented four times.

```
data VarInFormula f v = FR
  { fromVar  :: v -> f
  , fromVars :: Int -> v -> [ f ]
  }
```

Design

First, naïve implementation:

```
newtype Ands = Ands { lits :: [String] }
  deriving (Eq, Ord)

(&&&) :: Ands -> Ands -> Ands
(Ands lits1) &&& (Ands lits2) =
  Ands (lits1 ++ lits2)

newtype Formula = Formula { ands :: [Ands]}
```

Design

Second, using sets of ints and occurence maps:

```
type Literal = Int
newtype Ands = Ands { lits :: IS.IntSet }

{-# INLINABLE compAnds #-}
compAnds :: Ands -> Ands -> Ordering
compAnds (Ands a1) (Ands a2) =
  compare (IS.toAscList a1) (IS.toAscList a2)

(&&&) :: Ands -> Ands -> Ands
(Ands lits1) &&& (Ands lits2) =
  Ands (IS.union lits1 lits2)

-- raw XOR formulas
type XORFU = Map.Map Ands Int
-- Normalized XOR formulas, i.e occur 0 or 1 time
newtype Formula = Formula { ands :: MS.MultiSet Ands }
```

Design

Third, using bitmaps and occurence maps:

```
type Literal = Natural
newtype Ands = Ands { lits :: Literal }

{-# INLINABLE compAnds #-}
compAnds :: Ands -> Ands -> Ordering
compAnds (Ands a1) (Ands a2) = compare a1 a2

(&&&) :: Ands -> Ands -> Ands
(Ands lits1) &&& (Ands lits2) = Ands (lits1 .|. lits2)

type Occur = Int
-- Raw XOR formulas
type XORFU = Map.Map Ands Occur
-- Normalized XOR formulas, i.e occur 0 or 1 time
type XORF = MS.MultiSet Ands
```

Running Deutsch-Josza

- Retrodictive once on output measurement 0
- Output symbolic formula that decides $f(|\mathbf{x}\rangle) = 0$.
- f constant $\Rightarrow 0 = 0$ or $1 = 0$ regardless of circuit size.

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Sample outputs:

- $x_0 = 0$,
- $x_0 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 = 0$, and
- $1 \oplus x_3x_5 \oplus x_2x_4 \oplus x_1x_5 \oplus x_0x_3 \oplus x_0x_2 \oplus x_3x_4x_5 \oplus x_2x_3x_5 \oplus x_1x_3x_5 \oplus x_0x_3x_5 \oplus x_0x_1x_4 \oplus x_0x_1x_2 \oplus x_2x_3x_4x_5 \oplus x_1x_3x_4x_5 \oplus x_1x_2x_4x_5 \oplus x_1x_2x_3x_5 \oplus x_0x_3x_4x_5 \oplus x_0x_2x_4x_5 \oplus x_0x_2x_3x_5 \oplus x_0x_1x_4x_5 \oplus x_0x_1x_3x_5 \oplus x_0x_1x_3x_4 \oplus x_0x_1x_2x_4 \oplus x_0x_1x_2x_4x_5 \oplus x_0x_1x_2x_3x_5 \oplus x_0x_1x_2x_3x_4 = 0$.

How to decide? If it mentions a variable, it's balanced.
We tested all 12872 functions $\mathbb{B}^6 \to \mathbb{B}$.

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How to decide? If it mentions a variable, it's balanced.
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Running Grover

Recall: guess the position of the single 1 bit.

$n = 4$, w in the range $\{0..15\}$

$u = 0$ $1 \oplus x_3 \oplus x_2 \oplus x_1 \oplus x_0 \oplus x_2x_3 \oplus x_1x_3 \oplus x_1x_2 \oplus x_0x_3 \oplus x_0x_2 \oplus x_0x_1 \oplus x_1x_2x_3 \oplus x_0x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 1$ $x_0 \oplus x_0x_3 \oplus x_0x_2 \oplus x_0x_1 \oplus x_0x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 2$ $x_1 \oplus x_1x_3 \oplus x_1x_2 \oplus x_0x_1 \oplus x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 3$ $x_0x_1 \oplus x_0x_1x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 4$ $x_2 \oplus x_2x_3 \oplus x_1x_2 \oplus x_0x_2 \oplus x_1x_2x_3 \oplus x_0x_2x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 5$ $x_0x_2 \oplus x_0x_2x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 6$ $x_1x_2 \oplus x_1x_2x_3 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 7$ $x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 8$ $x_3 \oplus x_2x_3 \oplus x_1x_3 \oplus x_0x_3 \oplus x_1x_2x_3 \oplus x_0x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2x_3$

$u = 9$ $x_0x_3 \oplus x_0x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2x_3$

$u = 10$ $x_1x_3 \oplus x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_1x_2x_3$

$u = 11$ $x_0x_1x_3 \oplus x_0x_1x_2x_3$

$u = 12$ $x_2x_3 \oplus x_1x_2x_3 \oplus x_0x_2x_3 \oplus x_0x_1x_2x_3$

$u = 13$ $x_0x_2x_3 \oplus x_0x_1x_2x_3$

$u = 14$ $x_1x_2x_3 \oplus x_0x_1x_2x_3$

$u = 15$ $x_0x_1x_2x_3$

Shor

Problem

Factor a given N. Do this by using $f(x) = a^x \mod N$ for suitable a and $f : \mathbb{B}^Q \to \mathbb{B}^n$ with $Q = \lceil \log_2 (N^2) \rceil$, $n = \lceil \log_2 N \rceil$.

Factoring 15:

Base	Equations			Solution
$a = 11$	$x_0 = 0$			$x_0 = 0$
$a = 4, 14$	$1 \oplus x_0 = 1$	$x_0 = 0$		$x_0 = 0$
$a = 7, 13$	$1 \oplus x_1 \oplus x_0 x_1 = 1$	$x_0 x_1 = 0$	$x_0 \oplus x_1 \oplus x_0 x_1 = 0$	$x_0 \oplus x_0 x_1 = 0$	$x_0 = x_1 = 0$
$a = 2, 8$	$1 \oplus x_0 \oplus x_1 \oplus x_0 x_1 = 1$	$x_0 x_1 = 0$	$x_1 \oplus x_0 x_1 = 0$	$x_0 \oplus x_0 x_1 = 0$	$x_0 = x_1 = 0$

Auto-generated circuits: 56,538 generalized Toffoli gates.

Shor

Problem

Factor a given N. Do this by using $f(x) = a^x \mod N$ for suitable a and $f : \mathbb{B}^Q \to \mathbb{B}^n$ with $Q = \lceil \log_2 (N^2) \rceil$, $n = \lceil \log_2 N \rceil$.

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$a = 4, 14$	$1 \oplus x_0 = 1$	$x_0 = 0$		$x_0 = 0$
$a = 7, 13$	$1 \oplus x_1 \oplus x_0 x_1 = 1$	$x_0 x_1 = 0$	$x_0 \oplus x_1 \oplus x_0 x_1 = 0$	$x_0 \oplus x_0 x_1 = 0$	$x_0 = x_1 = 0$
$a = 2, 8$	$1 \oplus x_0 \oplus x_1 \oplus x_0 x_1 = 1$	$x_0 x_1 = 0$	$x_1 \oplus x_0 x_1 = 0$	$x_0 \oplus x_0 x_1 = 0$	$x_0 = x_1 = 0$

Auto-generated circuits: 56,538 generalized Toffoli gates.

For 3*65537=196611 (4,328,778 gates), 16 small equations that refer to just the four variables x_0, x_1, x_2, and x_3 constraining them to be all 0, i.e., asserting that the period is 16.

Timings: Deutsch-Jozsa

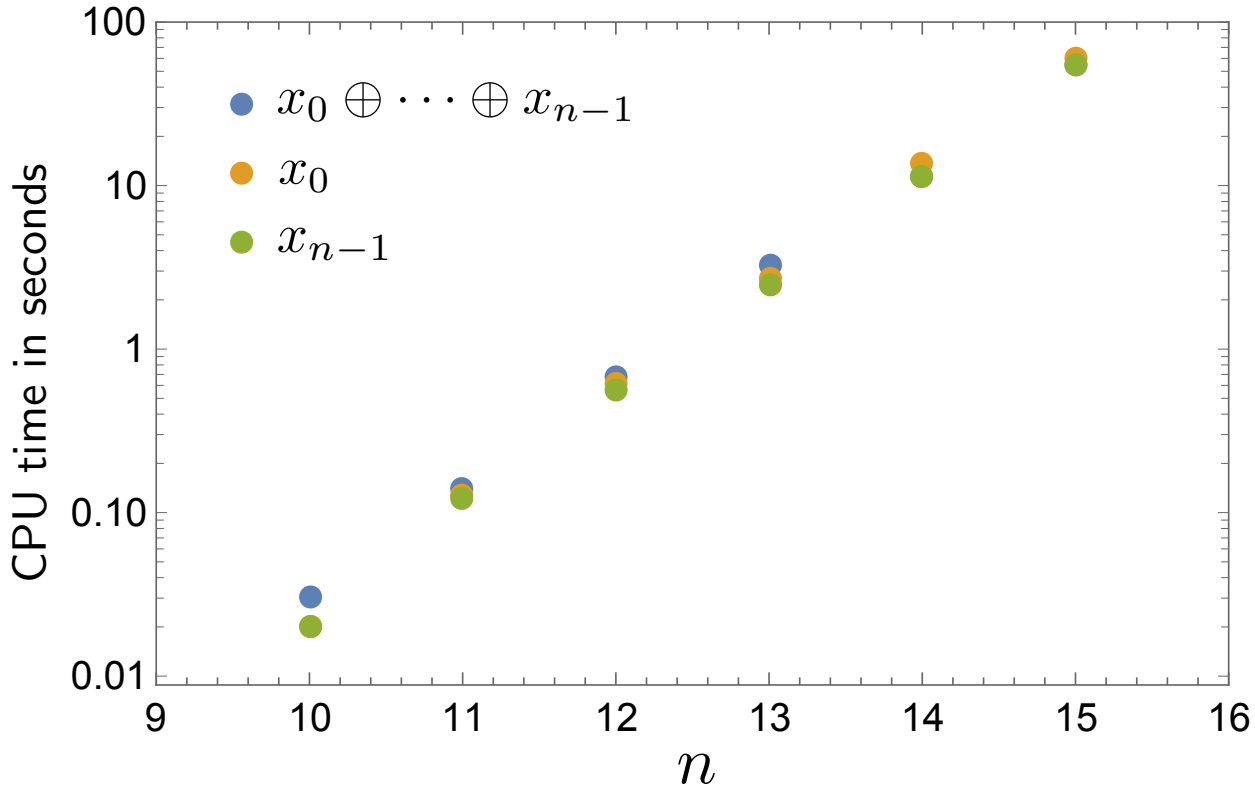

Figure: Retrodictive execution on 3 balanced functions at different sizes

Timings: Grover on different values

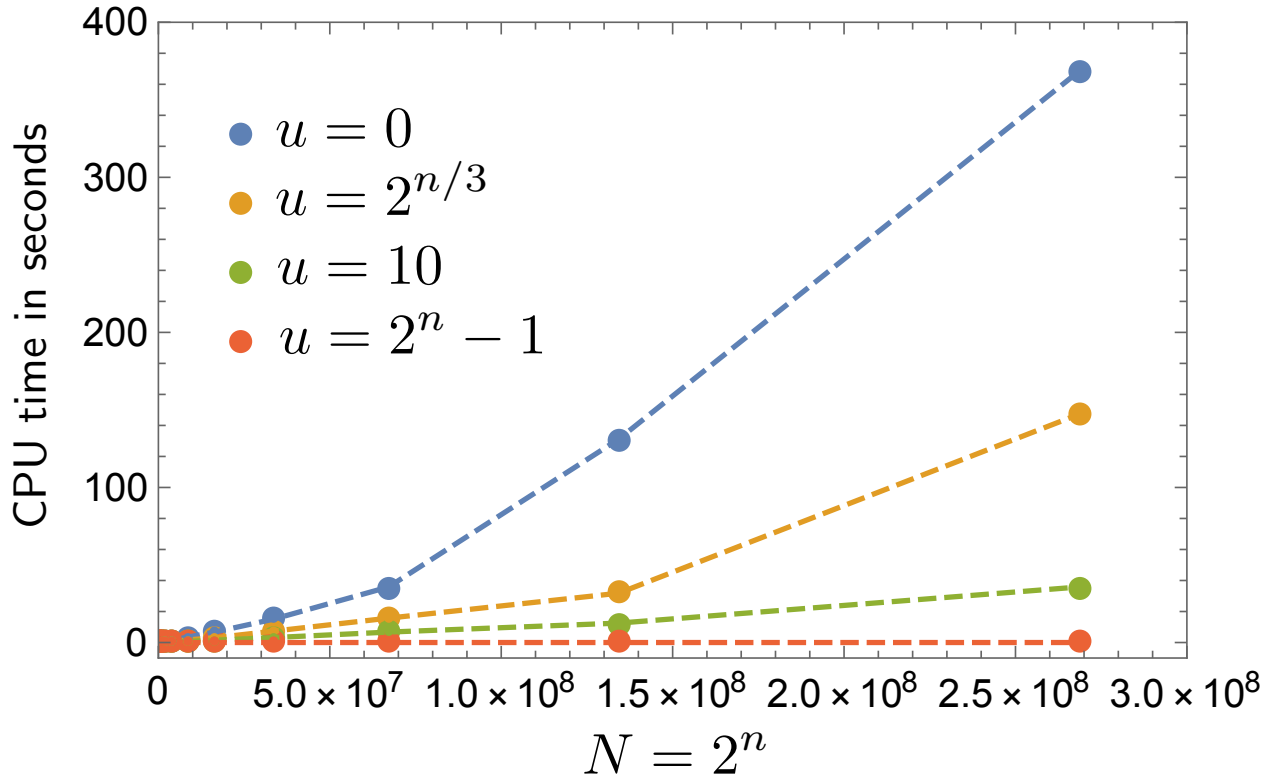

Figure: Retrodictive execution of the Grover algorithm on different "secret" values u at different sizes

Timings: Grover with different representations

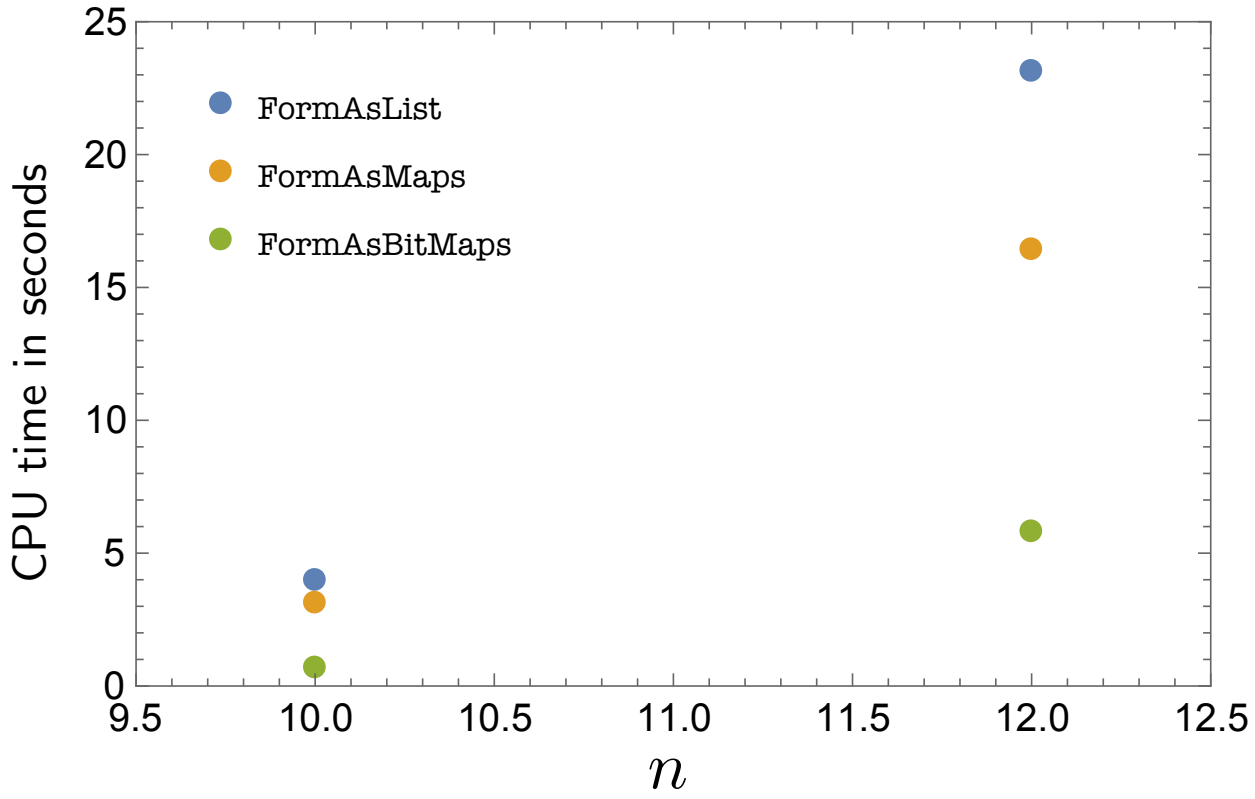

Figure: Retrodictive execution of the Grover algorithm on secret value $u = 0$ at two sizes but using different ANF representations

Complexity

Problem: circuit with T generalized Toffoli gates over $n+m$ qubits split into two registers (A,B).

1. Design oracle / circuit.
2. Let $A = |00\ldots0\rangle$ and $B = |00\ldots0\rangle$ and run classically. $\mathcal{O}(T)$. Leaves A intact, produce b in B.
3. Run backwards (symbolically) with $A = |x_{n-1}\ldots x_1 x_0\rangle$ and $B = |b\rangle$. Worst case, equations sized $\mathcal{O}(2^n)$, so $\mathcal{O}(Tm2^n)$.
4. Answer by inspecting/solving resulting m equations.

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Bottlenecks:

- step (3) at worst case $\mathcal{O}(Tm2^n)$,
- step (4) "solve", which is NP-complete.

However:

- Sometimes 'typical' case has expected run-time depending on bit-size of the information contained in the answer.

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Conclusion

- Different algorithms need different analyses to extract answer.
- (Not shown in slides) Shor seems to need qutrits/qudits in general.
- For a number of classical quantum algorithms, classical but symbolic execution works just as well.
- Even better: symbolic execute once instead of "enough" times to get probabilistic answer.
- Symbolic execution sometimes is really bad.

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- Symbolic execution sometimes is really bad.

- Speculation: some common quantum algorithms might not need quantum after all.

Cost Models

- White-box model: *We* implement U_f and the cost of implementing it and the cost of using it is counted as part of the overall complexity
- Black-box model: "Someone" implements U_f and gives us access to it; the complexity analysis only counts the number of times U_f is used. There are different cases based on what kind of access we are given:
  - Query U_f on a classical input
  - Query U_f on a quantum superposition
  - Query U_f on a *symbolic formula* (**NEW!**)