Probabilistic inference by program transformation in Hakaru (system description)*

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Abstract. We present Hakaru, a new probabilistic programming system that allows composable reuse of distributions, queries, and inference algorithms, all expressed in a single language of measures. The system implements two automatic and semantics-preserving program transformations—*disintegration*, which calculates conditional distributions, and *simplification*, which subsumes exact inference by computer algebra. We show how these features work together by describing the ideal workflow of a Hakaru user on two small problems. We highlight our composition of transformations and types in design and implementation.

1 Introduction

To perform probabilistic inference is to answer a query about a probability distribution. The longstanding enterprise of probabilistic programming aims to perform probabilistic inference in a modular way, so that the distribution, query, and inference algorithm can be separately reused, composed, and modified.

The modularity we envision is motivated by the typical machine-learning paper published today. Often the first section presents a problem, the second section presents a distribution and query, and the third section presents an inference algorithm that answers the particular query for the particular distribution. Just as the second section composes its content using words such as "mixture" and "condition", the third section composes its content using words such as "proposal" and "integrate out". From this description using English and math, a person skilled in the art of probabilistic inference can write the specialized code that reproduces the results of the paper.

We aim to automate this code-generation task, so that changes to programs that perform probabilistic inference become easier to try out and carry out. We focus on making inference compositional—that is, on making the third section of the typical machine-learning paper executable—because less is known about it.

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Contributions Hakaru is a new, proof-of-concept probabilistic programming system that achieves unprecedented modularity by two means:

1. a language of measures that represents distributions and queries as well as inference algorithms;
2. semantics-preserving program transformations based on computer algebra.

The two main transformations are

1. *disintegration*, which calculates conditional distributions and probability densities, and
2. *simplification*, which subsumes exact inference and supports approximate inference, by making use of Maple.

All our transformations take input and produce output in the same language, so we can compose them to express inference algorithms.

This paper shows how these features work together by describing the ideal workflow of a Hakaru user on two small problems.

2 Inference example on a discrete model

In Pearl's classic textbook on probabilistic reasoning [9], Example 1 (page 35) begins as follows:

> Imagine being awakened one night by the shrill sound of your burglar alarm. What is your degree of belief that a burglary attempt has taken place?

The workflow of a Hakaru user is that of Bayesian inference:

1. Model the world as a *prior* probability distribution on what is observed (alarm or not) and what is to be inferred (burglary or not). Hakaru defines a language of distributions that formalizes this modeling.
2. Turn the prior into a *conditional* distribution, which is a function that maps what is observed to a distribution on what is to be inferred. Hakaru provides transformations that automate this conditioning.
3. Apply the function to what is actually observed (true, the alarm did sound) to get the *posterior* distribution on what is to be inferred (burglary or not, given that the alarm did sound). Hakaru can show the distribution not only by generating a stream of samples, but also as a term in the language.

2.1 Modeling

We start with an example of step 1. The prior distribution given by Pearl can be expressed as the following Hakaru term:

```
model :: (Mochastic repr) => repr (HMeasure (HPair HBool HBool))
model = bern 0.0001 `bind` \burglary ->
        bern (if_ burglary 0.95 0.01) `bind` \alarm ->
        dirac (pair alarm burglary)
```

Hakaru is a language embedded in Haskell in "finally tagless" form [1], so the code above is actually parsed and type-checked by the Haskell compiler GHC. In the type signature, `(Mochastic repr) => repr` is due to the finally-tagless embedding, `HBool` is Hakaru's boolean type, `HPair` is Hakaru's product type constructor, and `HMeasure` turns a type of values into a type of distributions. The type constructor `HMeasure` is a monad [3, 10], whose unit and bind operations are spelled `dirac` and `bind`. In this embedding style the types of `dirac` and `bind` do not unify with Haskell's `return` and `>>=`. Although we could use `-XRebindableSyntax` to obtain `do` notation, we avoid doing so in this work.

As usual, the monad `HMeasure` is made interesting by the primitive operations Hakaru provides for it. In the `model` above, `bern 0.0001` is the distribution on booleans that is `true` 0.01% of the time and `false` the other 99.99% of the time. Its type is `(Mochastic repr) => repr (HMeasure HBool)`. The boolean randomly produced by this distribution is fed to Hakaru's `if_`, which models how `burglary` influences `alarm`.

Besides reading it, another way to understand this `model` is to run it as a sampler. Each run produces a `pair` of booleans: `runSample model` usually prints `Just (False,False)`. The sampler chooses `burglary`, the second component of the `pair`, followed by `alarm`, the first component.

2.2 Conditioning

To answer Pearl's question, we should focus on the portion of the distribution where `alarm` is `true`. We could run `model` over and over as a sampler and collect only the samples where `alarm` is `true`, but the vast majority of samples would have `alarm` be `false`, so it would take a long time to gather enough samples to answer Pearl's question with any accuracy. Instead, we move to step 2 of our workflow. We apply Hakaru's `disintegrate` transformation to obtain the `conditional` distribution of `burglary` *given* `alarm`:

```
conditional :: (Mochastic repr, Lambda repr)
            => repr (HBool :-> HMeasure HBool)
conditional = disintegrate model
```

In general, Hakaru's `disintegrate` transformation turns a Hakaru program of type `HMeasure (HPair a b)` into a Hakaru function (`:->`) from `a` to `HMeasure b`. The particular function produced in this case is shown in Figure 1. This is produced by Hakaru's pretty-printer.

The generated code is large and full of reducible expressions, such as

```
superpose [(1, weight (19/20) $
               weight (1/10000) $
               dirac true),
           (1, weight (1/100) $
               superpose [])]
```

To explain what this expression means and how it can be reduced, we need to describe the semantics of Hakaru.

```
lam $ \x1 ->
superpose [(1,
            superpose [(1,
                        superpose [(1,
                                    superpose [(1,
                                                if_ x1
                                                    (superpose [(1,
                                                                 weight (19/20) $
                                                                 weight (1/10000) $
                                                                 dirac true),
                                                                (1,
                                                                 weight (1/100) $ superpose [])])
                                                    (superpose [])),
                                               (1,
                                                superpose [(1, superpose []),
                                                           (1,
                                                            if_ x1 (superpose []) (superpose []))])]),
                                   (1,
                                    superpose [(1, if_ x1 (superpose []) (superpose [])),
                                               (1,
                                                superpose [(1, superpose []),
                                                           (1,
                                                            if_ x1
                                                                (superpose [])
                                                                (superpose [(1,
                                                                             weight (1/20) $
                                                                             weight (1/10000) $
                                                                             dirac true),
                                                                            (1,
                                                                             weight (99/100) $
                                                                             superpose [])])])])])]),
                       (1,
                        superpose [(1,
                                    superpose [(1,
                                                if_ x1
                                                    (superpose [(1, weight (19/20) $ superpose []),
                                                                (1,
                                                                 weight (1/100) $
                                                                 weight (9999/10000) $
                                                                 dirac false)])
                                                    (superpose [])),
                                               (1,
                                                superpose [(1, superpose []),
                                                           (1,
                                                            if_ x1 (superpose []) (superpose []))])]),
                                   (1,
                                    superpose [(1, if_ x1 (superpose []) (superpose [])),
                                               (1,
                                                superpose [(1, superpose []),
                                                           (1,
                                                            if_ x1
                                                                (superpose [])
                                                                (superpose [(1,
                                                                             weight (1/20) $
                                                                             superpose []),
                                                                            (1,
                                                                             weight (99/100) $
                                                                             weight (9999/10000) $
                                                                             dirac false)])])])])]),
           (1, superpose [])]
```

Fig. 1: The output of disintegrating the burglary model

A Hakaru program of `HMeasure` type can be understood in two ways: as an *importance sampler* and as a *measure*. An importance sampler is a random procedure that produces an outcome along with a weight (or fails, which is like producing the weight 0). A weight is a non-negative real number. For example, `dirac true` produces the outcome `true` with the weight 1. In general, `dirac` always produces the weight 1, whereas the weight produced by `m` `` `bind` `` `\x -> k x` is the product of the weights produced by `m` and `k x`. The weight produced by `weight (1/10000) $ m` is 1/10000 times the weight produced by `m`. A typical use of an importance sampler is to run it repeatedly while maintaining a running weighted average of some function of the outcome.

The syntax of `superpose` in Hakaru is that it takes a list of weight-measure pairs $[(w_1, m_1), \ldots, (w_n, m_n)]$ and produces a measure. What a sampler built with `superpose` does is to choose one of the measures m_i, with probability proportional to the weights w_i, and sample from it. The weight produced by `superpose` is $\sum_{i=1}^{n} w_i$ times the weight produced by m_i. If the list is empty (that is, $n = 0$) then `superpose` simply fails. Hence, the fragment displayed above produces the outcome `true` with weight $(1 + 1) \cdot (19/20) \cdot (1/10000)$ half of the time, and fails the other half of the time.

This sampler is *not* preserved when we simplify a Hakaru expression. What is preserved is the measure. A measure is like a probability distribution, but it doesn't necessarily sum to 1, thanks to `weight` and `superpose` in the language. The measure denoted by `weight (1/10000) $ m` is the measure denoted by `m` scaled by 1/10000. And `superpose` represents a linear combination of measures, so `superpose []` denotes the zero measure.

The following equations on measures are valid, like in linear algebra:

```
weight w $ superpose []                = superpose []
weight w $ weight w' $ m               = weight (w * w') $ m
superpose [(w, m), (w', m)]            = weight (w + w') $ m
superpose [(w, m), (w', superpose [])] = weight w $ m
```

Consequently, the fragment displayed above denotes the same measure as the simpler expression: `weight ((19/20) * (1/10000)) $ dirac true`.

This latter program as a sampler always produces the outcome `true` with weight $(19/20) \cdot (1/10000)$. This behavior is not the same but *better*, because a running weighted average would converge to the same result more quickly when we don't throw away half of our samples.

Instead of simplifying Figure 1 by hand, we can apply Hakaru's `simplify` transformation:

```
simplified = simplify conditional
```

The result produced by Hakaru's pretty-printer is:

```
lam $ \x1 ->
if_ x1
    (superpose [(19/200000, dirac true), (9999/1000000, dirac false)])
    (superpose [(1/200000, dirac true), (989901/1000000, dirac false)])
```

This result both runs more efficiently (because again, it fails less often) and reads more easily (because again, it is shorter). It is a Hakaru function (constructed using `lam`) that takes the observed `alarm` as input and returns a measure on `burglary`. In this instance, Hakaru performed linear-algebra-like reductions with the help of the computer algebra system Maple, and produced a compact representation of the conditional distribution. In this sense, simplification subsumes exact inference. We can easily read off that, if we `apply` this function to `true` in step 3 of our workflow, then we would get a posterior distribution that is 19/200000 parts burglary and 9999/1000000 parts no burglary. (These numbers do not sum to 1, nor do we expect them to.)

2.3 Sampling

The simplified posterior is a Hakaru program and it can be run as a sampler: `runSample (app simplified true)` usually prints `Just False`.

3 Inference example on a continuous model

We now turn to an example that involves random real numbers. Imagine the task of building thermometers to measure room temperatures. To build a reliable thermometer we would like to calibrate two attributes of the device. The first attribute is the amount of temperature noise—how much the room temperature fluctuates over time. The second attribute is the amount of measurement noise—how much the thermometer measures the same actual temperature as a different value each time it is used because of its imperfections. Calibrating a thermometer would mean approximating these attributes as best as possible. We can then build a thermometer that corrects its measurements with accurate knowledge of the temperature noise and measurement noise.

3.1 Modeling

To perform our calibration task, we would like to make several experimental measurements and then infer the noises from these measurements. We need step 2 of our workflow to produce a conditional distribution on noises given measurements. So we begin in step 1 by defining a distribution on pairs of measurements and noises:

```
thermometer :: (Mochastic repr)
            => repr (HMeasure (HPair (HPair HReal HReal)
                                     (HPair HProb HProb)))
thermometer =
    liftM unsafeProb (uniform 3 8) `bind` \noiseT ->
    liftM unsafeProb (uniform 1 4) `bind` \noiseM ->
    normal 21 noiseT `bind` \t1 ->
    normal t1 noiseM `bind` \m1 ->
    normal t1 noiseT `bind` \t2 ->
    normal t2 noiseM `bind` \m2 ->
    dirac (pair (pair m1 m2) (pair noiseT noiseM))
```

The type of `thermometer` shows it is a measure on pairs. The first component of the pair has type `HPair HReal HReal`. That is, we take only two measurements in this simplistic model. To determine the noises accurately, we should use thousands of measurements, not just two. That is why we need to add arrays to Hakaru. In order to handle arrays, the simplification and disintegration program transformations would have to be modified, which is the focus of ongoing work. Here we describe a system without container data-types and show its use on examples having a low number of dimensions.

The second component is a pair of non-negative reals, which are denoted by the `HProb` type in Hakaru. The `HProb` type is like `HReal` but records the knowledge that the number is non-negative. This knowledge is useful in at least two ways. First, knowing that `noiseT` is positive helps the simplification transformation produce `noiseT` instead of `sqrt(noiseT^2)`. Second, during sampling an `HProb` number is typically a probability and represented by its log in floating point. This alleviates the common probabilistic computation problem of underflow errors in extremely small probabilities.

In `thermometer` we express prior beliefs about how noises are distributed, which are seen in the calls to `uniform`. Often such beliefs about distributions (and their parameters such as $3, 8$ and $1, 4$ above) come from domain knowledge. Furthermore, we model temperatures and measurements as being Gaussian distributed with some noisy perturbations. This is expressed in Hakaru by `normal`:

```
normal :: (Mochastic repr) => repr HReal -> repr HProb -> repr (HMeasure HReal)
normal mu sd = lebesgue `bind` \x ->
               superpose [( exp_ (- (x - mu)^2
                                  / fromProb (2 * pow_ sd 2))
                             / sd / sqrt_ (2 * pi_)
                          , dirac x )]
```

The first argument to `normal` is the mean of the Gaussian distribution. The second argument is the standard deviation, which must be non-negative, as the type `HProb` above shows. Actually, it has to be positive. The term `lebesgue` denotes the Lebesgue measure on the reals.

Besides expressing these distributions, we model the network of influences among the random variables. First we draw the candidate noise values `noiseT` and `noiseM` from their respective distributions. We want to use these noises as standard deviations for the `normal` distributions. However, the `uniform` distribution is over `HReal` values since uniform distributions can, in general, produce negative real numbers. But, because the parameters to the `uniform` distributions are positive, we know that values drawn from them must in fact be positive. Thus, we use the `unsafeProb` construct (which is safe in this case) to produce the `HProb`-typed `noiseT` and `noiseM`.

The initial room temperature `t1` is centered around 21°C, and the later temperature `t2` is centered around `t1`. Both are drawn with standard deviation `noiseT`. We can think of this as a random walk starting at 21°C. Finally, measurements `m1` and `m2` are taken of each temperature, with standard deviation `noiseM`. These dependencies amount to a hidden Markov model, more specifically a linear dynamic model in one dimension.

3.2 Conditioning

For our inference goal we need to obtain a conditional distribution on the noises given the measurements. We can get it by applying the disintegration transformation, as in Section 2. A conditional distribution on noises given measurements is a function from measurements to a distribution on noises. This is precisely what the function type of `thermConditional` says.

```
thermConditional :: (Mochastic repr, Lambda repr)
                 => repr (HPair HReal HReal
                         :-> HMeasure (HPair HProb HProb))
thermConditional = disintegrate thermometer
```

Hakaru's pretty-printer, used on `thermConditional`, produces Figure 2. Of course, this expression is different from `thermometer`. The measurements `m1` and `m2`, which used to be drawn from `normal`s, are now the input variables `x2` and `x3` (bound by deconstructing the argument `x1` using `unpair`). In place of the two measurement calls to `normal`, the distribution is now `weight`ed by the *density* of each Gaussian at the corresponding measurement. Unlike a distribution, a density is a function from `HReal` to `HProb`.

```
lam $ \x1 ->
x1 `unpair` \x2 x3 ->
uniform 3 8 `bind` \x4 ->
normal 21 (unsafeProb x4) `bind` \x5 ->
uniform 1 4 `bind` \x6 ->
normal x5 (unsafeProb x4) `bind` \x7 ->
weight (exp_ (-(x3 - x7) * (x3 - x7)
              / fromProb (2 * pow_ (unsafeProb x6) 2))
        / unsafeProb x6
        / sqrt_ (2 * pi_)) $
weight (exp_ (-(x2 - x5) * (x2 - x5)
              / fromProb (2 * pow_ (unsafeProb x6) 2))
        / unsafeProb x6
        / sqrt_ (2 * pi_)) $
dirac (pair (unsafeProb x4) (unsafeProb x6))
```

Fig. 2: The `thermometer` model after disintegration

The reader might wonder why Figure 1 is rather large, given the simple model it came from, while Figure 2 is modest though the model it comes from seems more complex. This is because the burglar model contains discrete choices (calls to `bern`), which tend to inflate the output of disintegration, while the thermometer model is a straight-line program.

While `thermConditional` represents the correct posterior distribution, it is not yet efficient for inference. This is because of the two calls to `normal` that still exist. Our prior distribution `thermometer` does not return the variables drawn from these `normal`s, which are the temperatures `t1` and `t2`. The measure is a *marginal* distribution on only `m1`, `m2`, `noiseT`, and `noiseM`. Similarly, `thermConditional`, when given measurements, returns a distribution only on the noises, not on the variables `x5` and `x7`. Because the distribution uses the random variables `x5` and `x7` only internally, running it as a sampler amounts to naive numerical integration over them, which is inaccurate and slow.

It would be better to integrate over `x5` and `x7` exactly. We can use the simplification transformation to do it:

```
thermSimplified = simplify thermConditional
```

The pretty-printed output of simplifying the conditional distribution is shown in Figure 3. The remaining calls to `normal` have disappeared and all the weight

factors are combined into a single formula. By removing `x5` and `x7` and storing the intermediate factors that are the results of integrating these variables, Hakaru has performed what is known as marginalization or variable elimination.

```
lam $ \x1 ->
weight (recip pi_ * (1/6)) $
uniform 3 8 `bind` \x2 ->
uniform 1 4 `bind` \x3 ->
weight (exp_ ((x2 * x2
               * ((x1 `unpair` \x4 x5 -> x4) * (x1 `unpair` \x4 x5 -> x4))
               * 2
               + x2 * x2 * (x1 `unpair` \x4 x5 -> x4) * (x1 `unpair` \x4 x5 -> x5)
                 * (-2)
               + x2 * x2
                 * ((x1 `unpair` \x4 x5 -> x5) * (x1 `unpair` \x4 x5 -> x5))
               + x3 * x3
                 * ((x1 `unpair` \x4 x5 -> x4) * (x1 `unpair` \x4 x5 -> x4))
               + x3 * x3
                 * ((x1 `unpair` \x4 x5 -> x5) * (x1 `unpair` \x4 x5 -> x5))
               + x2 * x2 * (x1 `unpair` \x4 x5 -> x4) * (-42)
               + x3 * x3 * (x1 `unpair` \x4 x5 -> x4) * (-42)
               + x3 * x3 * (x1 `unpair` \x4 x5 -> x5) * (-42)
               + x2 * x2 * 441
               + x3 * x3 * 882)
              * recip (x2 * x2 * (x2 * x2) + x2 * x2 * (x3 * x3) * 3
                       + x3 * x3 * (x3 * x3))
              * (-1/2))
        * recip (sqrt_ (unsafeProb (x2 ** 4 + x2 ** 2 * x3 ** 2 * 3
                                    + x3 ** 4)))
        * 3) $
dirac (pair (unsafeProb x2) (unsafeProb x3))
```

Fig. 3: The result of simplifying `thermConditional`

Once again, simplification subsumes exact inference with help from computer algebra. While this code could be made yet more concise by let-binding, it already makes for an efficient sampler.

3.3 Sampling

The next step in the workflow of a Hakaru user is to sample from the posterior. In this example, the posterior distribution is only 2-dimensional, so it is easy to tune the noise parameters by importance sampling or by searching a grid exhaustively. In higher dimensions, most parameter combinations are very bad, and exhaustive search is intractable, so we often want to use a *Markov chain Monte Carlo* (MCMC) technique in order to get an answer in a reasonable amount of time. That is what we demonstrate here.

MCMC means that the sampler generates not a single random sample but a chain of them, each dependent on the previous one. Most MCMC algorithms require specifying an easy-to-sample *proposal distribution* that depends on the current element of the chain [7]. This distribution is sampled to propose a candidate for the next element in the chain. We show here the Metropolis–Hastings method (MH), a popular MCMC algorithm that compares the posterior and proposal densities at the current and proposed elements in order to decide whether

the next element of the chain should be the proposed element or a repetition of the current element. This decision mathematically ensures that the chain is composed of samples that represent the posterior accurately.

A good proposal distribution will propose samples that are representative of the posterior. In this sense, the proposal embodies a strategy for searching and approximating the posterior space. Hakaru lets us specify our own proposal distribution based on our understanding of the model. MH practitioners know that custom proposal distributions are an important way to improve MH performance. Here we show a proposal distribution that a Hakaru user could write for the current example.

```
proposal :: (Mochastic repr)
         => repr (HPair HReal HReal)
         -> repr (HPair HProb HProb)
         -> repr (HMeasure (HPair HProb HProb))
proposal _m1m2 ntne =
  unpair ntne $ \noiseTOld noiseEOld ->
  superpose [(1/2, uniform 3 8 `bind` \noiseT' ->
                   dirac (pair (unsafeProb noiseT') noiseEOld)),
             (1/2, uniform 1 4 `bind` \noiseE' ->
                   dirac (pair noiseTOld (unsafeProb noiseE')))]
```

This particular proposal distribution leaves one noise parameter unchanged and draws an update to the other noise parameter from a `uniform` distribution. Thus, once the Metropolis–Hastings sampler finds a good setting for one parameter, it can remember it and more or less leave it alone for a while as it fiddles with the other parameter. In contrast, importance sampling tries to hit upon good settings for both parameters at once, which is less likely to happen. On the other hand, the chain dependency of Metropolis–Hastings sampling means it can stay stuck in a globally sub-optimal part of the search space for a long time.

A generic MH algorithm can be defined as a reusable transformation on Hakaru terms. It has this type:

```
mh  :: (Mochastic repr, Integrate repr, Lambda repr,
        env ~ Expect' env, a ~ Expect' a, Backward a a)
    => (forall r'. (Mochastic r') => r' env -> r' a
                                    -> r' (HMeasure a))
    -> (forall r'. (Mochastic r') => r' env -> r' (HMeasure a))
    -> repr (env :-> (a :-> HMeasure (HPair a HProb)))
```

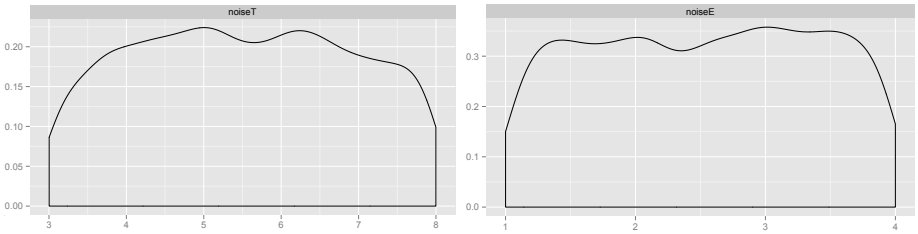
The various constraints in the type are a consequence of the finally tagless embedding. The `Expect`, `Integrate` and `Lambda` constraints require the interpretation (`repr`) to define operations for expectation, integration, and the lambda calculus. The Backward constraint is required for the density calculation step that forms a part of the MH procedure. The sampling interpretation satisfies these constraints.

We cam apply the generic `mh` to our custom `proposal` and the desired posterior `thermSimplified`:

```
mhKernel = mh proposal thermSimplified
```

This invocation generates a Metropolis–Hastings transition kernel that can be used to produce samples from the posterior. We first condition the kernel on observed temperature measurements – `pair 29 26`. We then run this sampler for 20000 iterations and use every 5^{th} sample (a process known as *thinning*) to produce the plot in Figure 4.

(a) Traceplot of sampled parameter values given observed temperatures (29°C,26°C)



(b) Density plots for the noise parameters

Fig. 4: Sampling from the conditioned posterior using Metropolis–Hastings

Knowing that the MH kernel likely contains large mathematical expressions from density calculation, and knowing that the `simplify` transformation is modular and reusable, we can attempt to speed up the computation at each iteration of the MH kernel by invoking `simplify mhKernel`. The performance gained from this reuse of algebraic simplification is shown in the middle row of Table 1.

Simplification here introduces efficiency with respect to wall clock time and not in the number of samples needed for convergence. The third program in Table 1 is a hand-coded Hakaru sampler written non-compositionally. For comparison, we also wrote a model of `thermometer` in the probabilistic language

Source of Hakaru code	Average run time
Generated by disintegrator	2,015 ± 4ms
Generated, then automatically simplified	569 ± 4
Written by hand	529 ± 10

Table 1: Time needed to draw 20,000 MH samples averaged over 10 runs.

WebPPL [5]; it generates a Metropolis-Hastings sampler that takes 948 ± 8ms. All measurements were produced on a quad-core Intel i5-2540M processor running 64-bit Ubuntu 14.04. Hakaru's samplers use GHC 7.8.3 `-O2`, and WebPPL's sampler is compiled to JavaScript and run on Node.js version 0.10.28.

4 Inference by composable program transformations

As the examples above show, Hakaru transforms a probabilistic program to other programs in the same language that generate samples or otherwise perform inference. This major design decision contrasts with most other probabilistic programming systems, which handle a probabilistic program either by producing code in a different language, or by generating samples or otherwise performing inference directly—without staging.

Because Hakaru transformations stay in the same language, we can compose them. For example, we use disintegration and simplification together to generate efficient densities, which are at the heart of Metropolis–Hastings sampling, as well as conditional distributions, which are at the heart of Gibbs sampling. Even after we apply an approximate inference technique such as Metropolis–Hastings to a problem, we can still inspect and optimize the generated solution before running it. We can also compose inference techniques (such as particle filtering and Metropolis–Hastings) as well as analyses (perhaps to estimate the running time or accuracy of a sampler).

4.1 Semantic specifications of transformations

Although all our transformations operate on Hakaru syntax, we specify what they do by referring to Hakaru semantics based on *integrators*. For example, for the models in Section 2 and Section 3, the specification of disintegration guarantees that the denotations of the model and the posterior are related as below.

```
model = superpose [(1, dirac true),
                   (1, dirac false)]
          `bind` \alarm ->
        app (disintegrate model) alarm
          `bind` \burglary ->
        dirac (pair alarm burglary)
```

```
thermometer = lebesgue `bind` \m1 ->
              lebesgue `bind` \m2 ->
              app (disintegrate thermometer)
                  (pair m1 m2)
                `bind` \noise ->
              dirac (pair (pair m1 m2) noise)
```

We describe elsewhere the implementation of this specification, which involves reordering integrals and computing a change of variables [12].

The specification of simplification is just that its output has the same measure semantics as the input. Implementing the specification involves translating to and from an integrator representation of measures, and improving the integral using computer algebra. We describe this process in detail elsewhere [2].

4.2 Comparison with other embeddings

Like us, Kiselyov and Shan [6] and Ścibior et al. [11] both embed a probabilistic language in a general-purpose functional language, respectively OCaml and Haskell. Like us, they both express and compose inference techniques as transformations that produce programs in the same language. But unlike our embedding, their embeddings are "shallower": the language defines a handful of constructs for manipulating distributions, and reuses the host languages' primitives for all other expressions.

On one hand, their transformations consequently cannot inspect most of the input source code, notably deterministic computations and the right-hand side of `>>=`. Thus, Hakaru can compute densities and conditional distributions in the face of deterministic dependencies, and Hakaru can generate Metropolis–Hastings samplers using a variety of proposal distributions. On the other hand, a shallow embedding ensures that any deterministic part of a probabilistic program runs at the full speed of the host language.

WebPPL is a probabilistic language embedded in Javascript, providing inference methods to transform input programs into a different language [5]. Venture [8] and Anglican [13] are probabilistic programming systems that build upon Lisp and Clojure respectively to define strict, impure languages for composing inference procedures. In all these works, the code – derived via transformations or building blocks – can perform direct inference but not stage any computation.

5 Expressing semantic distinctions by types

We make crucial use of types to capture semantic distinctions in Hakaru. These distinctions show up both in the implementation and in the language itself. Figure 6 illustrates the types of each construct or macro described in this paper, grouped by the interface (such as `Mochastic`) that an interpretation (such as `Sample`) would need to implement.

5.1 Distinguishing Hakaru from Haskell

The foremost distinction is between Hakaru's type system and Haskell's type system. That is, we distinguish between the universe [3] of Hakaru types, `Hakaru`, and the universe of Haskell types, `*`.

At first this distinction may seem unnecessary, since we can identify Hakaru types as those which are the argument to some `repr`. However, making the distinction eliminates two broad classes of bugs.

[3] We implement this universe of types by defining our own Haskell kind using GHC's `-XDataKinds` extension. Thus, we call `Hakaru` and `*` both "universes" and "kinds".

```
class (...) => Base (repr :: Hakaru * -> *) where
  pair   :: repr a -> repr b -> repr (HPair a b)
  unpair :: repr (HPair a b) -> (repr a -> repr b -> repr c) -> repr c
  true, false :: repr HBool
  if_         :: repr HBool -> repr c -> repr c -> repr c
  unsafeProb :: repr HReal -> repr HProb
  fromProb   :: repr HProb -> repr HReal
  exp_  :: repr HReal -> repr HProb
  pow_  :: repr HProb -> repr HReal -> repr HProb
  sqrt_ :: repr HProb -> repr HProb

class (Base repr) => Mochastic (repr :: Hakaru * -> *) where
  dirac :: repr a -> repr (HMeasure a)
  bind  :: repr (HMeasure a) ->
           (repr a -> repr (HMeasure b)) -> repr (HMeasure b)
  superpose :: [(repr HProb, repr (HMeasure a))] -> repr (HMeasure a)
  uniform   :: repr HReal -> repr HReal -> repr (HMeasure HReal)
  normal    :: repr HReal -> repr HProb -> repr (HMeasure HReal)

class Lambda (repr :: Hakaru * -> *) where
  lam  :: (repr a -> repr b) -> repr (a :-> b)
  app  :: repr (a :-> b) -> repr a -> repr b

bern   :: (Mochastic repr) => repr HProb -> repr (HMeasure HBool)
weight :: (Mochastic repr) => repr HProb -> repr (HMeasure w) -> repr (HMeasure w)
liftM  :: (Mochastic repr)
       => (repr a -> repr b) -> repr (HMeasure a) -> repr (HMeasure b)
```

Fig. 6: The types of language constructs and macros used in the examples

First, there are many types in `*` which we do not want to allow within `Hakaru`. For example, Hakaru has support for arbitrary user-defined regular recursive polynomial data types. However, Haskell's data types are far richer, including: non-regular recursive types, non-strictly-positive recursive types, exponential types, higher-rank polymorphism, GADTs, and so on. Consequently, we must not allow users to embed arbitrary Haskell data types into `Hakaru`. The kind `*` is much too large for Hakaru, so by introducing the `Hakaru` kind we can statically prohibit all these non-Hakaru types.

Second, distinguishing between `Hakaru` and `*` guarantees a form of hygiene in the implementation. Without the kind distinction it is easy to accidentally equivocate between Haskell's types and Hakaru's types. This equivocation introduces confusion about what exists within the Hakaru language itself, versus what exists within the interpretation of Hakaru programs. One example of this confusion is the `lub` operator used by disintegration to make a nondeterministic choice between two Hakaru programs that mean the same measure. When introducing the `Hakaru` kind we noticed that it was not entirely clear whether the `lub` operator is in the language or a mere implementation detail.

5.2 Distinguishing values and distributions

Hakaru's type system draws a hard distinction between individual values (of type `a`) and distributions on values (of type `HMeasure a`). To see why this is necessary, consider the pseudo-program "`x = uniform 0 1; x + x`". As written

it is unclear what this pseudo-program should actually mean. On one hand, it could mean that the value `x` is drawn from the distribution `uniform 0 1`, and then this fixed value is added to itself. On the other hand, it could mean that `x` is defined to be the distribution `uniform 0 1`, and then we draw two samples from this distribution and add them together.

To distinguish these two meanings, stochastic languages like Church [4] must introduce a memoization operator; however, it is often difficult to intuitively determine where to use the memoization operator to obtain the desired behavior. In contrast, Hakaru distinguishes these meanings by distinguishing between let-binding and monadic-binding:

```
sampleOnce  = uniform 0 1 `bind` \x -> dirac (x + x)
sampleTwice = uniform 0 1 `let_` \x -> liftM2 (+) x x
```

Importantly, there is no way to mix up the second lines of these programs. If `x` is monadically bound, and hence has type `HReal`, then the expression `liftM2 (+) x x` does not type check, because `HReal` is not a monad so it cannot be the type of arguments to `liftM2 (+)`. Whereas, if `x` is let-bound, and hence has type `HMeasure HReal`, then the expression `dirac (x + x)` does not type check, because we do not define `(+)` on measures; but even if we did define `(+)` on measures, then the expression would have type `HMeasure (HMeasure HReal)` which doesn't match the signature.

5.3 Distinguishing values in different domains

It is often helpful to distinguish between various numeric domains. Although the integers can be thought of as real numbers, it is often helpful to specify when we really mean integers. Similarly, although the natural numbers can be thought of as integers and the positive reals can be thought of as reals, it is often helpful to explicitly capture the restriction to non-negative numbers. Thus, Hakaru provides four primitive numeric types: `HNat`, `HInt`, `HProb`, and `HReal`.

There are at least four reasons for making these distinctions. First, it helps with human understanding to say what we mean. Second, many of the built-in distributions are only defined for non-negative parameters, thus the non-negative types `HNat` and `HProb` are necessary to avoid undefinedness. Third, capturing the integral and non-negative constraints helps with algebraic simplification since we do not have to worry about the non-occurring cases. Fourth, for the situations where we must actually compute values (e.g., the sampling interpretation), knowing that `HProb` is non-negative means that we can represent these values in the log-domain in order to avoid problems with underflowing.

5.4 Distinguishing different interpretations of Hakaru

We use a "finally tagless" embedding of Hakaru in Haskell [1]. Thus an *interpretation* of `Mochastic` is implemented as an `instance`. We have several such interpretations—a sampler, a pretty-printer, and two variants of embedding our

language into Maple. *Transformations* are also `instance`s, but where `repr` appears as a free variable. For example, we have an expectation transformation `Expect`, which takes a measure expression and returns its expectation functional. Another transformation implements disintegration, which is performed by lazy partial evaluation of measure terms.

As our use of `bind` indicates, we use Higher Order Abstract Syntax (HOAS) to encode Hakaru binding. In exchange for preventing scope extrusion, HOAS makes some manipulations of bindings difficult to express. Basically, one significant advantage of finally tagless is that if we can write your transformation compositionally, then it will compose beautifully with other such transformations. But some transformations are hard to write compositionally. For example, it is easy to write "macros" such as `liftM2`, as well as the `Expect` transformation, but it is hard to implement lazy partial evaluation.

Even for a compositional transformation, finally-tagless style makes our code hard to debug, because for example `Expect repr` is implemented in terms of an abstract `repr` that is not required to be an instance of `Show`. This rules out using `Debug.Trace.traceShow` in the middle of the implementation of `Expect`.

We love how the Haskell type of a Hakaru term tracks how the term will be interpreted, such as `Expect PrettyPrint`. The flip side is that to apply multiple interpretations to the same term (like to scale a distribution so it sums to 1), we must either create the "product" of two interpretations (and using it to interpret a term takes time exponential in the number of nested binders in the term), or the Haskell type of a term must be universally quantified over `repr`. The latter is very natural for experienced Haskell programmers, but is hard to explain to others, thus limiting the potential of Hakaru as a general-purpose probabilistic programming language. Another issue is that the very hygiene which is a strength of finally-tagless makes it awkward to have "free variables" (parameters) in a term; these must all be `lam` bound before a term can be disintegrated or simplified. This contrasts sharply with how easily a computer algebra system handles free variables in equivalent terms [2].

6 Conclusion

A major challenge faced by every probabilistic programming system is that probabilistic models and inference algorithms do not compose in tandem: just because a model we're interested in can be built naturally from two submodels does not mean a good inference algorithm for the model can be built naturally from good inference algorithms for the two submodels. Due to this challenge, many systems with good support for model composition resort to a fixed or monolithic inference algorithm and do their best to optimize it.

Hakaru demonstrates a new way to address this challenge. On one hand, Hakaru supports model composition like any other embedded monadic DSL does: on top of primitive combinators such as `dirac`, `bind`, and `superpose`, users can define Haskell functions to express common patterns of samplers and measures. On the other hand, because each inference building block is a transformation on

this DSL, Hakaru supports inference composition like a compiler construction kit or computer algebra system does: users can define Haskell functions to express custom pathways from models to inference.

We are working to extend the Hakaru language, to express high-dimensional models involving arrays and trees, and to express more inference algorithms, including parallel and streaming ones. We are also working to make Hakaru more usable: by representing the abstract syntax as a data type, by customizing the concrete syntax, and by inviting user interaction for transforming subexpressions.

References

[1] Carette, J., Kiselyov, O., Shan, C.c.: Finally tagless, partially evaluated: Tagless staged interpreters for simpler typed languages. Journal of Functional Programming 19(5), 509–543 (2009)

[2] Carette, J., Shan, C.c.: Simplifying probabilistic programs using computer algebra (2015), http://www.cs.indiana.edu/ftp/techreports/TR719.pdf

[3] Giry, M.: A categorical approach to probability theory. In: Banaschewski, B. (ed.) Categorical Aspects of Topology and Analysis: Proceedings of an International Conference. pp. 68–85. Springer (1982)

[4] Goodman, N.D., Mansinghka, V.K., Roy, D., Bonawitz, K., Tenenbaum, J.B.: Church: A language for generative models. In: Proceedings of the 24th Conference on Uncertainty in Artificial Intelligence. pp. 220–229. AUAI Press (2008)

[5] Goodman, N.D., Stuhlmüller, A.: The design and implementation of probabilistic programming languages. http://dippl.org (2014), accessed: 2015-11-20

[6] Kiselyov, O., Shan, C.c.: Embedded probabilistic programming. In: Proceedings of the Working Conference on Domain-Specific Languages. pp. 360–384. No. 5658 in Lecture Notes in Computer Science, Springer (2009)

[7] MacKay, D.J.C.: Introduction to Monte Carlo methods. In: Jordan, M.I. (ed.) Learning and Inference in Graphical Models. Kluwer (1998)

[8] Mansinghka, V.K., Selsam, D., Perov, Y.N.: Venture: a higher-order probabilistic programming platform with programmable inference. CoRR abs/1404.0099 (2014), http://arxiv.org/abs/1404.0099

[9] Pearl, J.: Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. Morgan Kaufmann (1988), revised 2nd printing, 1998

[10] Ramsey, N., Pfeffer, A.: Stochastic lambda calculus and monads of probability distributions. In: POPL '02: Conference Record of the Annual ACM Symposium on Principles of Programming Languages. pp. 154–165. ACM Press (2002)

[11] Ścibior, A., Ghahramani, Z., Gordon, A.D.: Practical probabilistic programming with monads. In: Proceedings of the 8th ACM SIGPLAN Symposium on Haskell. pp. 165–176. ACM Press (2015)

[12] Shan, C.c., Ramsey, N.: Symbolic Bayesian inference by lazy partial evaluation (2015), http://www.cs.tufts.edu/~nr/pubs/disintegrator-abstract.html

[13] Wood, F., van de Meent, J.W., Mansinghka, V.: A new approach to probabilistic programming inference. In: Proceedings of the 17th International conference on Artificial Intelligence and Statistics. pp. 1024–1032 (2014)