Online companion to "Design Principles for Flexible Systems"

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This companion contains most of the proofs of the results in the main paper.

1 Proof of Proposition 1

The proof follows from Theorem 1 of [1], where $\mu_{j,k}$, λ, and α_k in [1] are replaced by $\mu_{j,k} f_{j,k}$, γ, and $a_k \lambda_{i(k)}$, respectively, from this paper. Replacing α_k in [1] by $a_k \lambda_{i(k)}$ is equivalent to setting $p_{0,k}$ in [1] equal to $\lambda_{i(k)} p_{0,k} / \sum_{i=1}^{N} \lambda_i$ in this paper.

2 Proof of Proposition 2

In [1], a different system is considered, but Proposition 4 of [1] directly examines an LP ((3)-(5) in [1]) that is the same as our allocation LP, with γ, $\tilde{\lambda}_k$, and $f_{j,k}\mu_{j,k}$ here playing the roles of λ, α_k, and $\mu_{j,k}$ in [1], respectively. So, using the result of Proposition 4 of [1],

$$\gamma^*(\{1,\ldots,K\}) = \min_{\Gamma \subset \{1,\ldots,K\}} \frac{\sum_{j=1}^{M} \beta_j \mathbf{1}\{f_{j,k} = 1 \text{ for some } k \text{ in } \Gamma\}}{\sum_{k \in \Gamma} \tilde{\lambda}_k / \mu_k}, \tag{A}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. We first see that $\bar{\gamma} = \sum_{j=1}^{M} \beta_j / \left(\sum_{k \in \{1,\ldots,K\}} \tilde{\lambda}_k / \mu_k \right)$ as the numerator is $\sum_{j=1}^{M} \beta_j$ for all Γ under full flexibility and the denominator is maximized when $\Gamma = \{1,\ldots,K\}$.

Now, for the "2-chain" flexibility structure, the minimum in (A) is achieved for a set of the form $\Gamma_{i,n}$. To see this, suppose that Γ achieves the minimum in (A). We can write

$$\Gamma = \cup_{\ell=1}^{L} \Gamma_{i_\ell, n_\ell}, \tag{B}$$

where $i_\ell < i_{\ell+1}$ for $\ell = 1, \ldots, L-1$ and there is at least one task separating Γ_{i_ℓ,n_ℓ} and $\Gamma_{i_{\ell+1},n_{\ell+1}}$ (and Γ_{i_L,n_L} and Γ_{i_1,n_1} where tasks K and 1 are considered to be adjacent). Thus, the term in the minimum in (A) becomes

$$\frac{\sum_{\ell=1}^{L}\left(\sum_{j\in\Gamma_{i_\ell-1,n_\ell+1}} \beta_j\right)}{\sum_{\ell=1}^{L}\left(\sum_{k\in\Gamma_{i_\ell,n_\ell}} \tilde{\lambda}_k/\mu_k\right)},$$

where in particular each β_j appears at most once. Now, as for $b_\ell \geq 0$, $c_\ell > 0$, $\ell = 1, \ldots, L$,

$$\min_{\ell} \frac{b_\ell}{c_\ell} \leq \frac{\sum_{\ell=1}^{L} b_\ell}{\sum_{\ell=1}^{L} c_\ell},$$

we can conclude that the minimum is achieved by one of the sets Γ_{i_ℓ,n_ℓ} in (B). However, under (6), the minimum is uniquely achieved when $\Gamma = \{1, \ldots, K\}$ (note that (6) holds for all i when $n = K-1$) and is equal to $\bar{\gamma}$, so (4) holds. Finally, (5) holds as it suffices to consider sets Γ with $K-1$ elements (see the middle of page 15). In this case, the numerator in (A) remains unchanged, while the denominator decreases.

3 Proof of Proposition 3

Without loss of generality, suppose that it is $f_{M,1}$ that is changed to zero. Then we trivially have $\gamma^*(\{1\}) = \gamma^*(\{1, \ldots, K\}) = \mu/\tilde{\lambda}$, achieved with $\delta^*_{j,j} = 1$ for all j (recall that $\gamma^*(\{1, \ldots, K\}) \leq \gamma^*(\{1\})$).

4 Proof of Proposition 4

From Proposition 2 of [1], we know that under full flexibility (all $f_{j,k} = 1$), there exists an optimal solution to the allocation LP with no more than five values in the set $\{\delta^*_{j,k}\}$ greater than zero.

First note that if there are exactly three non-zero values, then each server j and each task k must have exactly one $\delta^*_{j,k}$ greater than zero (because $\mu_{j,k} > 0$ for all j, k), so we can relabel the servers and tasks such that $\delta^*_{1,1} = \delta^*_{2,2} = \delta^*_{3,3} = 1$ under full flexibility. So, setting $f_{1,1} = f_{2,2} = f_{3,3} = 1$ and $f_{j,k} = 0$ otherwise satisfies (4), while (5) is satisfied by the 2-chain constructed by adding $f_{1,2} = f_{2,3} = f_{3,1} = 1$.

If there are exactly four non-zero values under full flexibility, then as each server j and each task k must have at least one $\delta^*_{j,k}$ greater than zero, we can relabel the tasks such that $\delta^*_{1,1}$, $\delta^*_{1,2} > 0$ and $\delta^*_{2,2} = \delta^*_{3,3} = 1$. Setting $f_{1,1} = f_{1,2} = f_{2,2} = f_{3,3} = 1$ and $f_{j,k} = 0$ otherwise then satisfies (4), while (5) is satisfied by the 2-chain constructed by adding $f_{2,3} = f_{3,1} = 1$.

After relabelling servers and/or tasks as necessary, there are five potential cases for the sets $\{\delta^*_{j,k}\}$ with exactly five non-zero values. These sets of non-zero values are:

(i) $\{\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{2,2}, \delta^*_{2,3}, \delta^*_{3,3}\}$;

(ii) $\{\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{2,1}, \delta^*_{2,2}, \delta^*_{3,3}\}$;

(iii) $\{\delta^*_{1,1}, \delta^*_{2,1}, \delta^*_{3,1}, \delta^*_{2,2}, \delta^*_{3,3}\}$;

(iv) $\{\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{1,3}, \delta^*_{2,2}, \delta^*_{3,3}\}$;

(v) $\{\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{1,3}, \delta^*_{2,2}, \delta^*_{3,2}\}$.

For case (i), we have that if we choose a flexibility structure that has $f_{j,k} = 1$ for the j and k represented in the given set, then $\bar{\gamma}$ is still achieved with the same solution, so (4) is satisfied, but (5) may not be. Now, setting $f_{1,3} = 1$ leaves $\bar{\gamma}$ unchanged, but now (5) is trivially satisfied. The structure $\{f_{j,k}\}$ is a 2-chain.

For case (ii), applying Proposition 2 of [1] to servers 1 and 2 and classes 1 and 2 yield that one of $\{\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{2,1}, \delta^*_{2,2}\}$ is zero, so case (ii) is not possible.

For case (iii), note that $\delta^*_{1,1} = 1$. We have that the solution must satisfy

$$\begin{aligned} \bar{\mu}_{1,1} + \delta^*_{2,1}\bar{\mu}_{2,1} + \delta^*_{3,1}\bar{\mu}_{3,1} &= \bar{\gamma}, \\ \delta^*_{2,2}\bar{\mu}_{2,2} &= \bar{\gamma}, \\ \delta^*_{3,3}\bar{\mu}_{3,3} &= \bar{\gamma}. \end{aligned}$$

Clearly this is only possible if $\bar{\mu}_{1,1} \leq \min\{\bar{\mu}_{2,2}, \bar{\mu}_{3,3}\}$. We seek a condition under which either $\delta^*_{2,1}$ or $\delta^*_{3,1}$ is zero and thus case (iii) cannot occur. If

$$\bar{\mu}_{1,1} + \delta^*_{3,1}\bar{\mu}_{3,1} = (1 - \delta^*_{3,1})\bar{\mu}_{3,3},$$

then solving for $(1 - \delta^*_{3,1}) = \delta^*_{3,3}$ yields

$$1 - \delta^*_{3,1} = \frac{\bar{\mu}_{1,1} + \bar{\mu}_{3,1}}{\bar{\mu}_{3,3} + \bar{\mu}_{3,1}}.$$

So,

$$\bar{\mu}_{3,3}\left(\frac{\bar{\mu}_{1,1} + \bar{\mu}_{3,1}}{\bar{\mu}_{3,3} + \bar{\mu}_{3,1}}\right) \geq \bar{\mu}_{2,2} \tag{C}$$

implies that $\delta^*_{2,2} = 1$ and hence $\delta^*_{2,1} = 0$. The same argument for $\delta^*_{3,1} = 0$ yields the condition

$$\bar{\mu}_{2,2}\left(\frac{\bar{\mu}_{1,1} + \bar{\mu}_{2,1}}{\bar{\mu}_{2,2} + \bar{\mu}_{2,1}}\right) \geq \bar{\mu}_{3,3}. \tag{D}$$

So, (C) or (D) implies that case (iii) is not possible. Undoing the relabelling of servers and/or tasks yields that (8) implies that case (iii) is not possible.

Similar arguments yield that (9) is sufficient for case (iv) to be eliminated and (10) is sufficient for case (v) to be eliminated.

To show the second part, conditions (8)-(10) imply that a 2-chain satisfies (4) and (5). There are six possible 2-chains. In addition to the one in the Proposition statement, the remaining five are (only non-zero values of $f_{j,k}$ are given):

(a) $f_{1,1} = f_{1,3} = f_{2,2} = f_{2,3} = f_{3,1} = f_{3,2} = 1$;

(b) $f_{1,1} = f_{1,3} = f_{2,1} = f_{2,2} = f_{3,2} = f_{3,3} = 1$;

(c) $f_{1,2} = f_{1,3} = f_{2,1} = f_{2,3} = f_{3,1} = f_{3,2} = 1$;

(d) $f_{1,2} = f_{1,3} = f_{2,1} = f_{2,2} = f_{3,1} = f_{3,3} = 1$;

(e) $f_{1,1} = f_{1,2} = f_{2,1} = f_{2,3} = f_{3,2} = f_{3,3} = 1$.

We proceed by showing that for the 2-chains (a) through (e), the desired 2-chain has no worse value of $\bar{\gamma}$. This implies the desired 2-chain satisfies (4), and it also satisfies (5) because $\mu_{j,k} > 0$ for all j, k.

Suppose that chain (a) satisfies (4) and (5). Suppose first that $\delta^*_{1,3} > 0$ and $\delta^*_{3,2} > 0$. Suppose that we add ε to $\delta^*_{1,2}$ and subtract ε from $\delta^*_{1,3}$. Also, we add α to $\delta^*_{3,3}$ and subtract α from $\delta^*_{3,2}$. Since $\frac{\mu_{3,2}}{\mu_{3,3}} \leq \frac{\mu_{1,2}}{\mu_{1,3}}$, the inequalities

$$
\begin{aligned}
\varepsilon\mu_{1,2} - \alpha\mu_{3,2} &\geq 0, \\
\alpha\mu_{3,3} - \varepsilon\mu_{1,3} &\geq 0
\end{aligned}
$$

admit a solution satisfying $\alpha, \varepsilon > 0$. Therefore, we can decrease either $\delta^*_{1,3}$ or $\delta^*_{3,2}$ to zero without impacting (4). If both are zero, then we have that the desired 2-chain satisfies (4) and (5).

Now, suppose that $\delta^*_{3,2} > 0$ and $\delta^*_{1,3} = 0$. Suppose that we add α to $\delta^*_{3,3}$ and subtract α

from $\delta^*_{3,2}$. Also, we add ε to $\delta^*_{2,2}$ and subtract ε from $\delta^*_{2,3}$. Since $\frac{\mu_{3,2}}{\mu_{3,3}} \leq \frac{\mu_{2,2}}{\mu_{2,3}}$, the inequalities

$$\begin{aligned}\varepsilon\mu_{2,2} - \alpha\mu_{3,2} &\geq 0, \\ \alpha\mu_{3,3} - \varepsilon\mu_{2,3} &\geq 0\end{aligned}$$

admit a solution $\alpha, \varepsilon > 0$. Therefore, we can make either $\delta^*_{3,2}$ or $\delta^*_{2,3}$ zero. If $\delta^*_{3,2} = 0$, we have that the desired 2-chain satisfies (4) and (5). If $\delta^*_{2,3} = 0$, then we have that the flexibility structure with only $\delta^*_{1,1}, \delta^*_{1,2}, \delta^*_{2,2}, \delta^*_{3,1}, \delta^*_{3,2}, \delta^*_{3,3} \geq 0$ satisfies (4) and (5).

Assume that $\delta^*_{3,2} > 0$. If $\delta^*_{1,1} = 0$, we are done, as by relabeling the servers we have structure (v), which contradicts (10). If $\delta^*_{1,1} > 0$, we have

$$\begin{aligned}\mu_{1,1}\delta^*_{1,1} + \mu_{3,1}\delta^*_{3,1} &\geq \bar{\gamma}, \\ \mu_{1,2}\delta^*_{1,2} + \mu_{2,2}\delta^*_{2,2} + \mu_{3,2}\delta^*_{3,2} &\geq \bar{\gamma}, \\ \mu_{3,3}\delta^*_{3,3} &\geq \bar{\gamma}.\end{aligned}$$

Suppose that we decrease $\delta^*_{1,1}$ by α, increase $\delta^*_{1,2}$ by α, decrease $\delta^*_{3,2}$ by ε, and increase $\delta^*_{3,1}$ by ε. If $\frac{\mu_{1,1}}{\mu_{1,2}} \leq \frac{\mu_{3,1}}{\mu_{3,2}}$, the inequalities

$$\begin{aligned}\varepsilon\mu_{3,1} - \alpha\mu_{1,1} &\geq 0, \\ \alpha\mu_{1,2} - \varepsilon\mu_{3,2} &\geq 0\end{aligned}$$

admit a solution $\alpha, \varepsilon > 0$. Therefore, we can make either $\delta^*_{1,1}$ or $\delta^*_{3,2}$ zero. If $\delta^*_{3,2} = 0$, as before we are done. If $\delta^*_{1,1}$ is zero, then we have structure (v), a contradiction.

If $\delta^*_{1,3} > 0$ and $\delta^*_{3,2} = 0$, the proof is similar to the case when $\delta^*_{3,2} > 0$ and $\delta^*_{1,3} = 0$. This completes the proof for chain (a).

The proof for chains (b) through (e) is similar and is omitted in the interest of space.

5 Proof of Proposition 5

(i) The fact that $\bar{\gamma} = \mu M / \sum_{k=1}^{K} \tilde{\lambda}_k$ follows as in the proof of Proposition 2. It is not difficult to see that $\gamma^*(\{1,\ldots,K\}) = \bar{\gamma}$. Set $\delta^*_{1,1} = 1$ and $\delta^*_{j,j} = \tilde{\lambda}_j M / \sum_{k=1}^{K} \tilde{\lambda}_k$, $\delta^*_{j,1} = (\sum_{k=1}^{K} \tilde{\lambda}_k - M\tilde{\lambda}_j)/\sum_{k=1}^{K} \tilde{\lambda}_k$ for $j = 2,\ldots,M$. The conditions of the Proposition imply $\tilde{\lambda}_1 / \sum_{k=1}^{K} \tilde{\lambda}_k > (M-1)/M$, and hence

$$0 \leq \delta^*_{j,j} \leq \frac{M \sum_{j=2}^{M} \tilde{\lambda}_j}{\sum_{k=1}^{K} \tilde{\lambda}_k} = M \left(1 - \frac{\tilde{\lambda}_1}{\sum_{k=1}^{K} \tilde{\lambda}_k}\right) < 1,$$

for $j = 2,\ldots,M$. However, $\mu\delta^*_{j,j} = \bar{\gamma}\tilde{\lambda}_j$ for $j = 2,\ldots,M$, and similarly $\mu\delta^*_{1,1} + \mu\sum_{j=2}^{M} \delta^*_{j,1} = \bar{\gamma}\tilde{\lambda}_1$. This shows that (4) holds. Now, $\gamma^*(\{1,\ldots,K\} \setminus \{k\}) > \gamma^*(\{1,\ldots,K\})$ is trivial for $k = 1,\ldots,K$. Since (5) must only be verified for all subsets of $\{1,\ldots,K\}$ of size $K-1$, we have (i).

(ii) For $k = 1$, we have $\sum_{j=1}^{M} \delta_{j,1} f_{j,1} \leq 2$, which in turn implies that $\gamma^*(\{1,\ldots,K\}) \leq 2\mu/\lambda_1 < 2/(M-1) < \bar{\gamma}$ when $M > 2$.

6 Proof of Proposition 6

We will write the allocation LP for full flexibility in standard form,

$$\min c'x \text{ s.t.}$$

$$Ax = b,$$

$$x \geq 0,$$

where

$$b' = (0,0,0,1,1,1),$$

$$c' = (-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$x' = (\gamma, \delta_{1,1}, \delta_{1,2}, \delta_{1,3}, \delta_{2,1}, \delta_{2,2}, \delta_{2,3}, \delta_{3,1}, \delta_{3,2}, \delta_{3,3}, s_1, s_2, s_3, s_4, s_5, s_6),$$

$s_1, \ldots, s_6$ are slack variables, and

$$A = \begin{bmatrix} 1 & -\bar{\mu}_{1,1} & 0 & 0 & -\bar{\mu}_{2,1} & 0 & 0 & -\bar{\mu}_{3,1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\bar{\mu}_{1,2} & 0 & 0 & -\bar{\mu}_{2,2} & 0 & 0 & -\bar{\mu}_{3,2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\bar{\mu}_{1,3} & 0 & 0 & -\bar{\mu}_{2,3} & 0 & 0 & -\bar{\mu}_{3,3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To prove (4), it suffices to prove that the basis

$$B = \begin{bmatrix} 1 & -\bar{\mu}_{1,1} & -\bar{\mu}_{2,1} & 0 & -\bar{\mu}_{3,1} & 0 \\ 1 & 0 & 0 & -\bar{\mu}_{2,2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\bar{\mu}_{3,3} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

corresponding to the decision variables γ and $\delta_{j,k}$, where $(j,k) \in \{(1,1),(2,1),(3,1),(2,2),(3,3)\}$, is optimal. We will show that the corresponding basic solution is feasible and that the corresponding reduced costs are nonnegative (see for example Definition 3.3 in Bertismas and Tsitsiklis [2]). Let $c'_B = (-1,0,0,0,0,0)$. We will verify that the vectors $B^{-1}b$ and $c' - c'_B B^{-1} A$ are nonnegative. Algebra shows that the only terms in $B^{-1}b$ that could be negative are the terms corresponding to $\delta_{2,1}$ and $\delta_{3,1}$, which are given by

$$\frac{\bar{\mu}_{2,2}(\bar{\mu}_{3,1}+\bar{\mu}_{3,3}) - \bar{\mu}_{3,3}(\bar{\mu}_{1,1}+\bar{\mu}_{3,1})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}} \quad \text{and} \quad \frac{\bar{\mu}_{3,3}(\bar{\mu}_{2,1}+\bar{\mu}_{2,2}) - \bar{\mu}_{2,2}(\bar{\mu}_{1,1}+\bar{\mu}_{2,1})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}}, \qquad \text{(E)}$$

respectively, and both are nonnegative under condition (12). Similarly, the only terms in $c' - c'_B B^{-1} A$ that could be negative are the terms corresponding to $\delta_{1,2}$, $\delta_{1,3}$, $\delta_{2,3}$, and $\delta_{3,2}$, which are given by

$$\frac{\bar{\mu}_{3,3}(\bar{\mu}_{1,1}\bar{\mu}_{2,2} - \bar{\mu}_{1,2}\bar{\mu}_{2,1})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}}, \frac{\bar{\mu}_{2,2}(\bar{\mu}_{1,1}\bar{\mu}_{3,3} - \bar{\mu}_{1,3}\bar{\mu}_{3,1})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}},$$
$$\frac{\bar{\mu}_{2,2}(\bar{\mu}_{2,1}\bar{\mu}_{3,3} - \bar{\mu}_{2,3}\bar{\mu}_{3,1})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}}, \frac{\bar{\mu}_{3,3}(\bar{\mu}_{2,2}\bar{\mu}_{3,1} - \bar{\mu}_{2,1}\bar{\mu}_{3,2})}{\bar{\mu}_{2,2}\bar{\mu}_{3,1}+\bar{\mu}_{2,1}\bar{\mu}_{3,3}+\bar{\mu}_{2,2}\bar{\mu}_{3,3}},$$

respectively, and are all nonnegative under conditions (13) and (14) (note that (12) implies that $\mu_{2,2}, \mu_{3,3} > 0$). This proves that (4) holds.

In order to prove (5), it suffices to show that $\mu_{2,2}$, $\mu_{3,3}$, and the terms corresponding to $\delta_{2,1}$ and $\delta_{3,1}$ in $B^{-1}b$ are positive. This follows from condition (12), see (E).

7 Proof of Proposition 7

(i) The fact that $\bar{\gamma} = (K-1+d)\mu/K\lambda$ follows as in Proposition 2. To show that $\gamma^*(\{1,\dots,K\}) = \bar{\gamma}$, set $\delta^*_{1,1} = (d+K-1)/Kd \geq 0$ and $\delta^*_{1,k} = (d-1)/Kd \geq 0$, $k = 2,\dots,K$, so that $\sum_{k=1}^{K} \delta^*_{1,k} = 1$. Moreover, $d\mu\delta^*_{1,1} = \bar{\gamma}\lambda$ and similarly $d\mu\delta^*_{1,j} + \mu\delta^*_{j,j} = \bar{\gamma}\tilde{\lambda}$ for $j = 2,\dots,M$. This shows that (4) holds, and (5) is trivial.

For (ii), note that for $K > 2$, we have $\delta^*_{j,3} = 0$ for $j \neq 2, 3$, so that

$$\gamma^*(\{1,\dots,K\}) \leq \gamma^*(\{3\}) \leq \frac{2\mu}{\tilde{\lambda}} < \frac{K-1+d}{K} \times \frac{\mu}{\tilde{\lambda}} = \bar{\gamma}.$$

8 Proof of Proposition 8

The proof resembles the proof of Proposition 6, except that we now have

$$B = \begin{bmatrix} 1 & -\bar{\mu}_{1,1} & 0 & 0 & 0 & 0 \\ 1 & 0 & -\bar{\mu}_{1,2} & 0 & -\bar{\mu}_{2,2} & 0 \\ 1 & 0 & 0 & -\bar{\mu}_{1,3} & 0 & -\bar{\mu}_{3,3} \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Algebra shows that the only terms in $B^{-1}b$ that could be negative are the terms corresponding to $\delta_{1,2}$ and $\delta_{1,3}$, which are given by

$$\frac{\bar{\mu}_{1,1}(\bar{\mu}_{1,3}+\bar{\mu}_{3,3}) - \bar{\mu}_{2,2}(\bar{\mu}_{1,1}+\bar{\mu}_{1,3})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2}+\bar{\mu}_{1,1}\bar{\mu}_{1,3}+\bar{\mu}_{1,2}\bar{\mu}_{1,3}} \quad \text{and} \quad \frac{\bar{\mu}_{1,1}(\bar{\mu}_{1,2}+\bar{\mu}_{2,2}) - \bar{\mu}_{3,3}(\bar{\mu}_{1,1}+\bar{\mu}_{1,2})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2}+\bar{\mu}_{1,1}\bar{\mu}_{1,3}+\bar{\mu}_{1,2}\bar{\mu}_{1,3}}, \qquad \text{(F)}$$

respectively, and both are nonnegative under condition (15). Similarly, the only terms in $c' - c'_B B^{-1}A$ that could be negative are the terms corresponding to $\delta_{2,1}$, $\delta_{2,3}$, $\delta_{3,1}$, and $\delta_{3,2}$, which are given by

$$\frac{\bar{\mu}_{1,3}(\bar{\mu}_{1,1}\bar{\mu}_{2,2} - \bar{\mu}_{1,2}\bar{\mu}_{2,1})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2} + \bar{\mu}_{1,1}\bar{\mu}_{1,3} + \bar{\mu}_{1,2}\bar{\mu}_{1,3}}, \frac{\bar{\mu}_{1,1}(\bar{\mu}_{1,3}\bar{\mu}_{2,2} - \bar{\mu}_{1,2}\bar{\mu}_{2,3})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2} + \bar{\mu}_{1,1}\bar{\mu}_{1,3} + \bar{\mu}_{1,2}\bar{\mu}_{1,3}},$$
$$\frac{\bar{\mu}_{1,2}(\bar{\mu}_{1,1}\bar{\mu}_{3,3} - \bar{\mu}_{1,3}\bar{\mu}_{3,1})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2} + \bar{\mu}_{1,1}\bar{\mu}_{1,3} + \bar{\mu}_{1,2}\bar{\mu}_{1,3}}, \frac{\bar{\mu}_{1,1}(\bar{\mu}_{1,2}\bar{\mu}_{3,3} - \bar{\mu}_{1,3}\bar{\mu}_{3,2})}{\bar{\mu}_{1,1}\bar{\mu}_{1,2} + \bar{\mu}_{1,1}\bar{\mu}_{1,3} + \bar{\mu}_{1,2}\bar{\mu}_{1,3}},$$

respectively, and are all nonnegative under conditions (16) and (17). This proves that (4) holds.

In order to prove (5), it suffices to show that $\mu_{1,1}$ and the terms corresponding to $\delta_{1,2}$ and $\delta_{1,3}$ in $B^{-1}b$ are positive. This follows from condition (15), see (F).

9 Proof of Proposition 9

We first consider the full flexibility structure and make (1) and (2) tight for $k = 1, 2$:

$$\begin{aligned} \bar{\mu}_{1,1}\delta_{1,1} + \bar{\mu}_{2,1}\delta_{2,1} &= \gamma, \\ \bar{\mu}_{1,2}(1 - \delta_{1,1}) + \bar{\mu}_{2,2}(1 - \delta_{2,1}) &= \gamma, \end{aligned}$$

where $\bar{\mu}_{j,k} = \mu_{j,k}/\tilde{\lambda}_k$, $j, k = 1, 2$. Rewriting:

$$\begin{aligned} \delta_{1,1} &= -\frac{\bar{\mu}_{2,1} + \bar{\mu}_{2,2}}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}}\delta_{2,1} + \frac{\bar{\mu}_{1,2} + \bar{\mu}_{2,2}}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}}, \\ \gamma &= \left(\bar{\mu}_{2,1} - \bar{\mu}_{1,1}\left(\frac{\bar{\mu}_{2,1} + \bar{\mu}_{2,2}}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}}\right)\right)\delta_{2,1} + \frac{\bar{\mu}_{1,1}(\bar{\mu}_{1,2} + \bar{\mu}_{2,2})}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}}. \end{aligned}$$

So, the solution to the LP (1)-(3) satisfies $\delta^*_{2,1} = 0$, $\delta^*_{1,1} < 1$, and $\bar{\gamma} = \bar{\mu}_{1,1}(\bar{\mu}_{1,2} + \bar{\mu}_{2,2})/(\bar{\mu}_{1,1} + \bar{\mu}_{1,2})$ when

$$\bar{\mu}_{2,1} \leq \bar{\mu}_{1,1}\left(\frac{\bar{\mu}_{2,1} + \bar{\mu}_{2,2}}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}}\right) \tag{G}$$

and

$$\frac{\bar{\mu}_{1,2} + \bar{\mu}_{2,2}}{\bar{\mu}_{1,1} + \bar{\mu}_{1,2}} < 1. \tag{H}$$

The relation (G) reduces to $\mu_{2,1}\mu_{1,2} \leq \mu_{1,1}\mu_{2,2}$ and (H) reduces to $\tilde{\lambda}_1/\mu_{1,1} < \tilde{\lambda}_2/\mu_{2,2}$. The fact that $\delta^*_{2,1} = 0$ means that (4) holds for the "N" structure, and the fact that (5) holds follows from $\delta^*_{1,1} < 1$ and $\mu_{1,2} > 0$.

10 Proof of Proposition 10

As for the "N" structure in Proposition 9 we rewrite (1) and (2) with both constraints tight for full flexibility:

$$\begin{aligned}
\bar{\mu}_{1,1}\delta_{1,1} + \bar{\mu}_{2,1}\delta_{2,1} &= \gamma, \\
(1 - \delta_{1,1} - \delta_{1,3})\bar{\mu}_{1,2} + (1 - \delta_{2,1} - \delta_{2,3})\bar{\mu}_{2,2} &= \gamma, \\
\bar{\mu}_{1,3}\delta_{1,3} + \bar{\mu}_{2,3}\delta_{2,3} &= \gamma,
\end{aligned}$$

where $\bar{\mu}_{j,k} = \mu_{j,k}/\tilde{\lambda}_k$, $j = 1,2$, $k = 1,2,3$. This can be rewritten as

$$\begin{aligned}
\delta_{1,1} &= \frac{\gamma - \bar{\mu}_{2,1}\delta_{2,1}}{\bar{\mu}_{1,1}}, \\
\delta_{2,3} &= \frac{\gamma - \bar{\mu}_{1,3}\delta_{1,3}}{\bar{\mu}_{2,3}},
\end{aligned}$$

$$\left(\frac{\bar{\mu}_{2,1}\bar{\mu}_{1,2}}{\bar{\mu}_{1,1}} - \bar{\mu}_{2,2}\right)\delta_{2,1} + \left(\frac{\bar{\mu}_{1,3}\bar{\mu}_{2,2}}{\bar{\mu}_{2,3}} - \bar{\mu}_{1,2}\right)\delta_{1,3} + \bar{\mu}_{1,2} + \bar{\mu}_{2,2} = \gamma\left(1 + \frac{\bar{\mu}_{1,2}}{\bar{\mu}_{1,1}} + \frac{\bar{\mu}_{2,2}}{\bar{\mu}_{2,3}}\right).$$

So, the optimal solution to the LP (1)-(3) satisfies $\delta^*_{2,1} = 0$, $\delta^*_{1,3} = 0$, $\delta^*_{1,1} < 1$, $\delta^*_{2,3} < 1$, and

$$\bar{\gamma} = \frac{\bar{\mu}_{1,2}\bar{\mu}_{1,1}\bar{\mu}_{2,3} + \bar{\mu}_{2,2}\bar{\mu}_{1,1}\bar{\mu}_{2,3}}{\bar{\mu}_{1,1}\bar{\mu}_{2,3} + \bar{\mu}_{1,2}\bar{\mu}_{2,3} + \bar{\mu}_{2,2}\bar{\mu}_{1,1}}$$

when (18)-(20) hold. That the "W" structure satisfies (4) follows from $\delta^*_{2,1} = \delta^*_{1,3} = 0$, and (5) holds from $\delta^*_{1,1} < 1$, $\delta^*_{2,3} < 1$, $\mu_{1,2} > 0$, $\mu_{2,3} > 0$, $\mu_{2,2} > 0$ and $\mu_{1,1} > 0$.

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