On large sets of $v-1$ L-intersecting Steiner triple systems of order v

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Abstract

This paper presents four new recursive constructions for large sets of $v-1$ STS(v). These facilitate the production of several new infinite families of such large sets. In particular, we obtain for each $n \geq 2$ a large set of $3^n - 1$ STS(3^n) whose systems intersect in 0 or 3 blocks.

1 Introduction

First we recall the definitions of the basic concepts discussed in this paper. A *Steiner triple system of order* v (briefly STS(v)) is a pair $(V, \mathcal{B})$ where V is a set of cardinality v and $\mathcal{B}$ is a collection of 3-element subsets of V called *blocks*, such that each 2-element subset of V is contained in precisely one block. The elements of V are called *points*. Steiner triple systems of order v exist if and only if $v \equiv 1$ or 3 (mod 6). The number of blocks, $b = v(v-1)/6$. Some authors refer to the blocks as triples, but in this paper we wish to use the term "triple" for *any* 3-element subset of V.

A *large set* of STS(v) is a family $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_q)$ of q Steiner triple systems of order v, all on the same point set V, such that every triple is contained in at least one of the sets $\mathcal{B}_i$. In the case where every triple occurs in precisely one system, i.e. when $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$, $1 \le i < j \le q$, we have a large set of *mutually disjoint* (MD) Steiner triple systems. An easy counting argument shows that a large set of MD STS(v) contains precisely $v-2$ systems. These are known to exist for all $v \equiv 1$ or 3 (mod 6), $v \neq 7$, as a result of work by Lu and Teirlinck [5, 6, 7, 9]. In [4], Lindner and Rosa began the study of large sets of *mutually almost disjoint* (MAD) Steiner triple systems. These are large sets in which $|\mathcal{B}_i \cap \mathcal{B}_j| = 1$, $1 \le i < j \le q$. For $v \neq 7$, the number q of systems in a large set of MAD STS(v) equals v or $v+1$ (with one extra possibility, namely $q = 15$ for $v = 13$), [2]. Large sets of v MAD STS(v) are known to exist for all $v \equiv 1$ or 3 (mod 6), [4]. Large sets of $v+1$ MAD STS(v) are known for $v = 13$ and $v = 15$ but there is no such large set for $v = 9$, [2]. The question of existence of large sets of $v+1$ MAD STS(v) for other orders is a major unresolved problem.

In this paper we turn our attention to another question which is naturally suggested by the above discussion, namely: what can be said about large sets of $v-1$ STS(v)? Clearly such sets exist; we have only to take a large set of $v-2$ MD STS(v) and then adjoin a different STS(v) to obtain a large set of $v-1$ STS(v). But this construction is clearly not within the spirit of what is meant by a large set. In order to exclude it, we define a *minimal* large set to be one in which the removal of any STS(v) destroys the large set property. It is to be understood that subsequently throughout this paper the term "large set" means a "minimal large set" and, consequently $v \ge 7$.

Another simple construction also begins with a large set of $v-2$ MD STS(v) and relies on the concept of a *trade*. A trade is a pair $\{T_1, T_2\}$, where T_1 and T_2 are sets of triples covering precisely the same pairs of elements from a set of points V. In any one of the mutually disjoint systems, replace any collection of blocks T_1 which contribute to a trade $\{T_1, T_2\}$ by the triples of T_2. The blocks which have been replaced are completed to form a further STS(v). For small trades, there will normally be a considerable degree of choice over how this is done. If it can be done in an appropriate way, we may obtain a minimal large set of $v-1$ STS(v). For example, the method can certainly be applied to any large set of 11 MD STS(13) since both non-isomorphic STS(13) contain quadrilaterals (also known as Pasch configurations) and a trade $\{T_1, T_2\}$ is easily formed whenever T_1 is a quadrilateral. This construction alone indicates that, even for $v = 13$, there are likely to be many non-isomorphic large sets of $v-1$ STS(v). However, the nature of the construction is such that there is little control over the intersection numbers $|\mathcal{B}_i \cap \mathcal{B}_j|$, $1 \leq i < j \leq v-1$. In this paper we present a range of constructions in which the intersection numbers are controlled. Before proceeding to the constructions we review some basic theory and definitions.

Lemma 1.1 *In a large set of $v-1$ STS(v),*
(a) $v(v-1)(v-3)/6$ triples occur precisely once and $v(v-1)/6$ triples occur precisely twice, and
(b) the $v(v-1)/6$ triples which occur twice form an STS(v).

Proof. Consider any pair $\{a, b\}$ of points. There are $v-2$ distinct triples containing this pair and so one of these triples must occur precisely twice in the large set, and the others must occur once. Simple counting then gives (a). Since every pair of points appears in a unique repeated triple, these repeated triples must form an STS(v), which establishes (b). □

The STS(v) formed from the $v(v-1)/6$ triples which occur twice is known as the *cross* system of the large set.

Definition 1.1 *A large set of $v-1$ STS(v), $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_{v-1})$ is said to be L-intersecting if L is a subset of the non-negative integers, $|\mathcal{B}_i \cap \mathcal{B}_j| \in L$ for $1 \leq i < j \leq v-1$, and for each $l \in L$ there exists i' and j' ($i' \neq j'$) such that $|\mathcal{B}_{i'} \cap \mathcal{B}_{j'}| = l$.*

Lemma 1.2 *In a large set of $v-1$ L-intersecting STS(v), $0 \in L$.*

Proof. Suppose otherwise. Then every system of the large set intersects every other system in at least one block. Since every triple occurs either once or twice, the number of triples occurring twice is at least $(v-1)(v-2)/2$. But if $v>3$, this number exceeds $v(v-1)/6$ and we have a contradiction. □

As observed earlier, a large set of MD STS(v) contains precisely $v-2$ systems and so the set L must contain at least one positive integer. In other words, in a large set of $v-1$ STS(v), some pairs of distinct systems are disjoint and some have blocks in common. The most interesting cases are when the cardinality of L is small. We present four general constructions and give particular attention to the cases where $L=\{0,l\}$ or $L=\{0,l,m\}$. In the case $L=\{0,l\}$, we clearly require that $l|(v(v-1)/6)$. Our constructions are recursive and one of these proceeds from $(\{0,l\},v)$ to $(\{0,l\},3v)$; in this case we also require that $l|(3v(3v-1)/6)$, and these two requirements imply that $l|((3v(3v-1)-9v(v-1))/6)$, i.e. $l|v$. The current state of knowledge concerning large sets of the types just mentioned is fairly meagre and can be summarized as follows.

1. There exists, up to isomorphism, a unique large set of 6 $\{0,1\}$-intersecting STS(7), [2].
2. There exists, up to isomorphism, four large sets of 8 STS(9). One of these is $\{0,3\}$-intersecting and the other three are $\{0,1,3\}$-intersecting, [3].
3. There exists a large set of $v-1$ $\{0,1\}$-intersecting STS(v) for $v=2^n-1$, $n\geq 3$, [2].

In the case where $L=\{0,1\}$, a large set of L-intersecting STS(v) is called *nearly disjoint* (ND).

We conclude this introduction by giving brief definitions of some further items of terminology used in subsequent sections.

A *1-factor* of a graph G is a regular subgraph of G of degree 1 which includes all the vertices of G. A *1-factorization* of a graph G is a set $\mathcal{F}=\{F_1,F_2,\ldots,F_k\}$ of edge-disjoint 1-factors of G whose edge-sets partition the edge-set of G. If $G=K_n$, the complete graph on n vertices, then G has a 1-factorization if and only if n is even. A *near-1-factor* of a graph G is a subgraph of G which includes all the vertices of G and in which there is one isolated vertex and all other vertices have degree 1. A *near-1-factorization* of a graph G is a set $\mathcal{F}=\{F_1,F_2,\ldots,F_k\}$ of edge-disjoint near-1-factors of G whose edge-sets partition the edge-set of G. If $G=K_n$ then G has a near-1-factorization if and only if n is odd.

A *Steiner quadruple system of order* v (briefly SQS(v)) is a pair $(V, \mathcal{B})$ where V is a set of cardinality v and $\mathcal{B}$ is a collection of 4-element subsets of V called *blocks*, such that each 3-element subset of V is contained in precisely one block. An SQS(v) exists if and only if $v \equiv 2$ or $4 \pmod 6$. Given an SQS(v), by choosing a point $x \in V$, selecting all the blocks containing x, and deleting x from each of these, one obtains an STS($v-1$); this is called the *derived* triple system through x.

A *transversal design of order* v *and blocksize* 3 (briefly TD($3, v$)) is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of cardinality $3v$, $\mathcal{G}$ is a partition of V into 3 subsets called *groups* each of cardinality v, and $\mathcal{B}$ is a collection of 3-element subsets of V called *blocks*, such that each 2-element subset of V is either contained in precisely one group or is contained in precisely one block, but not both.

A *parallel class* of an STS(v) $= (V, \mathcal{B})$ is a set of blocks that partition the set V. The STS(v) is said to be *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes, and these classes are said to form a *resolution* of the system. An analogous definition applies to transversal designs. A *Kirkman triple system of order* v (briefly KTS(v)) is a resolvable STS(v) together with a specific resolution.

Our main constructions are recursive. The first of these takes a large set of $v-1$ STS(v) to a large set of $2v$ STS($2v+1$) ("doubling") and this is described in Section 2. This construction is more general than the doubling construction given in [2] which applies only to the cases $v = 2^n - 1$. The other constructions take a large set of $v-1$ STS(v) to a large set of $3v-1$ STS($3v$) ("tripling") and these are described in Section 3. If $L = \{l_i\}$ is a set of integers and if λ is an integer, then we use λL to denote the set $\{\lambda l_i\}$.

2 Doubling

Theorem 2.1 *Suppose that there exists a large set of* $v-1$ *L-intersecting STS(v). Then there exists a large set of* $2v$ *$(L \cup \{1\})$-intersecting STS($2v+1$).*

Proof. Let $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_{v-1})$ be a large set of L-intersecting STS(v) where $V = \{a_1, a_2, \ldots, a_v\}$. Put $W = V \cup Z_{v+1}$, and take $\mathcal{F} = \{F_1, F_2, \ldots, F_v\}$ to be a 1-factorization of K_{v+1} on Z_{v+1}. Let $\alpha : i \to i+1 \bmod v$ be the cyclic permutation $(1\ 2\ \ldots\ v)$ of the set $\{1, 2, \ldots, v\}$.

For $j = 1, 2, \ldots, v,\ i = 1, 2, \ldots, v-1$, and for $x, y \in Z_{v+1}$, put

$$C_{i,j} = \{\{x, y, a_{\alpha^i j}\} : [x, y] \in F_j\},$$

and define $\mathcal{C}_i = \bigcup_{j=1}^{v} C_{i,j}$. Clearly, for $i \neq i'$, $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ have no triples in common. Put $\mathcal{D}_i = \mathcal{B}_i \cup \mathcal{C}_i,\ i = 1, 2, \ldots, v-1$. Define now a collection of $v-1$ Steiner triple systems of order $2v+1$, $(W, \mathcal{D}_1), (W, \mathcal{D}_2), \ldots, (W, \mathcal{D}_{v-1})$. Clearly, $\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{B}_i \cap \mathcal{B}_j,\ i \neq j$, and so $(W, \mathcal{D}_1), (W, \mathcal{D}_2), \ldots, (W, \mathcal{D}_{v-1})$ is a set of $v-1$ L-intersecting STS$(2v+1)$.

Suppose now that $(Z_{v+1}, \mathcal{Q})$ is an SQS$(v+1)$, and let $(Z_{v+1} \setminus \{i\}, \mathcal{Q}_i)$ be the STS(v) derived from $(Z_{v+1}, \mathcal{Q})$ through the element $i \in Z_{v+1}$. For $i \in Z_{v+1}$, let $\mathcal{T}_i = \{\{i, j, a_k\} : j \in Z_{v+1} \setminus \{i\},\ [i, j] \in F_k\}$.

Take $\mathcal{G} = \{G_1, G_2, \ldots, G_v\}$ to be a near-1-factorization of K_v on $\{a_1, a_2, \ldots, a_v\}$. Assume, without loss of generality, that a_i is the isolated vertex of G_i. For $i \in Z_{v+1}$ and $s \in \{1, 2, \ldots, v\}$, put

$$P_{i,s} = \{\{a_x, a_y, j\} : [a_x, a_y] \in G_s,\ \{i, j, a_s\} \in \mathcal{T}_i\},$$

and define $\mathcal{P}_i = \bigcup_{s=1}^{v} P_{i,s}$. Further define $\mathcal{E}_i = \mathcal{Q}_i \cup \mathcal{T}_i \cup \mathcal{P}_i,\ i \in Z_{v+1}$, and consider the pairs $(W, \mathcal{E}_1), (W, \mathcal{E}_2), \ldots, (W, \mathcal{E}_{v+1})$.

We want to show that $(W, \mathcal{E}_i),\ i \in Z_{v+1}$, is an STS$(2v+1)$. There are $v(v-1)/6$ triples in $\mathcal{Q}_i$, there are v triples in $\mathcal{T}_i$, and there are $v(v-1)/2$ triples in $\mathcal{P}_i$, thus the total number of triples in $\mathcal{E}_i$ is $v(v-1)/6 + v + v(v-1)/2 = (2v+1)v/3$ which equals the number of blocks of an STS$(2v+1)$.

Consider a pair of distinct elements of W. If the two elements are both in V, i.e. they are, say, a_x and a_y, they are both contained in a triple of $\mathcal{P}_i$. Suppose next that the two elements are, say, m and n, both in $W \setminus V$. If one of m, n equals $i \in Z_{v+1}$, say, $m = i$, then the pair $\{i, n\}$ is contained in a triple of $\mathcal{T}_i$, namely in the triple $\{i, n, a_k\}$ provided $[i, n]$ is an edge of the 1-factor F_k of $\mathcal{F}$. If neither m nor n equals i then the pair $\{m, n\}$ is contained in $\mathcal{Q}_i$, the set of blocks of the derived STS(v) through the element i. Finally consider the case where one of the elements is $m \in Z_{v+1}$ and the other is $a_x \in V$. If $m = i$ then the pair $\{m, a_x\}$ is contained in a triple of $\mathcal{T}_i$. If $m \neq i$, determine s such that $[i, m] \in F_s$. If $a_x = a_s$ then the pair $\{m, a_x\}$ is contained in a triple of $\mathcal{T}_i$. If $a_x \neq a_s$ then G_s will contain an edge $[a_x, a_k]$ and hence $P_{i,s}$ will contain the triple $\{a_x, a_k, m\}$, and so the pair $\{m, a_x\}$ is contained in a triple of $\mathcal{P}_i$. Thus $(W, \mathcal{E}_i)$ is an STS$(2v+1)$ for each $i \in Z_{v+1}$.

Finally we show that $|\mathcal{E}_i \cap \mathcal{E}_j| = 1$ for $i \neq j,\ i, j \in Z_{v+1}$. Clearly, we have $\mathcal{Q}_i \cap \mathcal{Q}_j = \emptyset$, and $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$. On the other hand, consider $\mathcal{T}_i$ and $\mathcal{T}_j$, $i \neq j$. If the edge $[i, j]$ belongs to the 1-factor F_k of $\mathcal{F}$, then both $\mathcal{T}_i$ and $\mathcal{T}_j$ will contain the triple $\{i, j, a_k\}$; it is also clear that $\mathcal{T}_i$ and $\mathcal{T}_j$ cannot have any further triple in common (since all triples in $\mathcal{T}_i$ contain i, and all triples in $\mathcal{T}_j$

contain j), thus $|\mathcal{T}_i \cap \mathcal{T}_j| = 1$ for $i \neq j$. Consequently, $|\mathcal{E}_i \cap \mathcal{E}_j| = 1$ for $i \neq j$, $i, j \in Z_{v+1}$. It is also immediate that $\mathcal{D}_k \cap \mathcal{E}_m = \emptyset$ for any $k = 1, 2, \ldots, v-1$ and $m \in Z_{v+1}$, and the proof is complete. □

In fact the construction of Theorem 2.1 even works in the trivial case of two STS(3), both containing the triple $\{a_1, a_2, a_3\}$. Strictly speaking, these do not form a minimal large set of $\{0,1\}$-intersecting STS(3); however, the construction may still be applied, and what results is the large set of 6 $\{0,1\}$-intersecting STS(7) given in [2]. The Theorem may be re-applied to give the following Corollary.

Corollary 2.1 *There exists a large set of* $2^n - 2$ $\{0,1\}$*-intersecting STS(*$2^n - 1$*) for each* $n \geq 3$.

Proof. Apply the Theorem inductively, starting with the large set of 6 $\{0,1\}$-intersecting STS(7) given in [2]. □

We observe that the large set produced by Corollary 2.1 for $n \geq 4$ is different from the one given in [2]. To show this, define the *system intersection graph* of a large set of STS(v) to be the multigraph having the systems for vertices and with two distinct vertices joined by k edges if the two corresponding systems have k blocks in common. The large set of 6 STS(7) given in [2] has system intersection graph G_7 as shown in Figure 2.1.

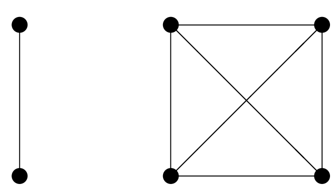

Figure 2.1: the graph G_7.

The graph G_7 is the disjoint union of K_2 and K_4. Examination of the proof of Theorem 2.1 shows that the corresponding graph G_{15} for the large set of 14 STS(15) given by Corollary 2.1 is the disjoint union of K_2, K_4 and K_8, a pattern which continues to higher values of n. For the large set of $2^n - 2$ $\{0,1\}$-intersecting STS($2^n - 1$), $n \geq 4$, given by Theorem 3.1 of [2], the system intersection graph is entirely different; for example it is connected.

Further corollaries to Theorem 2.1 are easily obtained. The following is an example.

Corollary 2.2 *There exists a large set of* $(5 \cdot 2^n - 2)$ $\{0,1,3\}$*-intersecting STS(*$5 \cdot 2^n - 1$*) for each* $n \geq 1$.

Proof. Apply the Theorem inductively, starting with a large set of 8 $\{0,1,3\}$-intersecting STS(9) given in [3]. □

3 Tripling

We start this section by formally stating the following useful result.

Lemma 3.1 *Suppose that* v *is odd. For* $i = 1,2,3$ *put* $G_i = Z_v \times \{i\} = \{0_i, 1_i, \ldots, (v-1)_i\}$, $U = \bigcup_{i=1}^{3} G_i$, *and* $\mathcal{G} = \{G_1, G_2, G_3\}$. *For each* $j \in Z_v$ *define*

$$\mathcal{C}_j = \{\{a_1, b_2, ((2b-a)+j)_3\} : a, b \in Z_v\}.$$

Then $(U, \mathcal{G}, \mathcal{C}_j)$ *is a TD(*$3, v$*) with groups* G_1, G_2 *and* G_3. *Also, given any triple* $\{a_1, b_2, c_3\}$ *with* $a, b, c \in Z_v$, *there exists precisely one value of* $j \in Z_v$ *for which* $\{a_1, b_2, c_3\} \in \mathcal{C}_j$. *Furthermore, if*

$$R_k = \{\{a_1, (k+a)_2, (2k+a)_3\} : a \in Z_v\} \qquad (k \in Z_v)$$

then $(U, \mathcal{G}, \mathcal{C}_0)$ *is resolvable into the parallel classes* R_k, $k \in Z_v$.

Proof. This is immediate. □

Theorem 3.1 *Suppose that there exists a large set of* $v-1$ L*-intersecting STS(*v*). Then there exists a large set of* $3v-1$ $(3L \cup \{v\})$*-intersecting STS(*$3v$*).*

Proof. We divide the proof into two parts; this subdivision will be useful later.
(a) Let $(Z_v, \mathcal{B}_1), (Z_v, \mathcal{B}_2), \ldots, (Z_v, \mathcal{B}_{v-1})$ be a large set of L-intersecting STS(v). For each $j = 1, 2, \ldots, v-1$ form an STS($3v$) on the point set $U = Z_v \times \{1,2,3\}$ by taking the blocks $\mathcal{C}_j$ of the TD($3, v$) given in Lemma 3.1, together with three copies of $\mathcal{B}_j$ (under the mappings $x \to x_i$) on $G_i =$

$Z_v \times \{i\}$ for $i = 1, 2, 3$. Denoting the resulting STS($3v$) by $(U, \mathcal{D}_j)$, it is clear that if $|\mathcal{B}_j \cap \mathcal{B}_{j'}| = l$ then $|\mathcal{D}_j \cap \mathcal{D}_{j'}| = 3l$. Thus $(U, \mathcal{D}_1), (U, \mathcal{D}_2), \ldots, (U, \mathcal{D}_{v-1})$ forms a set of $v - 1$ $(3L)$-intersecting STS($3v$).

(b) We next show how to construct a further $2v$ STS($3v$), say $(U, \mathcal{E}_1), (U, \mathcal{E}_2), \ldots, (U, \mathcal{E}_{2v})$ such that

(i) $\mathcal{E}_i \cap \mathcal{D}_j = \emptyset$ for $i = 1, 2, \ldots, 2v$ and $j = 1, 2, \ldots, v - 1$,

(ii) $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i, j = 1, 2, \ldots, 2v$ except when $\{i, j\}$ has the form $\{2k - 1, 2k\}$, in which case $|\mathcal{E}_i \cap \mathcal{E}_j| = v$, and

(iii) $(U, \mathcal{D}_1), (U, \mathcal{D}_2), \ldots, (U, \mathcal{D}_{v-1}), (U, \mathcal{E}_1), (U, \mathcal{E}_2), \ldots, (U, \mathcal{E}_{2v})$ form a large set.

Take the TD$(3, v)$ $(U, \mathcal{G}, \mathcal{C}_0)$ (this was not used in part (a)), and take R_k as in Lemma 3.1. From R_k, sets of triples $\mathcal{E}_{2k-1}$ and $\mathcal{E}_{2k}$ are constructed using the method of the Bose construction (see, for example [1]).

Let $\mathcal{A}_k$ denote the set of all triples of the forms

$$\left.\begin{array}{l} \{a_1, b_1, (k + \frac{a+b}{2})_2\} \\ \{(k + a)_2, (k + b)_2, (2k + \frac{a+b}{2})_3\} \\ \{(2k + a)_3, (2k + b)_3, (\frac{a+b}{2})_1\} \end{array}\right\} \qquad a < b \in Z_v.$$

The triples of $\mathcal{A}_k$ cover $3 \times 3 \times \binom{v}{2}$ distinct pairs of points from U and it is easy to see that these pairs are distinct from the pairs covered by the triples of R_k. Hence, if $\mathcal{E}_{2k-1} = R_k \cup \mathcal{A}_k$ then $(U, \mathcal{E}_{2k-1})$ is an STS($3v$).

The set of triples $\mathcal{E}_{2k}$ is formed in a similar fashion as $R_k \cup \mathcal{A}'_k$, where $\mathcal{A}'_k$ is the set of all triples of the forms

$$\left.\begin{array}{l} \{a_1, b_1, (2k + \frac{a+b}{2})_3\} \\ \{(k + a)_2, (k + b)_2, (\frac{a+b}{2})_1\} \\ \{(2k + a)_3, (2k + b)_3, (k + \frac{a+b}{2})_2\} \end{array}\right\} \qquad a < b \in Z_v.$$

The pair $(U, \mathcal{E}_{2k})$ is then also an STS($3v$).

Clearly $\mathcal{E}_i \cap \mathcal{D}_j = \emptyset$ for $i = 1, 2, \ldots, 2v$ and $j = 1, 2, \ldots, v - 1$. Also, since $\mathcal{A}_k \cap \mathcal{A}'_k = \emptyset$, it follows that $\mathcal{E}_{2k-1} \cap \mathcal{E}_{2k} = R_k$ and so $|\mathcal{E}_{2k-1} \cap \mathcal{E}_{2k}| = v$. Furthermore, it is easy to see that for any i and j, $\mathcal{A}_i \cap \mathcal{A}'_j = \emptyset$ and so $\mathcal{E}_{2i-1} \cap \mathcal{E}_{2j} = \emptyset$ if $i \neq j$. To prove that $\mathcal{E}_{2i-1} \cap \mathcal{E}_{2j-1} = \emptyset$ and that $\mathcal{E}_{2i} \cap \mathcal{E}_{2j} = \emptyset$ for $i \neq j$, it suffices to show that $\mathcal{A}_i \cap \mathcal{A}_j = \mathcal{A}'_i \cap \mathcal{A}'_j = \emptyset$ for $i \neq j$.

Suppose that a triple t lies in both $\mathcal{A}_i$ and $\mathcal{A}_j$. There are three cases to consider.

(a) $t = \{a_1, b_1, (i + \frac{a+b}{2})_2\} = \{a'_1, b'_1, (j + \frac{a'+b'}{2})_2\}$. This implies that $a+b = a'+b'$ and so, by considering the third element, $i = j$.

(b) $t = \{(i+a)_2, (i+b)_2, (2i + \frac{a+b}{2})_3\} = \{(j+a')_2, (j+b')_2, (2j + \frac{a'+b'}{2})_3\}$. This implies that $2i+a+b = 2j+a'+b'$ and so, again by considering the third element, $i = j$.

(c) $t = \{(2i+a)_3, (2i+b)_3, (\frac{a+b}{2})_1\} = \{(2j+a')_3, (2j+b')_3, (\frac{a'+b'}{2})_1\}$. This implies that $4i+a+b = 4j+a'+b'$ and so, again by considering the third element, $i = j$.

Hence $\mathcal{E}_{2i-1} \cap \mathcal{E}_{2j-1} = \emptyset$ if $i \neq j$ and, by a similar argument, $\mathcal{E}_{2i} \cap \mathcal{E}_{2j} = \emptyset$ if $i \neq j$.

To complete the proof, observe that $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{v-1}$ contain all triples of the form $\{a_i, b_i, c_i\}$ for $i = 1, 2, 3$ together with all triples of the form $\{a_1, b_2, c_3\}$ apart from those lying in $\mathcal{C}_0$. The sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{2v}$ contain all triples of the forms $\{a_i, b_i, c_{i+1}\}$ and $\{a_i, b_i, c_{i-1}\}$ (subscript arithmetic modulo 3 on $\{1, 2, 3\}$) together with all the triples $\{a_1, b_2, c_3\}$ lying in $\mathcal{C}_0$.

It follows that $(U, \mathcal{D}_1), (U, \mathcal{D}_2), \ldots, (U, \mathcal{D}_{v-1}), (U, \mathcal{E}_1), (U, \mathcal{E}_2), \ldots, (U, \mathcal{E}_{2v})$ form a large set of $(3L \cup \{v\})$-intersecting STS($3v$). □

In the notation of the Theorem, if $v/3 \in L$, then $3L \cup \{v\} = 3L$. Furthermore, if $L = \{0, v/3\}$ then $3L = \{0, v\}$. It is also easily seen from the proof that if the non-empty intersections of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{v-1}$ form parallel classes of the cross system, then so do those of $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{v-1}, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{2v}$. From [3] it is known that there exists a large set of 8 STS(9) intersecting in the parallel classes of the cross STS(9). In fact, the intersection graph G_9 is as shown below.

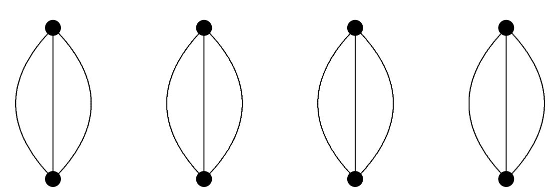

Figure 3.1: the graph G_9.

Consequently, we may state the following corollary to the Theorem.

Corollary 3.1 *There exists a large set of* $3^n - 1$ $\{0, 3^{n-1}\}$*-intersecting STS(*3^n*) for each* $n = 2, 3, \ldots,$ *whose intersections form the parallel classes of a cross KTS(*3^n*).*

Proof. Apply the Theorem inductively, starting with the large set of 8 $\{0, 3\}$-intersecting STS(9). □

Lemma 3.2 *Suppose that* v *is odd and that* $p < q \in Z_v$ *are such that* $(p, v) = (2p-1, v) = (q, v) = (2q-1, v) = 1$*. Referring to* $(U, \mathcal{G}, \mathcal{C}_0)$ *given in Lemma 3.1, define* $P_k, Q_k \subseteq \mathcal{C}_0$ *for each* $k \in Z_v$ *by*

$$P_k = \{\{a_1, (k + ap)_2, (2k + a(2p - 1))_3\} : a \in Z_v\},$$

$$Q_k = \{\{a_1, (k + aq)_2, (2k + a(2q - 1))_3\} : a \in Z_v\}.$$

Then $(U, \mathcal{G}, \mathcal{C}_0)$ *is resolvable into the parallel classes* $P_k,\ k \in Z_v$ *and also into the parallel classes* $Q_k,\ k \in Z_v$*. Furthermore, if* $l = (q-p, v)$ *then* $P_i \cap Q_j = \emptyset$ *unless* $i - j$ *is an integer multiple of* l*, in which case* $|P_i \cap Q_j| = l$*, and every triple of* $\mathcal{C}_0$ *appears in precisely one of the intersections* $P_i \cap Q_j$*.*

Proof. It is immediate that $P_k \subseteq \mathcal{C}_0$ for each $k \in Z_v$ and that $P_i \cap P_j = \emptyset$ for $i \neq j$. Since $(p, v) = (2p-1, v) = 1$ it also follows easily that each P_k is a parallel class. Hence $(U, \mathcal{G}, \mathcal{C}_0)$ is resolvable into parallel classes $P_k,\ k \in Z_v$. A similar argument deals with $Q_k,\ k \in Z_v$.

A triple t lies in both P_i and Q_j if and only if for some $a, a' \in Z_v$,

$$t = \{a_1, (i + ap)_2, (2i + a(2p - 1))_3\} = \{a'_1, (j + a'q)_2, (2j + a'(2q - 1))_3\}.$$

This holds if and only if $a = a'$ and $i - j = a(q - p)$ in Z_v. Thus $P_i \cap Q_j$ is empty if $i - j$ is not an integer multiple of l. If $i - j$ is an integer multiple of l, say $i - j = kl$, then $P_i \cap Q_j$ contains t provided $a(q - p) \equiv kl \pmod{v}$; this congruence gives l distinct values for a modulo v. Thus, when the intersection is non-empty, $|P_i \cap Q_j| = l$. For each fixed $i \in Z_v$ there will be v/l values of $j \in Z_v$ for which $i - j$ is an integer multiple of l; consequently there will be v/l classes Q_j having non-empty intersections of cardinality l with P_i and collectively these will contain all v distinct triples of P_i. By considering all possible values of i, we see that every triple of $\mathcal{C}_0$ appears in precisely one of the intersections. □

Before describing our next construction we define the *reduced intersection graph* of a large set of $\{0, \lambda\}$-intersecting STS(v). This is obtained from the system intersection graph by replacing each multiple edge (of multiplicity λ) by a single edge.

Theorem 3.2 *Suppose that there exists a large set of* $v-1$ $\{0,\lambda\}$*-intersecting STS(v) having a reduced intersection graph* H*, and that it is possible to pack* K_{v-1} *with three edge-disjoint copies of* H*. Then, if* v, p, q *satisfy the conditions of Lemma 3.2, there exists a large set of* $3v-1$ $\{0,\lambda,l\}$*-intersecting STS(3v), where* $l=(q-p,v)$*.*

Proof. Again we divide the proof into two parts.
(a) Let $(Z_v,\mathcal{B}_1),(Z_v,\mathcal{B}_2),\ldots,(Z_v,\mathcal{B}_{v-1})$ be a large set of $\{0,\lambda\}$-intersecting STS(v) having a reduced intersection graph H. For each $i=1,2,3$ let H_i denote a copy of H with vertex set $\{(Z_v\times\{i\},\mathcal{B}_{i,j}) : j=1,2,\ldots,v-1\}$, where $\mathcal{B}_{i,j}$ is a copy of $\mathcal{B}_j$ (under the mapping $x\to x_i$) on the point set $Z_v\times\{i\}$. Take K_{v-1} on $\{1,2,\ldots,v-1\}$. Since H_1,H_2 and H_3 may be packed into K_{v-1}, corresponding to each vertex k of K_{v-1} there will be a vertex $(Z_v\times\{i\},\mathcal{B}_{i,j(i,k)})$ of H_i ($i=1,2,3$) whose image is k under the mapping of vertices induced by the packing. For each $k=1,2,\ldots,v-1$ form an STS($3v$) on the point set $U=Z_v\times\{1,2,3\}$ by taking the blocks $\mathcal{C}_k$ of the TD($3,v$) given in Lemma 3.1, together with $\mathcal{B}_{i,j(i,k)}$ for $i=1,2,3$. Denote the resulting STS($3v$) by $(U,\mathcal{D}_k)$. For $k\neq k'$, the cardinality of $|\mathcal{D}_k\cap\mathcal{D}_{k'}|$ is given by $\sum_{i=1}^{3}|\mathcal{B}_{i,j(i,k)}\cap\mathcal{B}_{i,j(i,k')}|$. Of the three terms in this summation, at most one can be non-zero, and any non-zero term must have the value λ; this is because K_{v-1} can be packed with three edge-disjoint copies of H, and each edge of H represents λ common blocks. Thus $(U,\mathcal{D}_1),(U,\mathcal{D}_2),\ldots,(U,\mathcal{D}_{v-1})$ forms a set of $v-1$ $\{0,\lambda\}$-intersecting STS($3v$).
(b) We next show how to construct a further $2v$ STS($3v$), say $(U,\mathcal{E}_1),(U,\mathcal{E}_2),\ldots,(U,\mathcal{E}_v),(U,\mathcal{F}_1),(U,\mathcal{F}_2),\ldots,(U,\mathcal{F}_v)$ such that

(i) $\mathcal{E}_i\cap\mathcal{D}_j=\mathcal{F}_i\cap\mathcal{D}_j=\emptyset$ for $i=1,2,\ldots,2v$ and $j=1,2,\ldots,v-1$,

(ii) $\mathcal{E}_i\cap\mathcal{F}_j=\emptyset$ for $i,j=1,2,\ldots,v$ except when $i-j$ is an integer multiple of $l=(q-p,v)$, in which case $|\mathcal{E}_i\cap\mathcal{F}_j|=l$,

(iii) $\mathcal{E}_i\cap\mathcal{E}_j=\mathcal{F}_i\cap\mathcal{F}_j=\emptyset$ for $i,j=1,2,\ldots,v$ and $i\neq j$, and

(iv) $(U,\mathcal{D}_1),(U,\mathcal{D}_2),\ldots,(U,\mathcal{D}_{v-1}),(U,\mathcal{E}_1),(U,\mathcal{E}_2),\ldots,(U,\mathcal{E}_v),(U,\mathcal{F}_1),(U,\mathcal{F}_2),\ldots,(U,\mathcal{F}_v)$ form a large set.

Take the TD$(3, v)$ $(U, \mathcal{G}, \mathcal{C}_0)$ (this was not used in part (a)), and take P_k and Q_k as in Lemma 3.2. From P_k and Q_k, sets of triples $\mathcal{E}_k$ and $\mathcal{F}_k$ are constructed using the method of the Bose construction.

Let $\mathcal{A}_k$ denote the set of all triples of the forms

$$\left.\begin{array}{l} \{a_1, b_1, (k + p(\frac{a+b}{2}))_2\} \\ \{(k + pa)_2, (k + pb)_2, (2k + (2p-1)(\frac{a+b}{2}))_3\} \\ \{(2k + (2p-1)a)_3, (2k + (2p-1)b)_3, (\frac{a+b}{2})_1\} \end{array}\right\} \quad a < b \in Z_v.$$

The triples of $\mathcal{A}_k$ cover $3 \times 3 \times \binom{v}{2}$ distinct pairs of points from U and it is easy to see that these pairs are distinct from the pairs covered by the triples of P_k. Hence, if $\mathcal{E}_k = P_k \cup \mathcal{A}_k$ then $(U, \mathcal{E}_k)$ is an STS$(3v)$.

The set of triples $\mathcal{F}_k$ is formed in a similar fashion as $Q_k \cup \mathcal{A}'_k$, where $\mathcal{A}'_k$ is the set of all triples of the forms

$$\left.\begin{array}{l} \{a_1, b_1, (2k + (2q-1)(\frac{a+b}{2}))_3\} \\ \{(k + qa)_2, (k + qb)_2, (\frac{a+b}{2})_1\} \\ \{(2k + (2q-1)a)_3, (2k + (2q-1)b)_3, (k + q(\frac{a+b}{2}))_2\} \end{array}\right\} \quad a < b \in Z_v.$$

The pair $(U, \mathcal{F}_k)$ is then also an STS$(3v)$.

Clearly $\mathcal{E}_i \cap \mathcal{D}_j = \mathcal{F}_i \cap \mathcal{D}_j = \emptyset$ for $i = 1, 2, \ldots, v$ and $j = 1, 2, \ldots, v-1$. Also, since $\mathcal{A}_i \cap \mathcal{A}'_j = \emptyset$ for any i and j, it follows that $\mathcal{E}_i \cap \mathcal{F}_j = P_i \cap Q_j$. Thus $\mathcal{E}_i \cap \mathcal{F}_j = \emptyset$ unless $i - j$ is an integer multiple of $l = (q-p, v)$, in which case $|\mathcal{E}_i \cap \mathcal{F}_j| = l$. To prove that $\mathcal{E}_i \cap \mathcal{E}_j = \mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ for $i \neq j$, it suffices to show that $\mathcal{A}_i \cap \mathcal{A}_j = \mathcal{A}'_i \cap \mathcal{A}'_j = \emptyset$ for $i \neq j$; the proof of this follows in the same fashion as the proof of the corresponding section of Theorem 3.1.

To complete the proof, observe that $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{v-1}$ contain all triples of the form $\{a_i, b_i, c_i\}$ for $i = 1, 2, 3$ together with all triples of the form $\{a_1, b_2, c_3\}$ apart from those lying in $\mathcal{C}_0$. The sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_v, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_v$, contain all triples of the forms $\{a_i, b_i, c_{i+1}\}$ and $\{a_i, b_i, c_{i-1}\}$ (subscript arithmetic modulo 3 on $\{1, 2, 3\}$) together with all the triples $\{a_1, b_2, c_3\}$ lying in $\mathcal{C}_0$.

It follows that $(U, \mathcal{D}_1), (U, \mathcal{D}_2), \ldots, (U, \mathcal{D}_{v-1}), (U, \mathcal{E}_1), (U, \mathcal{E}_2), \ldots, (U, \mathcal{E}_v)$, $(U, \mathcal{F}_1), (U, \mathcal{F}_2), \ldots, (U, \mathcal{F}_v)$ form a large set of $\{0, \lambda, l\}$-intersecting STS$(3v)$.

□

We remark that if $\lambda = 1$, it is not possible to pack K_{v-1} with three copies of the reduced intersection graph H because in this case H is identical with the system intersection graph and has $v(v-1)/6$ edges.

In the case when $\lambda = l > 1$, Theorem 3.1 provides a recursive method for constructing large sets of $\{0, l\}$-intersecting systems. Of course, it is also necessary that all the conditions of the Theorem are satisfied at each stage. This can be achieved in certain cases, one of which is described in the following corollary.

Corollary 3.2 *There exists a large set of* $3^n - 1$ $\{0, 3\}$*-intersecting STS(*3^n*) for* $n \geq 2$.

Proof. We apply the construction described in the previous Theorem, starting with the large set of 8 $\{0, 3\}$-intersecting STS(9) given in [3]. Note that $v = 3^n, p = 1, q = 4$ satisfy the conditions of Lemma 3.2. In order to use an inductive argument, it is necessary to prove that at each stage it is possible to pack K_{v-1} with three edge-disjoint copies of the reduced intersection graph H^v. The initial graph H^9 has the form shown in Figure 3.2

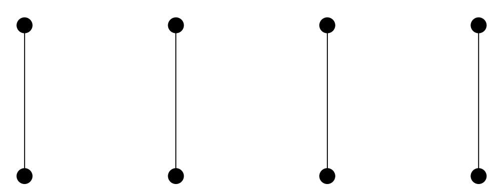

Figure 3.2: the reduced intersection graph H^9.

Three copies of this may be packed into K_8 as shown in Figure 3.3 where the three copies are identified by solid, dotted and dashed edges respectively.

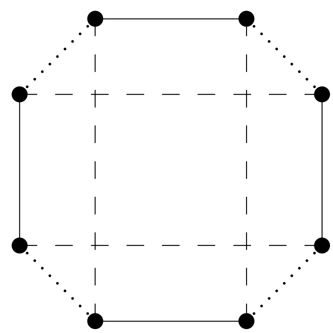

Figure 3.3: packing K_8 with three copies of H^9.

The inductive step is general to $\{0, l\}$-intersecting systems when $l \geq 3$ so, using the notation of the Theorem and assuming that the conditions of Lemma 3.2 are satisfied, suppose that it is possible to pack K_{v-1} with three copies of H^v. Denote the resulting subgraph of K_{v-1} by H^*. The large set of $3v-1$ $\{0, l\}$-intersecting STS($3v$) produced by the Theorem has reduced intersection graph H^{3v} illustrated in Figure 3.4.

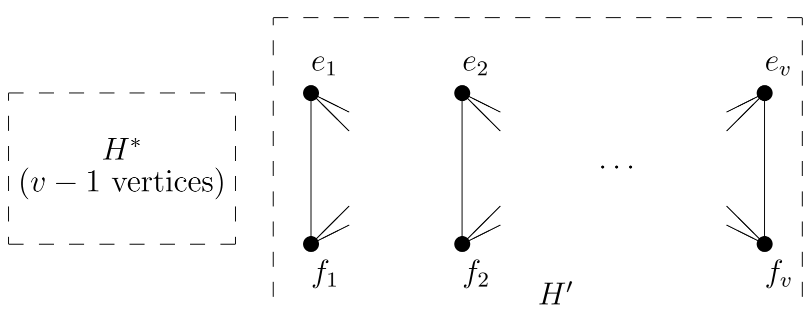

Figure 3.4: the graph H^{3v}.

The graph H^{3v} has two components, one of which is (a copy of) H^* and the second, here denoted by H', is a regular bipartite graph of degree v/l with bipartition (E, F), where $E = \{e_1, e_2, \ldots, e_v\}$ and $F = \{f_1, f_2, \ldots, f_v\}$ and where $\{e_i, f_j\}$ forms an edge if and only if $i - j$ is an integer multiple of l. To pack three copies of H^{3v} into K_{3v-1}, proceed as follows, where $V(G)$ denotes the vertex set of a graph G.

Take the first copy to be as shown in Figure 3.4. The second copy is formed by placing a copy of H^* onto the vertex set $\{e_1, e_2, \ldots, e_{v-1}\}$ and a copy of H' onto $(V(H^*) \cup \{e_v\}) \cup F$ with bipartition $(V(H^*) \cup \{e_v\}), F)$. This is possible, provided $l \geq 3$, since e_v is joined in H^{3v} to $v/l \leq v/3$ vertices of F. The third copy is formed in a similar fashion to the second by placing a copy of H^* onto $\{f_1, f_2, \ldots, f_{v-1}\}$ and a copy of H' onto $(V(H^*) \cup \{f_v\}) \cup E$ with bipartition $(V(H^*) \cup \{f_v\}), E)$. Again, it is easily possible to do this without creating repeated edges. □

Our final construction combines features of Theorems 3.1 and 3.2.

Theorem 3.3 *Suppose that there exists a large set of $v-1$ L-intersecting STS(v) and that v, p, q satisfy the conditions of Lemma 3.2. Then there exists a large set of $3v-1$ $(3L \cup \{l\})$-intersecting STS(3v), where $l = (q-p, v)$.*

Proof. Combine part (a) of the proof of Theorem 3.1 with part (b) of the proof of Theorem 3.2. Only very minor modifications are needed in order to check that the set of intersection sizes which result is $3L \cup \{l\}$. □

In fact, Theorem 3.1 can be regarded as a special case of Theorem 3.3 with $p = q = 1$ and $l = v$.

Corollary 3.3 *Suppose that $v = 3^m(4^n - 1)$, where $m \geq 1$ and $n \geq 2$. Then there exists a large set of $v-1$ $\{0, 3\}$-intersecting STS(v).*

Proof. Put $u = 4^n - 1$. Then, by Corollary 2.1, there exists a large set of $u-1$ $\{0, 1\}$-intersecting STS(u). Next apply Theorem 3.3 to produce a large set of $3u-1$ $\{0, 3\}$-intersecting STS($3u$); in order to do this it is necessary to specify appropriate values for p and q, which we do below. Then apply Theorem 3.2 recursively to produce a large set of $3^m u - 1$ $\{0, 3\}$-intersecting STS($3^m u$); in order to do this it is again necessary to specify appropriate values for p and q at each stage of the recursion. It is also necessary to check the condition regarding the packing of a complete graph with copies of the system intersection graph at each stage.

The choices of p and q are now described. Suppose that we are seeking to apply either Theorem 3.2 or Theorem 3.3 to construct a large set of $3w-1$ $\{0, 3\}$-intersecting STS($3w$) from an appropriate collection of STS(w), where $w = 3^r(4^n - 1)$ and $r \geq 0, n \geq 2$. We consider two cases.

(a) If $n \equiv 0$ or $2 \pmod 3$, put $p = 1$ and $q = 4^{n-1}$. Clearly $(p, w) = (2p - 1, w) = (q, w) = 1$. Put $x = (2q - 1, w)$, so that x is odd, $x|(2 \cdot 4^{n-1} - 1)$ and $x|(3^r(4^n - 1))$. Since $2 \cdot 4^{n-1} - 1 \equiv 1 \pmod 3$, x has no factor 3 and, consequently, $x|(4^n - 1)$. But then $x|(4^n - 2 \cdot 4^{n-1})$, i.e. $x|2 \cdot 4^{n-1}$, and so $x = 1$. Next put $y = (q - p, w)$, so that y is odd, $y|(4^{n-1} - 1)$ and $y|(3^r(4^n - 1))$. Since $n \equiv 0$ or $2 \pmod 3$, $4^{n-1} \equiv 4$ or $7 \pmod 9$ and, consequently, y has the form $y = 3z$, where z is odd and is not divisible by 3. But then $z|(4^{n-1} - 1)$ and $z|(4^n - 1)$, so $z|(4^n - 4^{n-1})$, i.e. $z|3 \cdot 4^{n-1}$ and so $z = 1$. Hence $(q - p, w) = 3$.

(b) If $n \equiv 1 \pmod 3$, put $p = 1$ and $q = w - 2 \cdot 4^{n-1}$. Clearly $(p, w) = (2p-1, w) = (q, w) = 1$. Put $x = (2q-1, w)$, so that x is odd, $x|(4^n+1)$ and $x|(3^r(4^n - 1))$. Since $4^n + 1 \equiv 2 \pmod 3$, x has no factor 3 and, consequently, $x|(4^n - 1)$. But then $x|(4^n + 1 - 4^n + 1)$, i.e. $x|2$, and so $x = 1$. Next put $y = (q - p, w)$, so that y is odd, $y|(2 \cdot 4^{n-1} + 1)$ and $y|(3^r(4^n - 1))$. Since $n \equiv 1 \pmod 3$, $4^{n-1} \equiv 1 \pmod 9$ and, consequently, y has the form $y = 3z$, where z is odd and is not divisible by 3. But then $z|(2 \cdot 4^{n-1} + 1)$ and $z|(4^n - 1)$, so $z|(4^n + 2 \cdot 4^{n-1})$, i.e. $z|6 \cdot 4^{n-1}$ and so $z = 1$. Hence $(q - p, w) = 3$.

With the above choices for p and q, Theorems 3.2 and 3.3 may be applied. The packing of a complete graph with copies of the reduced intersection graph necessary for each application of Theorem 3.2 may be achieved as described in the proof of Corollary 3.2. □

4 Concluding remarks

The recursive constructions described in Sections 2 and 3 above highlight the need to produce further small examples. We performed a computer search for a large set of 14 $\{0, 5\}$-intersecting STS(15). We assumed a cyclic automorphism c of order 7 acting on the large set, and tried to construct two "base" STS(15), S_1 and S_2, such that of the 65 orbits of triples under the action of c, the same 5 representatives of 5 orbits occur in both systems, while of the remaining 60 orbits, each is represented once in one of the two systems. In such a purported large set of 14 $\{0, 5\}$-intersecting STS(15), the cross-system must be 2-rotational. There are three such systems among the STS(15); in the standard listing [8] these are: No. 1 (=PG(3, 2)), No. 16, and No. 61.

There are many choices for the representatives of the 5 orbits to occur in both S_1 and S_2. *A priori*, any set of 5 blocks, one each from the five different orbits forming any one of the three 2-rotational systems, could be chosen. Our search was not exhaustive in that not all of these choices were considered. But of the several choices tested, including the unique one in which the 5 blocks form a parallel class (and then the cross-system would necessarily be the system No. 1), none produced a solution.

Clearly, the question of the existence of a large set of 14 $\{0, 5\}$- intersecting STS(15) remains open.

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