2-colorings of complete graphs with small number of monochromatic K_4 subgraphs.

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Abstract

Denote by $k_t(G)$ the number of cliques of order t in the graph G. Let $k_t(n) = min\{k_t(G) + k_t(\bar{G}) : |G| = n\}$, where $\bar{G}$ denotes the complement of G, and $|G|$ denotes the order of G. Let $c_t(n) = \frac{k_t(n)}{\binom{n}{t}}$, and let $c_t = \lim_{n\to\infty} c_t(n)$. An old conjecture of Erdös [E], related to Ramsey's theorem, states that $c_t = 2^{1-\binom{t}{2}}$. It was shown false by Thomason [T] for all $t \geq 4$. We present a class of simply describable Cayley graphs which also show the falsity Erdös's conjecture for $t = 4$. These graphs were found by a computer search and though of large orders ($2^{10} - 2^{14}$), they are rather simple and highly regular. The smallest upper bound for c_4 obtained by us is $0.976501\times\frac{1}{32}$, and is given by the graph on power set of 10 element set (and hence of order 2^{10}) determined by the configuration $\{1, 3, 4, 7, 8, 10\}$, and by the graph on power set of 11 elements (and hence of order 2^{11}) determined by the configuration $\{1, 3, 4, 7, 8, 10, 11\}$. It is also shown that the ratio of edges to non-edges in a sequence contradicting the conjecture for $t = 4$ may approach 1, unlike in the sequences of graphs Thomason used in [T].

1. Introduction.

Denote by $k_t(G)$ the number of cliques of order t in the graph G. Let $k_t(n) = min\{k_t(G) + k_t(\bar{G}) : |G| = n\}$, where $\bar{G}$ denotes the complement of G, and $|G|$ denotes the order of G. Let $c_t(n) = \frac{k_t(n)}{\binom{n}{t}}$, and let $c_t = \lim_{n\to\infty} c_t(n)$. Thus $c_t(n)$ denotes the minimum proportion of monochromatic K_t's in a coloring of the edges of K_n with two colors. An old conjecture of Erdös [E], related to Ramsey's theorem, states that $c_t = 2^{1-\binom{t}{2}}$. It follows from Goodman's work [G], that the conjecture is true for $t = 3$. Erdös and Moon showed in [EM] that the modified conjecture for complete bipartite subgraphs of bipartite graphs is true. Sidorenko [S] showed that the modified conjecture for cycles is true, and not true for certain incomplete subgraphs. Erdös's conjecture is obviously true for random graphs, and it follows from results of various people that it is also true for "pseudo-random" graphs (see [GS], [FRW], [T1]). For more details about the "modified conjecture" and "psuedo-random" graphs with respect to the conjecture see [FR]. Thomason [T] disproved the conjecture in general for all $t \geq 4$, producing an infinite sequence

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from a single underlying graph leading to a limit smaller than that which the conjecture stipulates. He obtained the following results: $c_4 < \frac{1}{33} < 0.976 \times \frac{1}{32}$, $c_5 < 0.906 \times 2^{1-\binom{5}{2}}$, and $c_t < 0.936 \times 2^{1-\binom{t}{2}}$ for $t > 5$. His underlying graphs are formed by vectors in orthogonal geometries. As for the lower bound, Giraud [Gi] showed that $c_4 > \frac{1}{46}$. On the other hand the authors showed in [FR] that the conjecture not only holds for "pseudo-random" graphs, but also for graphs obtained by "small perturbations" from "pseudo-random" graphs.

It is easy to realize that in order to obtain an infinite sequence $\{G_n\}_{n=0}^{\infty}$ of graphs with a given value of $\lim_{n\to\infty} \frac{k_4(G_n)+k_4(\bar{G}_n)}{\binom{|G_n|}{4}}$ it suffices to find just one graph that satisfies certain conditions (see Lemma 2 here, Lemma 1 in [T]). We are going to present an alternative way of obtaining Cayley graphs as the underlying graphs to produce counterexamples to the conjecture for $t = 4$. For a finite set X and a $F \subset \{1, 2, 3, 4, ..., |X|\}$ the graph $G_{X,F}$ has as its vertex set all subsets of X, two subsets $x, y \subset X$ are then connected by an edge if $|x \triangle y| \in F$ (where $x \triangle y$ is the symmetric difference of x and y). A computer was used to search for configurations of X, F to obtain a class of graphs which all lead to infinite sequences with c_4 smaller than $\frac{1}{32}$. The lowest upper bound for c_4 obtained by this method is $0.976501 \times \frac{1}{32}$ and is given by the sequence with $G_{X,F}$ as its underlying graph, where either $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $F = \{1, 3, 4, 7, 8, 10\}$, or $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $F = \{1, 3, 4, 7, 8, 10, 11\}$. Thomason's graphs exhibit ratio of edges to non-edges not approaching 1 as n tends to ∞ which (as Thomason remarked in [T]) runs counter to the received wisdom (prior to his work). As our results show the ratio of edges to non-edges in a sequence of graphs may approach 1: for instance consider the sequence of graphs determined by the graph $G_{X,F}$ where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ and $F = \{1, 4, 5, 8, 9, 11\}$ which leads to $c_4 \leq 0.987314 \times \frac{1}{32}$. The nth member of the sequence has $2^{12} \cdot 4095 \cdot n^2 + 2^{13} \cdot \binom{n}{2}$ edges and $2^{12} \cdot 4096 \cdot n^2$ non-edges, hence the ratio of edges to non-edges is $\frac{4096 - \frac{1}{n}}{4096}$ which approaches exactly 1. Also notice that for this sequence the neighborhood of every vertex of the nth member of the sequence has size $4095 \cdot n + n - 1$, thus the ratio of the size of the neighborhood to the order of the graph is $\frac{4096n-1}{2^{13} \cdot n}$ which approaches $\frac{1}{2}$ as n tends to ∞. We also obtained graphs giving rise to sequences in which the ratio of edges to non-edges does not approach 1 and/or in which the ratio of the size of the neighborhood to the order does not approach $\frac{1}{2}$.

2. Methods.

For a finite set X and $F \subset \{1, 2, ..., |X|\}$ $\bar{F}$ denotes $\{1, 2, ..., |X|\} - F$. It follows that $\bar{G}_{X,F} = G_{X,\bar{F}}$.

An ordered triple $\langle f_0, f_1, f_2\rangle$ is an X, F-triple, if $f_0, f_1, f_2 \subset X$, $|f_i| \in F$ for each $i \leq 2$, and $|f_i \triangle f_j| \in F$ for all $i \neq j \leq 2$. An ordered pair $\langle f_0, f_1\rangle$ is an X, F-pair, if $f_0, f_1 \subset X$, $|f_i| \in F$ for each $i \leq 1$, and $|f_i \triangle f_j| \in F$ for all $i \neq j \leq 1$. A singleton $\langle f_0\rangle$ is an X, F-singleton, if $f_0 \subset X$, an $|f_0| \in F$.

Lemma 1: Let X be a finite set, and let $F \subset \{1, 2, ..., |X|\}$. Let $tc(X, F)$ denote the number of X, F-triples, $pc(X, F)$ the number of X, F-pairs, and $sc(X, F)$ the number of X, F-singletons. Then $k_4(G_{X,F}) = \frac{2^{|X|}}{24} tc(X, F)$, $k_3(G_{X,F}) = \frac{2^{|X|}}{6} pc(X, F)$, and $k_2(G_{X,F}) = \frac{2^{|X|}}{2} sc(X, F)$.

Proof: Easy and hence left to the interested reader.

Similarly as Thomason did, we shall produce an infinite sequence of graphs from a single graph:

Def. 2: Let $G = \langle V, E\rangle$ be a graph, and let n be a positive integer. The graph $G_n = \langle V_n, E_n\rangle$ is defined as follows: let $\{B_v : v \in V\}$ be a system of mutually disjoint sets of size n. Then $V_n = \bigcup\{B_v : v \in V\}$. If $a, b \in B_v$, then $\{a, b\} \in E_n$, and if $a \in B_u$, $b \in B_v$, $u \neq v$, then $\{a, b\} \in E_n$ iff $\{u, v\} \in E$.

Lemma 2: If all graphs in an infinite sequence of graphs $\{G_n\}_{n=0}^{\infty}$ were obtained from a single graph G of size t as in Def. 2, then

$$\lim_{n\to\infty} \frac{k_4(G_n) + k_4(\bar{G}_n)}{\binom{tn}{4}} = \frac{24(k_4(G) + k_4(\bar{G})) + 36k_3(G) + 14k_2(G) + t}{t^4} .$$

Proof: The straight forward calculations are left to the reader.

We shall call the number $32 \cdot \frac{24(k_4(G)+k_4(\bar{G}))+36k_3(G)+14k_2(G)+t}{t^4}$ the Erdös number of the graph G.

Corollary 3: Let X be a finite set, and let $F \subset \{1, 2, ..., |X|\}$. Let $tc(X, F)$ denote the number of X, F-triples, $tc(X, \bar{F})$ denote the number of $X, \bar{F}$-triples, $pc(X, F)$ the number of X, F-pairs, and $sc(X, F)$ the number of X, F-singletons. Then the Erdös's number of $G_{X,F}$ =

$$\frac{tc(X, F) + tc(X, \bar{F}) + 6pc(X, F) + 7sc(X, F) + 1}{2^{3|X|-5}} .$$

Proof: Follows directly from the previous lemma.

Our task was to find such a set X and such a family $F \subset \{1, 2, ..., |X|\}$ so that the Erdös number of $G_{X,F}$ is less than 1. In the following we shall briefly describe the algorithm to compute the Erdös number for given X and F which was then used for an orderly search for suitable configurations.

Based on Lemma 3, it sufficed to compute the number of X, F-triples, X, F-pairs, and X, F-singletons:

Consider $\langle f_0, f_1, f_2\rangle$, an ordered triple of mutually distinct subsets of X. Denote $|f_i|$ as a_i $(i \leq 2)$, $|f_0 \triangle f_1|$ as a_3, $|f_0 \triangle f_2|$ as a_4, and $|f_1 \triangle f_2|$ as a_5. Let $x_{012} = f_0 \cap f_1 \cap f_2$, let $x_{01} = (f_0 \cap f_1) - x_{012}$, let $x_{02} = (f_0 \cap f_2) - x_{012}$, let $x_{12} = (f_1 \cap f_2) - x_{012}$, let $x_0 = f_0 - (f_1 \cup f_2)$, let $x_1 = f_1 - (f_0 \cup f_2)$, and let $x_2 = f_2 - (f_0 \cup f_1)$. Then $x_0, x_1, x_2, x_{01}, x_{02}, x_{12}, x_{012}$ are mutually disjoint and $f_0 \cup f_1 \cup f_2 = x_0 \cup x_1 \cup x_2 \cup x_{01} \cup x_{02} \cup x_{12} \cup x_{012}$. Let $m_0 = |x_0|$, $m_1 = |x_1|$, $m_2 = |x_2|$, $m_{01} = |x_{01}|$, $m_{02} = |x_{02}|$, $m_{12} = |x_{12}|$, and $m_{012} = |x_{012}|$. Since f_0, f_1 and f_2 are mutually distinct, $2 \leq |f_0 \cup f_1 \cup f_2|$, and so $2 \leq m_0 + m_1 + m_2 + m_{01} + m_{02} + m_{12} + m_{012} \leq |X|$. Thus

$$m_0 + m_{01} + m_{02} + m_{012} = a_0,$$

$$m_1 + m_{01} + m_{12} + m_{012} = a_1,$$

$$m_2 + m_{02} + m_{12} + m_{012} = a_2,$$

$$m_0 + m_{02} + m_1 + m_{12} = a_3,$$

$$m_0 + m_{01} + m_2 + m_{12} = a_4,$$

$$m_1 + m_{01} + m_2 + m_{02} = a_5.$$

For each solution of these equations calculate

$$\binom{|X|}{m_0} \cdot \binom{|X|-m_0}{m_1} \cdot \binom{|X|-m_0-m_1}{m_2} \cdot \binom{|X|-m_0-m_1-m_2}{m_{01}} \cdot \binom{|X|-m_0-m_1-m_2-m_{01}}{m_{02}} \cdot$$

$$\cdot \binom{|X|-m_0-m_1-m_2-m_{01}-m_{02}}{m_{12}} \cdot \binom{|X|-m_0-m_1-m_2-m_{01}-m_{02}-m_{12}}{m_{012}} \quad .$$

The sum of these numbers for all 5-tuples $\langle a_0, a_1, a_2, a_3, a_4, a_5\rangle$ then represents the number of all X, F-triples.

Similarly for the number of all X, F-pairs and X, F-singletons.

For a given $|X|$ the computer program calculates the Erdös number of $G_{X,F}$ for all possible families F in the lexicographical order. A result is output only when a new minimal value of the Erdös number is found. The Erdös numbers for $|X| = 10$, 11, 12, 13, and 14 were completely calculated in the above fashion.

The programs were written in the programming language C and the results were calculated on VAX 11/780 machine. The results were then later verified on SUN 4/280-S machine. For more on the programming aspects of the project and for complete list of results see [FR1].

References

[E] P.Erdös, On the number of complete subgraphs contained in certain graphs, *Publ. Math. Inst. Hung. Acad. Sci., VII, ser. A* 3 (1962), 459-464.

[G] A.W.Goodman, On sets of acquaintances and strangers at any party, *Amer. Math. Monthly*, 66 (1959), 778- 783.

[EM] P. Erdös and J.W.Moon, On subgraphs on the complete bipartite graph, *Canad. Math. Bull.* 7 (1964), 35-39.

[FR] F.Franek and V.Rödl, Erdös's conjecture on multiplicities of complete subgraphs is locally true, *submitted for publication.*

[FR1] F.Franek and V.Rödl, Disproving Erdös's conjecture on multiplicities of complete subgraphs using computer, *Tech. Rep., Dept. of Comp. Sci. & Systems, McMaster University, Hamilton, Ont., Canada*, 1988.

[FR2] F.Franek and V.Rödl, Erdös's conjecture on multiplicities of complete subgraphs is locally true, *Tech. Rep., Dept. of Comp. Sci. & Systems, McMaster University, Hamilton, Ont., Canada*, 1988.

[FRW] P.Frankl, V.Rödl, R.M.Wilson, The number of submatrices of given type in a Hadamard matrix and related results, *J. Comb. Theory*, 44 (1988), 317-328.

[Gi] G.Giraud, Sur le probleme de Goodman pour les quadrangles et la majoration des nombres de Ramsey, *J. Combin. Theory Ser. B* 30 (1979),237-253.

[GS] R.L.Graham, J.H.Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* 14 (1971), 45-48.

[S] A.F.Sidorenko, Cycles in Graphs and Functional inequalities, *manuscript in Russian*, and also private communication.

[T] A.Thomason , A disproof of a conjecture of Erdös in Ramsey theory, *preprint.*

[T1] A.Thomason , Pseudo-random graphs, in "Proceedings of Random Graphs, Poznan, '85", (M. Karonski, ed.), *Discrete Applied Math.*, 307-331.