A note on Erdös's conjecture on multiplicities of complete subgraphs

Lower upper bound for cliques of size 6 *

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1 The purpose of the note

Denote by $k_t(G)$ the number of cliques of order t in graph G. Let $k_t(n) = min\{k_t(G)+k_t(\overline{G}) : |G| = n\}$, where $\overline{G}$ denotes the complement of G, and $|G|$ denotes the order of G. Let $c_t(n) = k_t(n)/\binom{n}{t}$, and let $c_t = \lim_{n\to\infty} c_t(n)$. An old conjecture of Erdös, related to Ramsey's theorem, states that $c_t = 2^{1-\binom{t}{2}}$. It was shown false by Thomason for all $t \geq 4$ ([3],[4]). Franek and Rödl ([1]) presented a simpler counterexample to the conjecture for $t = 4$ derived from a simple Cayley graph of order 2^{10} obtained by a computer search giving essentially the same upper bound for c_4 as Thomason's. In this note we show that the same graph gives rise to two sequences of graphs, one a counteraxmple for $t = 5$ and the other for $t = 6$ improving the original Thomason's $c_5 < 0.906{\cdot}2^{-9}$ to $c_5 \leq 0.885834{\cdot}2^{-9}$ (though Jagger, Thomason, and Šťovíček [2] obtained a better $c_5 \leq 0.8801{\cdot}2^{-9}$), and Thomason's

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original $c_6 < 0.936{\cdot}2^{-14}$ to $c_6 \leq 0.744514{\cdot}2^{-14}$ (though meanwhile [2] gave a bit worse $0.7641{\cdot}2^{-14}$). If *weak Rödl's conjecure* that $c_t 2^{\binom{t}{2}} \to 0$ is true, then the bounds of the Ramsey number $r(t,t)$ improve, while if the *strong Rödl's conjecture* that $c_t 2^{\binom{t}{2}} \to 0$ *exponentially fast* is true, then the bounds of $r(t,t)$ improve exponentially. The interesting aspects of the new and previous bounds for c_5 and c_6 is that they corroborate Rödl's conjecture. It is interesting to mention that the referee of this note obtained $c_7 \leq 0.715527$ for the same graph, though it had not been verified yet.

2 A brief description of the method

The method from [1] was used again. The vertices of graph G are all subsets of $X = \{1,2,3,4,5,6,7,8,9,10\}$. $x, y \subset X$ form an edge if and only if $|x \triangle y| \in F = \{1,3,4,7,8,10\}$, where $\triangle$ denotes the operation of symmetric difference. A sequence $\{nG\}$ of graphs is constructed from G in the same way as described in [4] or [1]. It is not then hard to verify that

$$c_5 \leq \lim_{n\to\infty} \frac{k_5(nG)+k_5(\overline{nG})}{\binom{|nG|}{5}} = \frac{120(k_5(G)+k_5(\overline{G}))+240k_4(G)+150k_3(G)+30k_2(G)+|G|}{|G|^5}$$

and

$$c_6 \leq \lim_{n\to\infty} \frac{k_6(nG)+k_6(\overline{nG})}{\binom{|nG|}{6}} = \frac{720(k_6(G)+k_6(\overline{G}))+1800k_5(G)+1560_4(G)+540k_3(G)+62k_2(G)+|G|}{|G|^6}$$

Since we cannot compute $k_t(G)$ directly, we instead computed a number of (ordered) sequences of subsets of X, $\langle x_1, \cdots, x_t \rangle$, so that $|x_i| \in F$ and $|x_i \triangle x_j| \in F$. This is based on an observation that $k_{t+1}(G) = \frac{2^{10}}{(t+1)!} s_t(F)$ (and $k_{t+1}(\overline{G}) = \frac{2^{10}}{(t+1)!} s_t(\overline{F})$), where s_t is the number of such sequences of length t. The sequences were counted by being generated by a computer program (see [1]). Thus,

$$c_5 \leq \frac{s_4(F)+s_4(\overline{F})+10s_3(F)+25s_2(F)+15s_1(F)+1}{2^{40}}$$

and

$$c_6 \leq \frac{s_5(F)+s_5(\overline{F})+15s_4(F)+65s_3(F)+90s_2(F)+31s_1(F)+1}{2^{50}}$$

Since computer-generated results that cannot be easily verified are always suspect, an utmost care was used in checking the programs. First, the routines to calculate $s_t(F)$ can be checked (and were) whether they work properly by using $F = \{1,2,3,4,5,6,7,8,9,10\}$ which should lead to a value of $(2^{10}-1)(2^{10}-2)...(2^{10}-t)$. Second, the values were calculated by a two independently written set of programs in a span of three years. Thus, we can be reasonably confident in the results. The results were obtained using various large SUN machines, the first set of programs was written in C and the other was written in C++. The calculations required use of arbitrary precision numbers, however with the exception of the computations of $s_t(F)$, they all can be done manually.

3 Results

Upper bound for c_5:
cardinality family: $F = \{1, 3, 4, 7, 8, 10\}$
$s_1 = 506$, $s_2 = 125730$, $s_3 = 14734170$, $s_4 = 742203000$
complementary cardinality family: $\overline{F} = \{2, 5, 6, 9\}$
$s_4 = 1009617840$
numerator=1902313381
denominator=2147483648 ($2^{31} = 2^{40-9}$)
result=0.88583369785919785499572753906250

Upper bound for c_6:
cardinality family: $F = \{1, 3, 4, 7, 8, 10\}$
$s_1 = 506$, $s_2 = 125730$, $s_3 = 14734170$, $s_4 = 742203000$, $s_5 = 13677741000$
complementary cardinality family: $\overline{F} = \{2, 5, 6, 9\}$
$s_5 = 25382760480$
numerator=51162598917
denominator=68719476736 ($2^{36} = 2^{50-14}$)
result=0.744513802303117699921131134033203125

References

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