Solutions to the Oberwolfach problem for orders 18 to 40

A. Deza[1], F. Franek[1], W. Hua[1], M. Meszka[2], and A. Rosa[3]

[1] Department of Computing and Software
McMaster University, Hamilton, Ontario, Canada L8S 4K1
email: deza/franek/huaw@mcmaster.ca
[2] Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30 30-059 Kraków, Poland
email: meszka@agh.edu.pl
[3] Department of Mathematics and Statistics
McMaster University, Hamilton, Ontario, Canada L8S 4K1
email: rosa@mcmaster.ca

October 10, 2008

Abstract. The Oberwolfach problem (OP) asks whether K_n (for n odd) or K_n minus a 1-factor (for n even) admits a 2-factorization where each 2-factor is isomorphic to a given 2-factor F. The order n and the type of the 2-factor F are the parameters of the problem. For $n \leq 17$, the existence of a solution has been resolved for all possible parameters. There are also many special types of 2-factors for which solutions to OP are known. We provide solutions to OP for all orders n, $18 \leq n \leq 40$. The computational results for higher orders were obtained using the SHARCNET high-performance computing cluster.

1 Introduction

A *2-factor* of a graph G is a 2-regular spanning subgraph of G. A *2-factorization* of G is an edge-disjoint partition of the edge set of G into 2-factors. Determining if the complete graph K_{2k+1} has a 2-factorization where the 2-factors are isomorphic to each other is known as the Oberwolfach problem. The problem, when generalized to graphs of even order, asks if $K_{2k} \setminus I$, where I is a 1-factor, has a 2-factorization where the 2-factors are isomorphic to each other.

More specifically, an instance $\mathrm{OP}(n; a_1, \ldots, a_m)$ of the Oberwolfach problem asks if there is a 2-factorization of K_n ($K_n \setminus I$ for even n) such that each 2-factor is isomorphic to $C_{a_1} \cup \ldots \cup C_{a_m}$. Decomposing a graph of order n into 2-factors necessitates that we have $\sum a_i = n$, $a_i \geq 3$.

The problem was first introduced by Gerhard Ringel and named the Oberwolfach problem as it was inspired by a question whether participants at a mathematical meeting at the Oberwolfach Institute could be seated during various dinners at the conference so that everybody would sit next to any other participant exactly once.

Since the problem was introduced, many papers on the topic have appeared. With an exception of four cases (OP(6; 3^2), OP(12; 3^4), OP(9; 4, 5), and OP(11; 3^2, 5)) for which solutions are known *not* to exist, solutions were produced for all orders $n \leq 17$ (see [AB06]) and for many special cases (for instance $\mathrm{OP}(n; r^k, n-rk)$ for all $n \geq 6kr - 1$, see [HJ01]). A comprehensive survey by B. Alspach can be found in [CD96], with more up-to-date results in [CD06].

2 Methods

If we consider looking for solutions computationally, the naive brute-force approach runs in $\mathrm{O}((n!)^{\lfloor \frac{n-1}{2} \rfloor})$ time, and the problem is already intractable for $n \geq 18$. We present five methods of constructing possible solutions for different instances, all but one depending only on n. These methods facilitate significant reduction of the search and were successfully used to construct solutions for all orders between 18 and 40. More precisely, we were able to construct solutions for $\mathrm{OP}(n; \cdot)$, $18 \leq n \leq 40$, with the exception of $\mathrm{OP}(18; 3^6)$ and $\mathrm{OP}(33; 3^{11})$; however, solutions for both of these instances are well known since these correspond to an NKTS(18), a nearly Kirkman triple system of order 18 (cf., e.g., [VS93,MR01]), and to a KTS(33), a Kirkman triple system of order 33 (cf., e.g., [RW71]).

2.1 The method used for $n \equiv 3 \pmod 4$

This case is the simplest, and most of the remaining cases are variations of this one. Rather than search for $k = \frac{n-1}{2}$ edge-disjoint 2-factors of K_n, we seek a single *base* 2-factor satisfying certain properties. Developing then this base 2-factor according to a prescribed group (i.e. letting this group act on the base 2-factor) produces the remaining 2-factors of the 2-factorization.

We identify the vertex set of K_n with the set $V = Z_k \times \{1, 2\} \cup \{\infty\}$, and let $\alpha : V \to V$ be such that $\alpha(\infty) = \infty$, $\alpha(j_i) = j_i + 1 \pmod k$, $j \in Z_k, i = 1, 2$. We then apply Bose's well-known "method of pure and mixed differences" (cf., e.g., [MH86]).

An edge of F between $s, t \in Z_k \times \{i\}$ $(i = 1, 2)$, is of pure difference j, $j \leq \frac{k}{2}$, of type i if and only if $t - s \equiv j \pmod k$ or $t - s \equiv -j \pmod k$. An edge of F between $s \in Z_k \times \{1\}$, $t \in Z_k \times \{2\}$ is of mixed difference j if and only if $t - s \equiv j \pmod k$. Let us call an edge joining ∞ to an element of $Z_k \times \{i\}$ an *i-infinity edge.*

Necessary and sufficient conditions for the base 2-factor F to produce a 2-factorization are:

1. F contains exactly one i-infinity edge for $i = 1, 2$
2. F contains exactly one edge of pure difference j of type i for $i = 1, 2$, $1 \leq j \leq \frac{k-1}{2}$
3. F contains exactly one edge of mixed difference j for $0 \leq j < k$

With the given α which is an automorphism of the resulting 2-factorization, and the conditions on the base 2-factor, exhibiting solutions for these instances is much faster because of the reduced search space. Finding a base 2-factor at this point can be done using brute-force backtracking with reasonable pruning.

This method was successfully used for orders 19, 23, 27, 31, 35, and 39.

2.2 The method for $n \equiv 0 \pmod 4$

Similarly to the first case, we seek a base 2-factor which yields, upon an action of a group of order $k = \frac{n-2}{2}$ on it, a 2-factorization of $K_n \setminus I$. Here we identify the vertex set of K_n with $Z_k \times \{1,2\} \cup \{\infty_1, \infty_2\}$, and let $\alpha : V \to V$ be given by $\alpha(\infty_i) = \infty_i$, $\alpha(j_i) = j_i + 1 \pmod k$, $j \in Z_k, i = 1, 2$.

The conditions on the base 2-factor F for a 2-factorization remain the same, except that exactly one mixed difference is forbidden from F; the edges with this mixed difference, together with the edge $\{\infty_1, \infty_2\}$ form the 1-factor I deleted from K_n. In our computations for this case, we forbade the mixed difference 1.

This method was successfully used for orders 20, 24, 28, 32, 36, and 40.

2.3 The method for $n \equiv 2 \pmod 4$

This method is the same as in the previous case, except that instead of forbidding a mixed difference, we forbid the pure differences $\frac{k}{2}$ of type i, $i = 1, 2$.

This method was successfully used for orders 18, 22, 26, 30, 34 and 38 with the exception of $\mathrm{OP}(18; 3^6)$ which, as stated before, is already known.

2.4 The method for $n \equiv 1 \pmod 4$

The method from the first case, where $n \equiv 3 \pmod 4$, can be generalized to obtain a method for $n \equiv 1 \pmod 4$. After choosing an infinity element ∞, the remaining vertices are partitioned into r sets of size $w = \frac{n-1}{r}$, where r is the largest power of two that divides $n - 1$.

Let the r sets be $Z_w \times \{i\}$, $1 \le i \le r$. We now seek $\frac{r}{2}$ base 2-factors in K_n satisfying certain conditions outlined below such that under the automorphism α given by $\alpha(\infty) = \infty$, $\alpha(j_i) = j_i + 1 \pmod w$, $j \in Z_w, i = 1, \ldots, r$, a 2-factorization of K_n (with all 2-factors isomorphic) results. Such a 2-factorization is said to be r-*rotational*. In this sense, a solution in the case where $n \equiv 3 \pmod 4$ is a 2-rotational 2-factorization.

Generalizing from section 2.1, we say that an edge between $s \in Z_w \times \{i\}, t \in Z_w \times \{j\}$ has mixed difference k of type (i, j) provided $t - s \equiv k \pmod w$.

Necessary and sufficient conditions on the 2-factors for an r-rotational solution are:

1. Exactly one edge in the union of the 2-factors is an i-infinity edge, $1 \le i \le r$
2. Exactly one edge in the union of the 2-factors has a pure difference j of type i, $1 \le i \le r$, $1 \le j \le \frac{w-1}{2}$

3. Exactly one edge in the union of the 2-factors has a mixed difference k of type (i,j), $1 \leq i < j \leq r$, $0 \leq k < w$

This method was successfully used for orders 21, 25, 29, and 37. One will note that for $n = 33$, we have $r = 32$, and so the method here is reduced to an exhaustive search. In this case, we show, by a different method, how to exhibit solutions for all instances except for $\mathrm{OP}(33; 3^{11})$.

Different approach for $n \equiv 1 \pmod 4$ Another method can be used to deal with the case $n \equiv 1 \pmod 4$. Suppose that $w = \frac{n-1}{2}$; hence w is even. After choosing the infinity element ∞, partition the remaining vertices into two sets $Z_w \times \{i\}, i = 1, 2$. To construct a 2-factor F, consider two cases.

Suppose first that F contains a cycle of length at least 5. We need F to satisfy the following conditions:

1. F has exactly one i-infinity edge for $i = 1, 2$
2. One component of F of size at least 5 contains the sequence of vertices $(0_1, (\frac{w}{2})_1, (\frac{w}{2})_2, 0_2)$
3. F contains exactly one edge of each pure difference of type 1 and exactly one edge of each pure difference of type 2
4. Every mixed difference except $\frac{w}{2}$ appears exactly once in F

The action of the permutation α on the vertex set of F produces w 2-factors $F_0, F_1, \ldots, F_{w-1}$, each isomorphic to F. To get a required 2-factorization, in the 2-factor F_k, for $k = \frac{w}{2}, \frac{w}{2}+1, \ldots, w-1$, we have to replace the path $k_1, (\frac{w}{2}+k)_1, (\frac{w}{2}+k)_2, k_2$ with $k_1, (\frac{w}{2}+k)_2, (\frac{w}{2}+k)_1, k_2$. This operation replaces two edges of pure difference $\frac{w}{2}$ with two edges of mixed difference $\frac{w}{2}$ in each 2-factor F_k, and yields a proper 2-factorization.

Consider now the case when F contains one cycle of length 4 and one cycle of length 3. We want to find F' satisfying the following conditions.

1. F' contains the cycle $0_1, \infty, (\frac{w}{2})_1, j_2, 0_1$, where $j \neq 0, \frac{w}{2}$
2. F' contains the cycle $0_2, (\frac{w}{2})_2, i_1, 0_2$, where $i \neq 0, \frac{w}{2}$
3. F' contains exactly one edge of each pure difference in $1, 2, \ldots, \frac{w}{2}-1$ of type 1, and exactly one edge of each pure difference in $1, 2, \ldots, \frac{w}{2}$ of type 2
4. Every mixed difference appears exactly once in F'

Similarly to the above, α produces w 2-factors $F'_0, F'_1, \ldots F'_{w-1}$. To get a proper 2-factorization, in the 2-factor F'_k, for $k = \frac{w}{2}, \frac{w}{2}+1, \ldots, w-1$, we have to replace the path $k_1, \infty, (\frac{w}{2}+k)_1$ with the edge $k_1, (\frac{w}{2}+k)_1$ and moreover the edge $k_2, (\frac{w}{2}+k)_2$ with the path $k_2, \infty, (\frac{w}{2}+k)_2$. Notice that such replacement does not change the structure of F'_k.

The presented method was successfully used for $n = 33$ in all cases except in one case where the 2-factor has to have all its cycles of length 3.

3 Conclusion

Using variations on the 2-rotational approach, we presented five methods for restricting a search for possible solutions for the Oberwolfach problem. The methods were successful and we were able to obtain by computer search solutions for all orders $18 \leq n \leq 40$, with two exceptions, both of which have known solutions. For the higher orders we used SHARCNET computing facilities, due to the size of the search. Our results further substantiate the conjecture that OP(6; 3^2), OP(9; 4, 5), OP(11; 3^2, 5), OP(12; 3^4) are the only instances which have no solutions.

Since each order has many types of 2-factors to be considered (and the bigger the order, the more types to deal with, see Table 1), a complete listing of results is too large for inclusion in this paper. In the appendix, we list some selected results. A comprehensive set of results can be found available at `http://optlab.mcmaster.ca/~oberwolfach/`.

Acknowledgments

Research supported in part by the Natural Sciences and Engineering Research Council of Canada through the Discovery Grants, NSERC/MITACS, Canada Research Chairs, and Canada Graduate Scholarships programs.

This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca). We would like to thank them for their generous allocation of CPU time on their shared memory and distributed clusters for this project.

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A Appendix

A.1 Selected Results

Only selected results are listed here; a comprehensive listing can be found at `http://optlab.mcmaster.ca/~oberwolfach/`.

n	# of instances	n	# of instances
18	33	30	331
19	39	31	391
20	49	32	468
21	60	33	556
22	73	34	660
23	88	35	779
24	110	36	927
25	130	37	1087
26	158	38	1284
27	191	39	1510
28	230	40	1775
29	273		

Table 1. Number of OP instances by order

$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 1_2\ 3_1\ 4_2\ 5_2\ 7_2\ \infty_2\ 5_1\ 2_2\ 6_1\ 6_2\ 3_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 1_2\ 3_1\ 4_2\ 6_2\ 7_2\ 2_2\ 6_1\ 3_2)\ (5_1\ 5_2\ \infty_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 1_2\ 3_1\ 6_2\ 5_1\ 5_2\ 7_2\ 4_2)\ (6_1\ 3_2\ 2_2\ \infty_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 1_2\ 3_1\ 6_2\ 6_1\ 7_2\ 4_2)\ (5_1\ 2_2\ 3_2\ 5_2\ \infty_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 1_2\ 3_1\ 6_2\ 3_2\ 4_2)\ (5_1\ 2_2\ \infty_2\ 6_1\ 7_2\ 5_2)$

$$\vdots$$

$(0_1\ \infty_1\ 0_2\ 1_1\ 3_1\ 1_2)\ (2_1\ 5_1\ 7_2)\ (4_1\ 4_2\ \infty_2)\ (6_1\ 7_1\ 2_2)\ (3_2\ 5_2\ 6_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 1_2\ 7_1\ 3_2)\ (4_1\ 2_2\ 5_1\ 6_2\ \infty_2)\ (4_2\ 5_2\ 7_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 2_2\ 3_2\ 6_2)\ (4_1\ 4_2\ 7_1\ 5_2)\ (5_1\ 7_2\ 1_2\ \infty_2)$
$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 3_2\ 5_1\ 6_2)\ (4_1\ 7_1\ 1_2)\ (6_1\ 2_2\ \infty_2)\ (4_2\ 5_2\ 7_2)$
$(0_1\ \infty_1\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 2_2)\ (3_1\ 5_2\ 7_2\ \infty_2)\ (5_1\ 1_2\ 6_2)\ (6_1\ 3_2\ 4_2)$

Fig. 1. Base 2-factors for $n = 18$

$$(0_1\ \infty\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 5_2\ 8_1\ 2_2\ 7_2\ 6_1\ 8_2\ 6_2\ 3_2\ 4_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 7_2\ 6_1\ 6_2\ 5_2\ 8_1\ 3_2)\ (2_2\ 4_2\ 8_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 6_1\ 6_2\ 5_1\ 7_2\ 8_2\ 3_2)\ (8_1\ 4_2\ 2_2\ 5_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 5_2\ 8_1\ 2_2\ 4_2)\ (6_1\ 7_2\ 6_2\ 3_2\ 8_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 5_2\ 4_2\ 6_2)\ (6_1\ 7_2\ 3_2\ 8_1\ 2_2\ 8_2)$$

$$\vdots$$

$$(0_1\ \infty\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 1_2\ 2_2\ 5_2)\ (4_1\ 8_1\ 4_2\ 7_1\ 7_2)\ (5_1\ 3_2\ 8_2\ 6_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 3_2\ 5_1\ 7_2)\ (4_1\ 8_1\ 4_2)\ (7_1\ 1_2\ 8_2)\ (2_2\ 5_2\ 6_2)$$
$$(0_1\ \infty\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 4_2\ 8_1)\ (4_1\ 1_2\ 7_1\ 8_2)\ (5_1\ 5_2\ 7_2)\ (2_2\ 3_2\ 6_2)$$
$$(0_1\ \infty\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 5_2)\ (3_1\ 8_1\ 3_2\ 4_2)\ (5_1\ 1_2\ 7_2\ 2_2)\ (6_1\ 6_2\ 8_2)$$
$$(0_1\ \infty\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1)\ (3_1\ 1_2\ 4_2)\ (5_1\ 7_2\ 8_2)\ (6_1\ 2_2\ 6_2)\ (8_1\ 3_2\ 5_2)$$

Fig. 2. Base 2-factors for $n = 19$

$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 2_2\ 8_1\ 3_2\ 5_2\ 6_2\ 6_1\ 8_2\ 4_2\ 7_2\ \infty_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 2_2\ 8_1\ 3_2\ 7_2\ 4_2\ 5_2\ \infty_2)\ (6_1\ 6_2\ 8_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 2_2\ 8_1\ 8_2\ 6_1\ \infty_2\ 4_2)\ (3_2\ 6_2\ 5_2\ 7_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 2_2\ 8_1\ 3_2\ 5_2\ \infty_2)\ (6_1\ 6_2\ 7_2\ 4_2\ 8_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 1_2\ 5_1\ 2_2\ 8_1\ 3_2\ \infty_2)\ (6_1\ 6_2\ 5_2\ 7_2\ 4_2\ 8_2)$$

$$\vdots$$

$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 1_2\ 5_1\ 3_2)\ (4_1\ 8_1\ 2_2\ 6_2)\ (7_1\ 4_2\ \infty_2)\ (5_2\ 7_2\ 8_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 6_1\ 1_2\ 7_1)\ (4_1\ 6_2\ 7_2\ \infty_2)\ (5_1\ 2_2\ 5_2\ 3_2)\ (8_1\ 4_2\ 8_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1\ 2_1)\ (3_1\ 7_1\ 1_2)\ (4_1\ 6_2\ 8_2)\ (5_1\ 8_1\ 5_2)\ (6_1\ 2_2\ \infty_2)\ (3_2\ 4_2\ 7_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 2_2)\ (3_1\ 8_1\ 4_2\ \infty_2)\ (5_1\ 3_2\ 6_1\ 8_2)\ (1_2\ 6_2\ 5_2\ 7_2)$$
$$(0_1\ \infty_1\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 2_2)\ (3_1\ 8_1\ 5_2)\ (5_1\ 1_2\ 8_2)\ (6_1\ 4_2\ \infty_2)\ (3_2\ 6_2\ 7_2)$$

Fig. 3. Base 2-factors for $n = 20$

$$\{\ (0_1\ \infty\ 0_3\ 1_1\ 2_1\ 4_1\ 0_2\ 3_1\ 1_2\ 2_2\ 4_2\ 1_3\ 3_2\ 2_3\ 0_4\ 3_3\ 4_4\ 4_3\ 3_4\ 1_4\ 2_4),$$
$$(0_2\ \infty\ 0_4\ 0_1\ 4_2\ 4_1\ 0_3\ 2_1\ 2_3\ 1_2\ 1_3\ 4_3\ 3_3\ 1_1\ 2_4\ 3_1\ 1_4\ 2_2\ 4_4\ 3_2\ 3_4)\ \}$$
$$\{\ (0_1\ \infty\ 0_3\ 1_1\ 2_1\ 4_1\ 0_2\ 3_1\ 1_2\ 2_2\ 4_2\ 1_3\ 3_2\ 2_3\ 0_4\ 4_3\ 1_4\ 4_4)\ (3_3\ 2_4\ 3_4),$$
$$(0_2\ \infty\ 0_4\ 0_1\ 4_2\ 4_1\ 0_3\ 2_3\ 2_1\ 4_4\ 3_1\ 1_4\ 3_2\ 2_4\ 2_2\ 3_4\ 1_2\ 1_3)\ (1_1\ 3_3\ 4_3)\ \}$$

$$\vdots$$

$$\{\ (0_1\ \infty\ 0_3\ 1_1)\ (2_1\ 4_1\ 0_2\ 1_2)\ (3_1\ 3_2\ 1_3\ 4_3)\ (2_2\ 2_3\ 1_4)\ (4_2\ 0_4\ 2_4)\ (3_3\ 3_4\ 4_4),$$
$$(0_2\ \infty\ 0_4\ 3_2)\ (0_1\ 2_2\ 3_3)\ (1_1\ 1_4\ 2_1\ 3_4)\ (3_1\ 1_2\ 0_3)\ (4_1\ 4_3\ 2_4)\ (4_2\ 1_3\ 2_3\ 4_4)\ \}$$
$$\{\ (0_1\ \infty\ 0_3)\ (1_1\ 2_1\ 0_2)\ (3_1\ 3_2\ 4_2)\ (4_1\ 1_3\ 0_4)\ (1_2\ 3_3\ 4_4)\ (2_2\ 1_4\ 3_4)\ (2_3\ 4_3\ 2_4),$$
$$(0_2\ \infty\ 0_4)\ (0_1\ 3_3\ 4_3)\ (1_1\ 3_2\ 2_3)\ (2_1\ 4_1\ 4_4)\ (3_1\ 1_4\ 2_4)\ (1_2\ 1_3\ 3_4)\ (2_2\ 4_2\ 0_3)\ \}$$

Fig. 4. Base 2-factors for $n = 21$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 9_1\ 14_1\ 5_1\ 1_2\ 6_1\ 3_2\ 10_1\ 4_2\ 13_1\ 2_2\ 15_1\ 5_2\ 6_2\ 10_2\ 12_1\ 13_2\ 11_1\ 15_2\ 9_2\ 11_2\ 14_2\ 7_2\ 12_2\ \infty)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 9_1\ 14_1\ 5_1\ 1_2\ 6_1\ 3_2\ 10_1\ 4_2\ 13_1\ 2_2\ 15_1\ 5_2\ 6_2\ 11_2\ 7_2\ 14_2\ 12_1\ 13_2\ 10_2\ 12_2\ \infty)\ (11_1\ 9_2\ 15_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 9_1\ 14_1\ 5_1\ 1_2\ 6_1\ 3_2\ 10_1\ 4_2\ 13_1\ 2_2\ 15_1\ 5_2\ 6_2\ 11_2\ 13_2\ 9_2\ 15_2\ 11_1\ 12_2\ \infty)\ (12_1\ 10_2\ 7_2\ 14_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 9_1\ 14_1\ 5_1\ 1_2\ 6_1\ 3_2\ 10_1\ 4_2\ 13_1\ 2_2\ 15_1\ 5_2\ 6_2\ 12_2\ 9_2\ 11_1\ 15_2\ 10_2\ \infty)\ (12_1\ 13_2\ 11_2\ 7_2\ 14_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1\ 2_1\ 4_1\ 7_1\ 3_1\ 9_1\ 14_1\ 5_1\ 1_2\ 6_1\ 3_2\ 10_1\ 4_2\ 13_1\ 2_2\ 15_1\ 5_2\ 6_2\ 12_2\ 9_2\ 11_2\ 7_2\ 14_2)\ (11_1\ 13_2\ 12_1\ \infty\ 10_2\ 15_2)$

$$\vdots$$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 3_1\ 9_1)\ (5_1\ 10_1\ 1_2\ 2_2)\ (6_1\ 7_2\ 14_1\ 10_2)\ (11_1\ 6_2\ 12_1\ 14_2)\ (13_1\ 3_2\ 13_2\ 11_2)\ (15_1\ 4_2\ 15_2\ \infty)\ (5_2\ 9_2\ 12_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 3_1\ 9_1)\ (5_1\ 10_1\ 1_2\ 7_2)\ (6_1\ 2_2\ 12_1\ 15_2)\ (11_1\ 5_2\ \infty)\ (13_1\ 10_2\ 14_2)\ (14_1\ 9_2\ 12_2)\ (15_1\ 3_2\ 4_2)\ (6_2\ 11_2\ 13_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 11_1)\ (3_1\ 9_1\ 14_1\ 1_2)\ (5_1\ 2_2\ 6_1\ 7_2)\ (10_1\ 5_2\ 10_2\ 14_2)\ (12_1\ 3_2\ 13_1\ 6_2)\ (15_1\ 4_2\ 11_2\ \infty)\ (9_2\ 12_2\ 13_2\ 15_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 11_1)\ (3_1\ 9_1\ 14_1\ 1_2)\ (5_1\ 2_2\ 6_1\ 12_2)\ (10_1\ 11_2\ 10_2\ 14_2)\ (12_1\ 5_2\ 7_2)\ (13_1\ 15_2\ \infty)\ (15_1\ 4_2\ 9_2)\ (3_2\ 6_2\ 13_2)$

$(0_1\ 8_1\ 8_2\ 0_2\ 1_1)\ (2_1\ 4_1\ 7_1\ 11_1)\ (3_1\ 9_1\ 4_2)\ (5_1\ 10_1\ 1_2)\ (6_1\ 9_2\ 15_2)\ (12_1\ 14_2\ \infty)\ (13_1\ 2_2\ 7_2)\ (14_1\ 11_2\ 12_2)\ (15_1\ 3_2\ 5_2)\ (6_2\ 10_2\ 13_2)$

Fig. 5. Base 2-factors for $n = 33$, with some $a_i \geq 5$

$(0_1\ \infty\ 8_1\ 14_2)\ (2_1\ 3_1\ 5_1\ 9_1)\ (4_1\ 7_1\ 13_1\ 15_2)\ (6_1\ 2_2\ 12_2\ 11_2)\ (10_1\ 15_1\ 3_2\ 10_2)\ (11_1\ 4_2\ 7_2\ 5_2)\ (1_1\ 0_2\ 8_2)\ (12_1\ 9_2\ 13_2)\ (14_1\ 1_2\ 6_2)$

$(0_1\ \infty\ 8_1\ 14_2)\ (2_1\ 3_1\ 5_1\ 9_1)\ (4_1\ 7_1\ 1_2\ 14_1)\ (6_1\ 7_2\ 10_2)\ (10_1\ 15_1\ 12_2)\ (11_1\ 6_2\ 11_2)\ (12_1\ 4_2\ 5_2)\ (13_1\ 2_2\ 9_2)\ (3_2\ 13_2\ 15_2)\ (1_1\ 0_2\ 8_2)$

Fig. 6. Base 2-factors for $OP(33; 4^6, 3^3)$ and $OP(33; 4^3, 3^7)$