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On a conjecture of Erdős for multiplicities of cliques

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ABSTRACT

Denote by $k_t(G)$ the number of cliques of order t in a graph G having n vertices. Let $k_t(n) = \min\{k_t(G) + k_t(\overline{G})\}$ where $\overline{G}$ denotes the complement of G. Let $c_t(n) = k_t(n)/\binom{n}{t}$ and c_t be the limit of $c_t(n)$ for n going to infinity. A 1962 conjecture of Erdős stating that $c_t = 2^{1-\binom{t}{2}}$ was disproved by Thomason in 1989 for all $t \geqslant 4$. Tighter counterexamples have been constructed by Jagger, Šťovíček and Thomason in 1996, by Thomason for $t \leqslant 6$ in 1997, and by Franek for $t = 6$ in 2002. We investigate a computational framework to search for tighter upper bounds for small t and provide the following improved upper bounds for $t = 6, 7$ and 8: $c_6 \leqslant 0.7445 \times 2^{1-\binom{6}{2}}$, $c_7 \leqslant 0.6869 \times 2^{1-\binom{7}{2}}$, and $c_8 \leqslant 0.7002 \times 2^{1-\binom{8}{2}}$. The constructions are based on a large but highly regular variant of Cayley graphs for which the number of cliques and cocliques can be expressed in closed form. Note that if we consider the quantity $e_t = 2^{\binom{t}{2}-1}c_t$, the new upper bound of 0.687 for e_7 is the first bound for any e_t smaller than the lower bound of 0.695 for e_4 due to Giraud in 1979.

1. Introduction

Denote by $k_t(G)$ the number of cliques of order t in a graph G having n vertices. Let $k_t(n) = \min\{k_t(G) + k_t(\overline{G})\}$ where $\overline{G}$ denotes the complement of G. The cliques in $\overline{G}$ are referred to as cocliques. If we want to be specific about their sizes, we talk of t-cliques and t-cocliques. Let $c_t(n) = k_t(n)/\binom{n}{t}$ and $c_t = \lim_{n\to\infty} c_t(n)$. Since we can view G and $\overline{G}$ as a 2-colouring of the edges of the complete graph K_n, $c_t(n)$ denotes the minimum proportion of monochromatic t-cliques and t-cocliques for all 2-colourings of the edges of K_n.

A conjecture of Erdős related to Ramsey's Theorem [2], states that $c_t = 2^{1-\binom{t}{2}}$. The conjecture is clearly true for $t = 2$, and using Goodman's approach [8], one can show that the conjecture holds for $t = 3$. One of the motivations behind the conjecture is the fact that the conjecture holds for any t for random graphs. Erdős and Moon [3] showed that a modified conjecture for complete bipartite subgraphs of bipartite graphs is true. Sidorenko [11] showed that a modified conjecture for cycles is true, but not true for certain incomplete subgraphs. Franek and Rödl [5] showed that the original conjecture for $t = 4$ is true for nearly quasirandom, and hence quasirandom graphs.

Thomason [12] disproved the conjecture for $t \geqslant 4$ by exhibiting constructions achieving low numbers of cliques and cocliques. Thomason's upper bounds from [12] were: $c_4 \leqslant 0.976 \times 2^{1-\binom{4}{2}}$, $c_5 \leqslant 0.906 \times 2^{1-\binom{5}{2}}$, and $c_t \leqslant 0.936 \times 2^{1-\binom{t}{2}}$ for $t \geqslant 6$. These bounds were further improved in [13] to $c_4 \leqslant 0.9693 \times 2^{1-\binom{4}{2}}$ and $c_5 \leqslant 0.8801 \times 2^{1-\binom{5}{2}}$, in [4] to $c_6 \leqslant 0.7446 \times 2^{1-\binom{6}{2}}$, and in [9] to $c_t \leqslant 0.835 \times 2^{1-\binom{t}{2}}$ for $t \geqslant 7$. The construction used in [4] to bound c_6 is based on the approach used by Franek and Rödl [6], who tied the best upper bound for c_4. It improves the best upper bound for $t = 7$ to $c_7 \leqslant 0.7156 \times 2^{1-\binom{7}{2}}$. This bound for c_7 was mentioned in a referee report but never formally put forward.

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In this paper we investigate a computational framework to search for tighter upper bounds for small t and give improved upper bounds for $t = 6, 7$ and 8: $c_6 \leqslant 0.74444 \times 2^{1-\binom{6}{2}}$, $c_7 \leqslant 0.6869 \times 2^{1-\binom{7}{2}}$, and $c_8 \leqslant 0.7002 \times 2^{1-\binom{8}{2}}$.

Concerning the lower bound, see Conlon [1] for a recent improvement over Erdős's original application of Ramsey's Theorem, and Giraud [7] who showed that $c_4 \geqslant 0.695 \times 2^{1-\binom{4}{2}}$. Note that if we consider the quantity $e_t = 2^{\binom{t}{2}-1} c_t$, the new upper bound for e_7 is in fact smaller than Giraud's lower bound for e_4; this is the first such upper bound for any e_t.

2. Constructing counterexamples

In order to improve the upper bound for c_t for small t, we follow the approach used in [4,6] and work with graphs for which the number of cliques and cocliques can be expressed in closed form. This allows a viable search among them for the ones that exhibit the lowest numbers of cliques and cocliques.

For a positive integer X and $F \subseteq \{1, 2, \dots, X\}$, we consider the graph $G_{X,F}$ whose vertices are all 2^X subsets of $\{0, 1, \dots, X-1\}$, and two subsets x_i and x_j of $\{0, 1, \dots, X-1\}$ are connected by an edge if and only if $|x_i \bigtriangleup x_j| \in F$, where $\bigtriangleup$ denotes the operation of symmetric difference. We clearly have $\overline{G}_{X,F} = G_{X,\overline{F}}$ where $\overline{F} = \{1, 2, \dots, X\} - F$.

Since it would be too complicated to count cliques in $G_{X,F}$, we introduce the notion of (X, F)-tuples and count the (X, F)-tuples instead. Lemma 1 recalls the straightforward relationship between these quantities. For $m \geqslant 1$, an ordered m-tuple $\langle x_0, x_1, \dots, x_{m-1}\rangle$ is an (X, F)-*m-tuple* if $x_i \subseteq \{0, 1, \dots, X-1\}$, $|x_i| \in F$ for $i < m$, and $|x_i \bigtriangleup x_j| \in F$ for all $i \neq j < m$.

Lemma 1. *The number $k_{m+1}(G_{X,F})$ of cliques of size $m+1$ in the graph $G_{X,F}$ satisfies $k_{m+1}(G_{X,F}) = \frac{2^n}{(m+1)!} S_m(X, F)$ where $S_m(X, F)$ is the number of (X, F)-m-tuples.*

Proof. We simply illustrate the cases $m = 1$ and $m = 2$. Case $m = 1$: let $\{x_i, x_j\}$ be a 2-clique; i.e. an edge, in $G_{X,F}$. Clearly $\langle x_i \bigtriangleup x_j\rangle$ and $\langle x_j \bigtriangleup x_i\rangle$ are (X, F)-singletons, so we have 2 distinct (X, F)-singletons for each 2-clique and $k_2(G_{X,F}) = \frac{2^n}{2!} S_1(X, F)$. Case $m = 2$: let $\{x_i, x_j, x_k\}$ be a 3-clique in $G_{X,F}$. Clearly $\langle x_i \bigtriangleup x_j, x_i \bigtriangleup x_k\rangle$ is an (X, F)-pair of distinct elements. Considering the permutations of i, j, k we have 3! distinct (X, F)-pairs for each 3-clique and $k_3(G_{X,F}) = \frac{2^n}{3!} S_2(X, F)$. $\square$

For a positive integer d and a graph G of order n, the graph G^d is obtained by replacing each vertex of G by a d-clique; therefore G^d has dn vertices. Besides the edges within the created d-cliques, there is an edge between two vertices v_i and v_j of G^d if and only if an edge existed in G between the two vertices corresponding to the d-cliques containing v_i and v_j, $i \neq j$.

Lemma 2. *We have*

$$\lim_{d\to\infty} \frac{k_7(G^d) + k_7(\overline{G^d})}{\binom{dn}{7}} = \frac{5040(k_7(G) + k_7(\bar{G})) + 15120k_6(G) + 16800k_5(G) + 8400k_4(G) + 1806k_3(G) + 126k_2(G) + k_1(G)}{n^7}.$$

Proof. A 7-clique in G^d may arise from the following seven cases which correspond to the possible partitioning of number 7: [{7}], [{1, 6}{2, 5}{3, 4}], [{1, 1, 5}{1, 2, 4}{1, 3, 3}{2, 2, 3}], [{1, 1, 1, 4}{1, 1, 2, 3}{1, 2, 2, 2}], [{1, 1, 1, 1, 3}{1, 1, 1, 2, 2}], [{1, 1, 1, 1, 1, 2}], [{1, 1, 1, 1, 1, 1, 1}] – we have grouped them by the number of partitions. For illustration, we count the number $Q_2(d)$ of cliques corresponding to the decompositions [{1, 6}{2, 5}{3, 4}]. We have:

$$Q_2(d) = \left(\binom{2}{1}\binom{d}{1}\binom{d}{6} + \binom{2}{1}\binom{d}{2}\binom{d}{5} + \binom{2}{1}\binom{d}{3}\binom{d}{4}\right)k_2(G) = \big(2L_1(d) + 6L_2(d) + 10L_3(d)\big)\frac{d^7}{6!}k_2(G)$$

where $L_1(d) = (1-\frac{1}{d})(1-\frac{2}{d})(1-\frac{3}{d})(1-\frac{4}{d})(1-\frac{5}{d})$, $L_2(d) = (1-\frac{1}{d})^2(1-\frac{2}{d})(1-\frac{3}{d})(1-\frac{4}{d})$ and $L_3(d) = (1-\frac{1}{d})^2(1-\frac{2}{d})^2(1-\frac{3}{d})$. To derive similar formulas for the other partitions is straightforward, giving

$$\lim_{d\to\infty} \frac{k_7(G^d)}{\binom{dn}{7}} = \frac{5040k_7(G) + 15120k_6(G) + 16800k_5(G) + 8400k_4(G) + 1806k_3(G) + 126k_2(G) + k_1(G)}{n^7}.$$

A 7-coclique can only arise in one way, and thus for the number of 7-cocliques, we get

$$\lim_{d\to\infty} \frac{k_7(\overline{G^d})}{\binom{nd}{7}} = \frac{5040k_7(\overline{G})}{n^7}. \qquad \square$$

Remark. In general, the coefficients $\alpha_{m,t}$ for $k_m(G)$ in the formula reducing the computation of $\lim_{d\to\infty} \frac{k_t(G^d)+k_t(\overline{G^d})}{\binom{dn}{t}}$ to counting cliques and cocliques in the underlying graph G follow a pattern similar to the Pascal triangle equality as we have $\alpha_{m,t} = m(\alpha_{m,t-1} + \alpha_{m-1,t-1})$. See Lemma 2 for the case $t = 7$. The coefficients for $k_m(G)$ and other auxiliary results are available online at [10].

We can set $G = G_{X,F}$ and then substitute $k_m(G_{X,F})$ by $S_{m-1}(X,F)$ using Lemma 1, and restate Lemma 2 as:

Lemma 3. *For a given pair (X,F), we have*

$$\lim_{d\to\infty} \frac{k_7(G^d_{X,F}) + k_7(\overline{G^d_{X,F}})}{\binom{d2^n}{7}}$$
$$= \frac{S_6(X,F) + S_6(X,\bar{F}) + 21S_5(X,F) + 140S_4(X,F) + 350S_3(X,F) + 301S_2(X,F) + 63S_1(X,F) + 1}{2^{6n-20}}.$$

The approach used in [6] is based on an exhaustive search for a pair (X,F) achieving a low number of cliques and cocliques for $t = 4$. The identified best pair $(10, \{1,3,4,7,8,10\})$ yields a tie for the best upper bound for c_4 and was used to achieve $c_6 \leqslant 0.7446 \times 2^{1-\binom{6}{2}}$. The referee's report for [6] mentioned that the same pair yields $c_7 \leqslant 0.7156 \times 2^{1-\binom{7}{2}}$ but this bound was never formally put forward. In this paper we improve the bounds for c_t for $t = 6, 7$ and 8.

3. Computational framework

Lemma 3 provides a closed formula for computing a limit of a special sequence of graphs determined by a given pair (X,F). If this limit is small enough, it constitutes a counterexample to the conjecture of Erdős. Thus, the computational framework consists of a routine to compute all the required $S_i(X,F)$'s for a given pair (X,F) and a routine performing a search for the best (X,F). First, in Section 3.1 we discuss the approach for computing $S_i(X,F)$ that was used previously in [4,6]. This approach is rather slow and cannot be employed for $t > 4$. That is why only a single pair $(10, \{1,3,4,7,8,10\})$ was used in [4]. Then, in Sections 3.2 and 3.3 we discuss a different approach to the computation of $S_i(X,F)$'s referred to as m-approach, and a further enhancement based on symmetry. These techniques provide a significant speedup allowing an exhaustive search for $t = 6$ and 7 that was previously intractable.

3.1. Straightforward computation of S_i

For simplicity, for a given X, $\hat{X}$ denotes the set $\{0, 1, \ldots, X-1\}$.

3.1.1. Straightforward computation of $S_1(X,F)$

Generate all possible $x_0 \subseteq \hat{X}$ so that $|x_0| \in F$; then

$$S_1(X,F) = \sum_{|x_0|\in F} \binom{X}{|x_0|}.$$

3.1.2. Straightforward computation of $S_2(X,F)$

Consider an ordered pair $\langle x_0, x_1\rangle$ of mutually distinct subsets of $\hat{X}$. Clearly, $x_0 \cap (x_0 \triangle x_1)$, $x_1 \cap (x_0 \triangle x_1)$ and $x_0 \cap x_1$ are mutually disjoint. Let $m_0 = |x_0 \cap (x_0 \triangle x_1)|$, $m_1 = |x_1 \cap (x_0 \triangle x_1)|$ and $m_{01} = |x_0 \cap x_1|$. We have $m_0 + m_{01} = |x_0|$, $m_1 + m_{01} = |x_1|$, and $m_0 + m_1 = |x_0 \triangle x_1|$. In addition, we have $|x_0|$, $|x_1|$ and $|x_0 \triangle x_1| \in F$. Thus, once generating all possible valid solutions $\langle m_0, m_1, m_{01}\rangle$, we obtain the value of $S_2(X,F)$ by:

$$S_2(X,F) = \sum_{\text{all valid } \langle m_0,m_1,m_{01}\rangle} \binom{X}{m_0}\binom{X-m_0}{m_1}\binom{X-m_0-m_1}{m_{01}}.$$

3.1.3. Straightforward computation of $S_i(X,F)$ for $i > 2$

Similar computations, with increasing computation time, are performed to obtain the values of $S_i(X,F)$. We need to consider an ordered i-tuple $\langle x_0, x_1, x_2, \ldots, x_{i-1}\rangle$ of mutually distinct subsets of $\hat{X}$, and find all the valid solutions $\langle m_0, m_1, m_2, \ldots\rangle$. Then we can compute the sum of the corresponding binomial coefficients using a dynamically expanded and maintained Pascal triangle. Notice that the total number of the solutions increases rather quickly. In general, we have to consider $(2^i - 1)$ solutions to compute $S_i(X,F)$.

Table 1
Ordering of the x_i's and corresponding coefficients for S_4 computation.

x_i ordering	Coefficient
$\lvert x_0\rvert > \lvert x_1\rvert > \lvert x_2\rvert > \lvert x_3\rvert$	$4!$
$\lvert x_0\rvert > \lvert x_1\rvert > \lvert x_2\rvert = \lvert x_3\rvert$	$2\binom{4}{2}$
$\lvert x_0\rvert > \lvert x_1\rvert = \lvert x_2\rvert > \lvert x_3\rvert$	$2\binom{4}{2}$
$\lvert x_0\rvert = \lvert x_1\rvert > \lvert x_2\rvert > \lvert x_3\rvert$	$2\binom{4}{2}$
$\lvert x_0\rvert > \lvert x_1\rvert = \lvert x_2\rvert = \lvert x_3\rvert$	$\binom{4}{3}$
$\lvert x_0\rvert = \lvert x_1\rvert > \lvert x_2\rvert = \lvert x_3\rvert$	$\binom{4}{2}$
$\lvert x_0\rvert = \lvert x_1\rvert = \lvert x_2\rvert > \lvert x_3\rvert$	$\binom{4}{3}$
$\lvert x_0\rvert = \lvert x_1\rvert = \lvert x_2\rvert = \lvert x_3\rvert$	1

3.2. The m-approach to computing S_i

In Section 3.1, the S_i was obtained by finding all valid solutions and computing the sum of the corresponding binomial coefficients, a procedure with an $O(2^{iX})$ worst-case complexity. Therefore, a more efficient approach is required to speed up the computation.

The following example illustrates how knowing and storing solutions for S_{i-1}'s can be used to faster obtain solutions for S_i. For the illustration, we consider computing a solution for S_3 while having $m^* = \langle m_0^*, m_1^*, m_{01}^*\rangle$ a valid solution for S_2. We could generate a valid solution $m = \langle m_0, m_1, m_2, m_{01}, m_{02}, m_{12}, m_{012}\rangle$ for S_3 by reusing m^*, since $m_0 + m_{02} = m_0{}^*$, $m_1 + m_{12} = m_1{}^*$ and $m_{01} + m_{012} = m_{01}{}^*$. The following constraints can be used to check the validity: $0 \leqslant m_0 \leqslant m_0{}^*$, $0 \leqslant m_1 \leqslant m_1{}^*$, and $0 \leqslant m_{01} \leqslant m_{01}{}^*$. Recall that $|x_2|$ should be in F, and thus we can calculate m_2 directly: as $m_2 = z - m_{12} - m_{02} - m_{012}$ for some $z \in F$. We also need to check the symmetric difference relationships among the x_i's. However, we only need to check $|x_0 \triangle x_2| \in F$ and $|x_1 \triangle x_2| \in F$.

Remark. If m^* is a valid solution for S_i, and m is the corresponding valid solution for S_{i+1},

$$Y^* = \binom{X}{m_0{}^*}\binom{X - m_0{}^*}{m_1{}^*}\binom{X - m_0{}^* - m_1{}^*}{m_2^*}\binom{X - m_0{}^* - m_1{}^* - m_2{}^*}{m_3{}^*}\cdots$$

is the corresponding product of the binomial coefficients for m^*, and

$$Y = \binom{X}{m_0}\binom{X - m_0}{m_1}\binom{X - m_0 - m_1}{m_2}\binom{X - m_0 - m_1 - m_2}{m_3}\cdots$$

is the corresponding product of the binomial coefficients for m, then

$$Y = Y^*\binom{m_0{}^*}{m_0}\binom{m_1{}^*}{m_1}\cdots\binom{m_{01\cdots i}{}^*}{m_{01\cdots i}}\binom{X - m_0{}^* - m_1{}^* - m_{01}{}^* - \cdots}{m_i}.$$

Similarly, to compute S_i we only need to consider 2^{i-1} m's, if we reuse the results from the computation of S_{i-1}.

3.3. Exploiting symmetry

This technique to further speed up the computation of S_i relies on the inherent symmetries of the m_i's. We shall illustrate the technique on S_2: if $\langle m_0, m_1, m_{01}\rangle$ is a valid solution for S_2 with $m_0 \neq m_1$, then $\langle m_1, m_0, m_{01}\rangle$ is also a valid solution. Since the products of the corresponding binomial coefficients for those two solutions are the same, we only need to compute the product of the binomial coefficients for one solution and multiply it by 2.

Similarly, the symmetries can be exploited for computing S_i for $i \geqslant 2$. Thus, one can fix the order of the x_i and take into account multiplicities by multiplying by the corresponding coefficients. We therefore need, for example for the computation of S_7, to consider only about 1% of the total number of solutions. Table 1 shows the coefficients used for S_4. The coefficients for other S_i's are available online at [10].

Note that while the determination of S_i and $\bar{S}_i$ for the first i's is very fast even without exploiting the symmetry, the computational gain increases with i. Table 2 shows the number of solutions that need to be computed when we used the pair $(X, F) = (11, \{3, 4, 7, 8, 10, 11\})$ to compute S_4, S_5 and S_6.

4. New upper bounds for c_6, c_7, and c_8

Using the approach described in Sections 3.2 and 3.3 we performed an exhaustive search on (X, F) for $X = 9, 10, 11$ and 12 for $t = 6$ and 7, using code written in C++.

Table 2
Exploiting symmetry for $(X, F) = (11, \{3, 4, 7, 8, 10, 11\})$.

i	# of solutions in S_i	# of solutions exploiting symmetry	Ratio	# of solutions in $\bar{S}_i$	# of solutions exploiting symmetry	Ratio
4	15,668	1813	3.0%	4477	794	5.9%
5	377,196	17,625	0.5%	86,978	8214	1.7%
6	9,104,496	160,626	0.08%	1,145,103	55,803	0.46%

Table 3
The values of $S_i(X, F)$ and $S_i(X, \bar{F})$ when $(X, F) = (10, \{1, 3, 4, 7, 8\})$.

i	1	2	3	4	5
$S_i(X, F)$	505	125,010	14,562,090	726,780,600	13,191,935,400
$S_i(X, \bar{F})$	518	135,726	17,463,606	1,028,265,840	26,106,252,480

Table 4
The values of $S_i(X, F)$ and $S_i(X, \bar{F})$ when $(X, F) = (11, \{3, 4, 7, 8, 10, 11\})$.

i	1	2	3	4	5	6
$S_i(X, F)$	1002	490,050	113,148,090	11,590,147,800	506,500,533,000	14,677,396,549,200
$S_i(X, \bar{F})$	1045	556,842	146,860,362	17,896,958,640	950,437,303,200	21,359,851,904,160

Table 5
The values of $S_i(X, F)$ and $S_i(X, \bar{F})$ when $(X, F) = (12, \{1, 3, 4, 7, 8, 11, 12\})$.

i	1	2	3	4	5	6	7
$S_i(X, F)$	2027	2,030,562	986,934,042	223,874,343,000	21,997,023,741,000	868,195,804,568,400	23,207,044,770,478,800
$S_i(X, \bar{F})$	2068	2,158,860	1,120,464,444	279,763,013,640	32,608,321,954,560	1,762,344,151,444,800	47,296,455,155,389,440

Besides the usual testing and verification, we also computationally checked the new program by recomputing previously known values as well as theoretically known ones. We first computed the values of $S_1, \dots, S_6$ for all the previously used pairs (X, F) and obtained the same results, using a tiny fraction of the computation time previously required. We then computed the values of $S_1, \dots, S_7$ for *full families* because for such a family $\{1, 2, \dots, X\}$ the number *i-tuples* can be expressed using Lemma 1 with a closed form formula $S_i = \frac{(2^X-1)!}{(2^X-i-1)!}$. The computed and theoretical values coincided, which is a strong indication that the generation of valid solutions is both sound and complete.

The best results were achieved for $t = 6$ by $(X, F) = (10, \{1, 3, 4, 7, 8\})$ yielding $c_6 \leqslant 0.74444 \times 2^{1-\binom{6}{2}}$, see Table 3. For $t = 7$ by $(X, F) = (11, \{3, 4, 7, 8, 10, 11\})$ yielding $c_7 \leqslant 0.6869 \times 2^{1-\binom{7}{2}}$, see Table 4.

Representing the pair (X, F) as the characteristic vector of F as a subset of $\{1, 2, \dots, X\}$, one notices the best results for $t = 6$, respectively $t = 7$, are obtained with $(X, F) = [1011001100]$, respectively $(X, F) = [00110011011]$, so a natural candidate to consider for $t = 8$ is $(X, F) = [101100110011]$. Setting accordingly $(X, F) = (12, \{1, 3, 4, 7, 8, 11, 12\})$ indeed yielded an improved upper bound $c_8 \leqslant 0.7002 \times 2^{1-\binom{8}{2}}$, see Table 5.

Proposition 1. *We have* $c_6 \leqslant 0.7445 \times 2^{1-\binom{6}{2}}$, $c_7 \leqslant 0.6869 \times 2^{1-\binom{7}{2}}$, *and* $c_8 \leqslant 0.7002 \times 2^{1-\binom{8}{2}}$.

5. Conclusion and future work

We presented a computational framework for computing the ratio of monochromatic t-cliques and the number of all t-subsets for a specific Cayley graph determined by a pair (X, F). The program allows for searching for counterexamples to a 1960 Erdős's conjecture on multiplicities of complete subgraphs. We described a significant speedup obtained by the so-called m-approach and considering inherent symmetries. As a result, we were able to improve the known upper bounds for $t = 6, 7$ and 8.

The computational framework presented lends itself to straightforward parallelisations. A parallel version of our program will allow us to explore larger t's and also to enlarge the search space for smaller values of t. The first task thus will be to search for a better pair (X, F) for $t = 8$ to improve the upper bound for c_8, and to redo the searches for $t = 4, 5, 6$ and 7 in larger search spaces.

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