TWO-FACTORIZATIONS OF SMALL COMPLETE GRAPHS II: THE CASE OF 13 VERTICES

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Abstract. We establish that for each of the 5005 possible types of 2-factorizations of the complete graph K_{13}, there exists at least one solution. We also enumerate all nonisomorphic solutions to the Oberwolfach problem OP(13;3,3,3,4).

1. Introduction

A *2-factor* of a graph G is a spanning subgraph of G which is regular of degree 2. A *2-factorization* of G is an edge-disjoint decomposition of G into 2-factors.

The *type of a 2-factor* F in an n-vertex graph is a partition π whose parts are the lengths of the components of F. The *type of a 2-factorization* $\mathcal{F}$ of K_{2m+1} is a sequence $T = (t_1, t_2, \ldots, t_m)$ of types of 2-factors which we assume to be nondecreasing with respect to some ordering of types of 2-factors (usually lexicographic).

If the number of types of 2-factors of K_{2m+1} is x, it is easily seen that the number of distinct types of 2-factorizations of K_{2m+1} is $\binom{m+x-1}{m}$.

The main existence problem for 2-factorizations of K_{2m+1} is to characterize those types T for which a 2-factorization of type T exists. If all 2-factors in a 2-factorization have the same type $\pi = (\pi_1, \ldots, \pi_s)$, we have an instance of the well known Oberwolfach problem OP$(2m+1; \pi_1, \ldots, \pi_s)$ first formulated by G. Ringel in a meeting in Oberwolfach in 1967 (see [1]).

In [3], all nonisomorphic 2-factorizations were enumerated by types for $n \leq 9$. There are 252 types of 2-factorizations for K_{11}, and it was shown in [3] that at least one 2-factorization exists of each type, with the exception of a single type $T = (3, 3, 5)$ for which no 2-factorization exists.

In this note we continue the investigations of [3] for the case of the complete graph K_{13}. There are 5005 types of 2-factorizations in this case. We establish that at least one solution exists for *each* type. Also, we enumerate all nonisomorphic solutions to the Oberwolfach problem OP$(13; 3, 3, 3, 4)$.

2. Two-factorizations of K_{13}

There are 10 distinct types of 2-factors in K_{13}:
$a = (3, 3, 3, 4)$, $b = (3, 3, 7)$, $c = (3, 4, 6)$, $d = (3, 5, 5)$, $e = (4, 4, 5)$, $f = (3, 10)$, $g = (4, 9)$, $h = (5, 8)$, $i = (6, 7)$, $j = (13)$. There are $\binom{6+10-1}{6} = 5005$ possible types of 2-factorizations.

It has been known previously that there is at least one solution for each instance of the Oberwolfach problem when $n = 13$, i.e. for types $aaaaaa, bbbbbb, \ldots, jjjjjj$. We have used the computer to establish that for each of the 5005 types of two-factorizations of K_{13}, there is at least one solution. A list of sample solutions for each type can be found at
http://www.cas.mcmaster.ca/˜franek.

We search for solutions using an exhaustive recursion-based backtracking algorithm avoiding redundant tests. At the beginning we generate all nonisomorphic types of 2-factorizations $T = (t_1, t_2, t_3, t_4, t_5, t_6)$, $t_i \in \{a, b, c, d, e, f\}$ such that $t_i \leq t_{i+1}$, $1 \leq i < 6$. For each of these types we search for a solution that is stored.

In order to avoid testing isomorphic partial 2-factorizations that cannot be extended to full ones we introduce the following rules: (i) The first vertex of the first cycle of the first 2-factor is always 1. (ii) The lowest vertices of the first cycles of all 2-factors form a nondecreasing sequence. (iii) For each triangle (i.e. 3-cycle) $\tau = (v_1, v_2, v_3)$, $v_1 < v_2 < v_3$. (iv) For all triangles $\tau_i = (v_{i,1}, v_{i,2}, v_{i,3})$ in a 2-factor of type a ($i \in \{1, 2, 3\}$) or b ($i \in \{1, 2\}$) we enforce $v_{i,1} < v_{i+1,1}$. (v) For each cycle $u = (v_1, v_2, \ldots, v_j)$, $j > 3$ we require that $v_1 < v_i$, $1 < i \leq j$ and $v_2 < v_j$ (which prevents tracing of the same cycle in two different directions).

We achieved a significant speed-up of the execution of the program when we started from the largest cycles in each 2-factor (i.e. in order $t_6, t_5, \ldots, t_1$).

It might be tempting to conjecture that for sufficiently large orders $n > n_0$ the situation is similar to that for $n = 13$. However, the number of possible types of 2-factorizations increases very rapidly, and already for $n = 15$ there are $\binom{17+7-1}{7} = 245,157$ possible types of 2-factorizations. On the other hand, surely $n_0 > 13$: by [2], there are at least 5 types of 2-factorizations of K_{15} for which no solution exists. In each of these 5 types, there are six triangle-factors (i.e. 2-factors of type $(3, 3, 3, 3, 3)$) and the seventh 2-factor is of one of the following types:
$(3, 3, 4, 5), (3, 5, 7), (4, 4, 7), (5, 5, 5), (7, 8)$ (cf. [2]).

3. Solutions to OP$(13; 3, 3, 3, 4)$

Enumerating all solutions to the Oberwolfach problem OP$(13; 3, 3, 3, 4)$ is of some interest, for the similarity of this instance of the Oberwolfach problem to the famous Kirkman's 15-schoolgirls problem. We established that there are exactly 8 nonisomorphic solutions.

To obtain an exhaustive list of solutions of OP$(13; 3, 3, 3, 4)$ we utilized basically the same program as for finding the individual solutions (see above). The lexicographic ordering of the generated solutions together with the rules used to prevent the repeating of unsuccessful tests with permutations of 2-factors in a partial 2-factorization guarantee a small number of isomorphic solutions is generated. We proceeded as follows. If F, G are two disjoint 2-factors in K_{13} (each of type $(3, 3, 3, 4)$, of course), let $H = H_{FG}$ be the graph which is their union. Starting in turn with each of the 12 nonisomorphic graphs H, we generated in an orderly manner all solutions to OP$(13; 3, 3, 3, 4)$ corresponding to the particular graph H. Only 7 of the 12 nonisomorphic graphs H actually lead to at least one solution. The duplicates obtained from different starts were then eliminated using simple invariants, to be described below.

Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_6\}$ is a solution to OP$(13; 3, 3, 3, 4)$. The first invariant is based on the graph $\Gamma_{\mathcal{F}}$ which is the union of the *six 4-gons* in $\mathcal{F}$: it is the vector $V = (v_0, v_1, v_2, v_3, v_4)$ where v_i is the number of vertices of degree $2i$ in $\Gamma_{\mathcal{F}}$. This invariant turned out to be complete (see Table 1).

The second invariant is based on the graph Q_{ij} that is formed by the union of the 4-gons of two distinct 2-factors F_i, F_j in $\mathcal{F}$. The two 4-gons may have either two (types 1 and 2), or one (type 3), or zero (type 4) vertices in common, and if they have two vertices in common, then either the two vertices are adjacent in one of the 4-gons (type 1), or they are nonadjacent in both 4-gons (type 2). The invariant is then the vector $B = (b_1, b_2, b_3, b_4)$ where b_j is the number of pairs of 2-factors of type j, and where $b_1 + b_2 + b_3 + b_4 = 15$, of course. This invariant also turned out to be complete (see Table 1).

Table 1 which contains also the information about the order of the automorphism group of each of the solutions, is followed by the complete listing of the 8 nonisomorphic solutions.

Table 1

Type	v_0	v_1	v_2	v_3	v_4	b_1	b_2	b_3	b_4	$\|G\|$
No. 1	4	0	6	0	3	0	6	9	0	6
No. 2	2	4	4	0	3	0	8	5	2	2
No. 3	3	3	0	7	0	0	9	6	0	3

No. 4	1	6	3	0	3	0	9	3	3	6
No. 5	2	2	7	0	2	1	9	5	0	2
No. 6	2	2	6	2	1	2	8	3	2	2
No. 7	1	4	4	4	0	3	8	4	0	4
No. 8	1	0	12	0	0	3	12	0	0	24

Each solution contains the 2-factor (0 1 2 3)(4 5 6)(7 8 9)(10 11 12) and the following five 2-factors:

No. 1 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 5 11 7)(1 6 8)(2 4 12)(3 9 10)
(0 10 1 11)(2 5 9)(3 4 8)(6 7 12)
(1 3 7 5)(0 8 12)(2 6 10)(4 9 11)
(1 4 10 7)(0 6 9)(2 8 11)(3 5 12)
No. 2 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 6 12 8)(1 4 10)(2 9 11)(3 5 7)
(0 10 7 12)(1 5 11)(2 6 8)(3 4 9)
(2 4 12 5)(0 7 11)(1 3 8)(6 9 10)
(2 10 3 12)(0 5 9)(1 6 7)(4 8 11)
No. 3 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 5 1 11)(2 4 8)(3 9 10)(6 7 12)
(1 3 5 7)(0 8 12)(2 6 10)(4 9 11)
(2 9 5 11)(0 7 10)(1 6 8)(3 4 12)
(3 7 11 8)(0 6 9)(1 4 10)(2 5 12)
No. 4 (0 5 2 8)(1 4 10)(3 7 12)(6 9 11)
(1 5 3 8)(0 6 12)(2 9 10)(4 7 11)
(2 4 3 6)(0 7 10)(1 9 12)(5 8 11)
(2 7 5 12)(0 4 9)(1 3 11)(4 8 12)
(3 9 5 10)(0 2 11)(1 6 7)(4 8 12)
No. 5 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 5 9 11)(1 3 8)(2 4 10)(6 7 12)
(0 9 4 12)(1 6 10)(2 8 11)(3 5 7)
(1 4 11 7)(0 6 8)(2 5 12)(3 9 10)
(3 4 8 12)(0 7 10)(1 5 11)(2 6 9)
No. 6 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 5 9 10)(1 4 11)(2 6 8)(3 7 12)
(0 9 6 12)(1 3 8)(2 4 10)(5 7 11)
(1 5 3 10)(0 6 7)(2 9 11)(4 8 12)
(1 6 10 7)(0 8 11)(2 5 12)(3 4 9)
No. 7 (0 2 7 4)(1 9 12)(3 6 11)(5 8 10)
(0 6 8 12)(1 4 10)(2 9 11)(3 5 7)
(3 4 9 10)(0 8 11)(1 6 7)(2 5 12)
(3 9 6 12)(0 7 10)(1 5 11)(2 4 8)

(4 11 7 12)(0 5 9)(1 3 8)(2 6 10)
No. 8 (0 4 8 10)(1 5 7)(3 6 11)(2 9 12)
(1 6 7 11)(0 8 12)(2 4 10)(3 5 9)
(2 5 8 11)(0 6 9)(1 4 12)(3 7 10)
(3 4 7 12)(0 5 11)(1 9 10)(2 6 8)
(5 10 6 12)(0 2 7)(1 3 8)(4 9 11)

Let us note that W. Piotrowski in his extensive unpublished manuscript [4] gives two solutions to $\mathrm{OP}(13; 3, 3, 3, 4)$; these are isomorphic to No. 3 and No. 4 above, respectively.

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