Two squares canonical factorization

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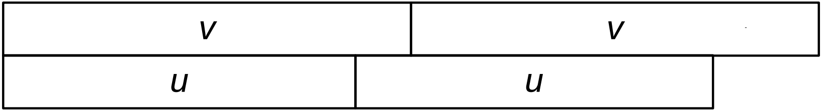
Outline

Introduction

a configuration of two ***proportional*** squares $\boldsymbol{u}^2$ and $\boldsymbol{v}^2$



has been investigated in many different contexts:

- *Smyth et al.*: investigating three squares with intention to find a position for amortization argument for the runs conjecture a unique factorization of the type

$$\boldsymbol{u} = \boldsymbol{u_1 u_2 u_1}, \ \boldsymbol{v} = \boldsymbol{u_1 u_2 u_1 u_1 u_2} \iff \frac{3|\boldsymbol{u}|}{2} < |\boldsymbol{v}| < 2|\boldsymbol{u}|$$

- in a computational framework for computations of $\sigma_d(n)$ developed by *Deza*, *F.*, and *Jiang*: such configurations are used in *Liu*'s PhD thesis to speed up computation of certain values $\sigma_d(n)$ in the $(d, n{-}d)$ table

$\sigma_d(n)$ denotes the maximum number of distinct squares in a string of length n with d distinct symbols

- *Lam*: two rightmost squares $\boldsymbol{u}^2 \lhd \boldsymbol{v}^2$ in $\boldsymbol{x}$ have a very particular structure $\boldsymbol{u} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}$ and $\boldsymbol{v} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}\boldsymbol{u_1}^{e_2}$ a primitive $\boldsymbol{u_1}$ and a non-trivial proper prefix $\boldsymbol{u_2}$ of $\boldsymbol{u_1}$, and $e_1 \geq e_2 \geq 1$.

  note that two rightmost squares in $\boldsymbol{x}$ are necessarily proportional

- *Deza, F., Thierry*: two proportional squares $\boldsymbol{u}^2 \lhd \boldsymbol{v}^2$ form a *factorizable double square* if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is primitive or $\boldsymbol{u}^2$ is rightmost in $\boldsymbol{v}^2$

  A factorizable double square has a *unique factorization* $(\boldsymbol{u_1}, \boldsymbol{u_2}, e_1, e_2)$: $\boldsymbol{u} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}$ and $\boldsymbol{v} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}\boldsymbol{u_1}^{e_2}$

where $\boldsymbol{u_1}$ is primitive, $\boldsymbol{u_2}$ a non-trivial proper prefix of $\boldsymbol{u_1}$, and $e_1 \geq e_2 \geq 1$.

moreover, there are some additional constrains to the structure of a factorizable double square

in this contribution we generalize and extend the factorization to all pairs of proportional squares starting at the same position

Basic notions

x is primitive $\iff x \neq y^p$ for any string y and any integer $p \geq 2$

Ex: *aab aab* is not primitive, while *aabaaba* is

primitive root of x: the smallest y s.t. $x = y^p$ for some integer $p \geq 1$ *(is unique and primitive)*

u^2 is primitively rooted $\iff u$ is a primitive string

x and y are *conjugates* if $x = uv$ and $y = vu$ for some u, v

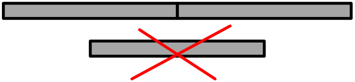
$x \lhd y \iff x$ is a proper prefix of y

A *double square* $\mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})$: $\boldsymbol{u}^2 \lhd \boldsymbol{v}^2$ and $|\boldsymbol{u}| < |\boldsymbol{v}| < 2|\boldsymbol{u}|$.

*Note that in Deza, F., Thierry this would be called a **balanced double square***

Lemma (*Synchronization Principle*)

Given a primitive string $\boldsymbol{x}$, a proper suffix $\boldsymbol{y}$ of $\boldsymbol{x}$, a proper prefix $\boldsymbol{z}$ of $\boldsymbol{x}$, and $m \geq 0$, there are exactly m occurrences of $\boldsymbol{x}$ in $\boldsymbol{y}\boldsymbol{x}^m\boldsymbol{z}$.



Lemma (*Common Factor Lemma*)

For any strings x and y, if a non-trivial power of x and a non-trivial power of y have a common factor of length $|x|+|y|$, then the primitive roots of x and y are conjugates.

In particular, if x and y are primitive, then x and y are conjugates.

Note that both x and y must repeat at least twice

these lemmas are really a folklore, but we included proofs as we did not know of a published proof of the Common Factor Lemma

A simple corollary of Common Factor Lemma:

Corollary (Uniqueness Lemma)

$\boldsymbol{x}$ and $\boldsymbol{y}$ primitive strings, $p, q \geq 1$:

(a) $\boldsymbol{x}^p = \boldsymbol{y}^q \;\Rightarrow\; \boldsymbol{x} = \boldsymbol{y} \;\&\; p = q$

(b) $p, q \geq 2, \quad \boldsymbol{x_1} \lhd \boldsymbol{x}, \quad \boldsymbol{y_1} \lhd \boldsymbol{y}$

$$\boldsymbol{x}^p \boldsymbol{x_1} = \boldsymbol{y}^q \boldsymbol{y_1} \;\Rightarrow\; \boldsymbol{x} = \boldsymbol{y} \;\&\; \boldsymbol{x_1} = \boldsymbol{y_1} \;\&\; p = q$$

(a) $x^p = y^q$

- $p = 1$

  then $\boldsymbol{x} = \boldsymbol{y}^q$, $\boldsymbol{x}$ primitive $\Rightarrow$ $q = 1$ and $\boldsymbol{x} = \boldsymbol{y}$

- $p, q \geq 2$

  $\boldsymbol{x}^p$ and $\boldsymbol{y}^q$ have a common factor of length $\geq |\boldsymbol{x}|+|\boldsymbol{y}|$, by the Common Factor Lemma $\boldsymbol{x} \sim \boldsymbol{y}$, hence $\boldsymbol{x} = \boldsymbol{y}$

(b) $\boldsymbol{x}^p\boldsymbol{x_1} = \boldsymbol{y}^q\boldsymbol{y_1}$, $p, q \geq 2$

$\boldsymbol{x}^p\boldsymbol{x_1} = \boldsymbol{y}^q\boldsymbol{y_1}$ have a common factor of length $|\boldsymbol{x}|+|\boldsymbol{y}|$, hence $x = y$

the requirement $p, q \geq 2$ is essential – for instance:

$\boldsymbol{x} = aabb$, $\boldsymbol{x_1} = aa$ and $p = 2$:

$$\boldsymbol{x}^2\boldsymbol{x_1} = aabbaabbaa$$

$\boldsymbol{y} = aabbaabba$, $\boldsymbol{y_1} = a$ and $q = 1$:

$$\boldsymbol{y}^1\boldsymbol{y_1} = aabbaabbaa$$

$$\boldsymbol{x}^2\boldsymbol{x_1} = \boldsymbol{y}^1\boldsymbol{y_1}$$

Main results

Lemma (Two Squares Factorization Lemma)

$\forall\ \mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})\ \exists$ *unique* $\boldsymbol{u_1}, \boldsymbol{u_2}, e_1, e_2$ *such that*
$\boldsymbol{u} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}$ *and* $\boldsymbol{v} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}\boldsymbol{u_1}^{e_2}$

$\boldsymbol{u_1}$ *primitive*

$\boldsymbol{u_2}$ *is a possibly trivial proper prefix of* $\boldsymbol{u_1}$

$e_1 \geq e_2 \geq 1$

Moreover,

(a) $|\boldsymbol{u_2}| = 0 \ \Rightarrow\ e_1 > e_2 \geq 1$

(b) $|\boldsymbol{u_2}| > 0 \ \Rightarrow\ \boldsymbol{v}$ *primitive*

(c) $|\boldsymbol{u_2}| > 0\ \&\ e_1 \geq 2 \ \Rightarrow\ \boldsymbol{u}$ *primitive*

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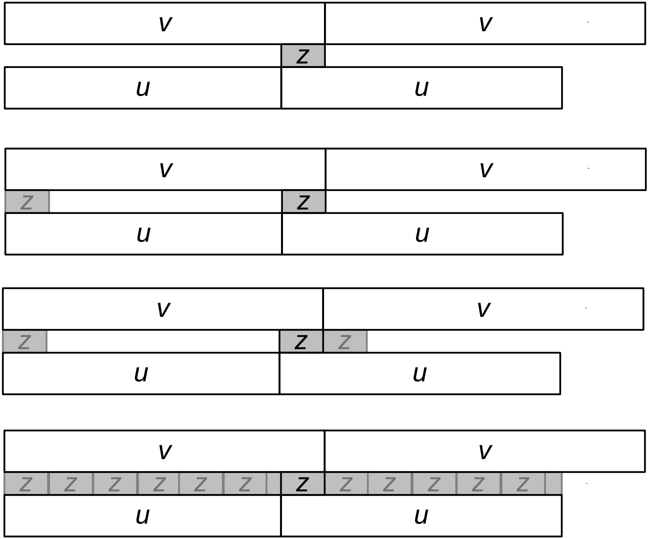
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$\exists\ k \geq 1$ s.t. $\boldsymbol{u} = \boldsymbol{z}^k \boldsymbol{z'}$ for some proper prefix $\boldsymbol{z'}$ of $\boldsymbol{z}$

$\boldsymbol{u_1}$ primitive root of $\boldsymbol{z}$

$\boldsymbol{z} = \boldsymbol{u_1}^{e_2}$ for some $e_2 \geq 1$

for some $e_1 \geq e_2 k$ and some prefix $\boldsymbol{u_2}$ of $\boldsymbol{u_1}$,

$\boldsymbol{u} = \boldsymbol{u_1}^{e_1} \boldsymbol{u_2}$ and $\boldsymbol{v} = \boldsymbol{uz} = \boldsymbol{u_1}^{e_1} \boldsymbol{u_2} \boldsymbol{u_1}^{e_2}$

now the uniqueness and other properties:

(i) $|\boldsymbol{u_2}| = 0$

$\boldsymbol{u} = \boldsymbol{u_1}^{e_1}$ and $\boldsymbol{v} = \boldsymbol{u_1}^{e_1+e_2} \Rightarrow \boldsymbol{v} = \boldsymbol{u_1}^{2(e_1+e_2)}$

$|\boldsymbol{v}| < 2|\boldsymbol{u}|$ and $e_1 \geq e_2 \Rightarrow e_1 > e_2$

uniqueness of $\boldsymbol{u_1}$ is a consequence of Uniqueness Lemma (a)

(ii) $|\boldsymbol{u_2}| > 0$

assume $(\boldsymbol{w_1}, \boldsymbol{w_2}, f_1, f_2)$

$\boldsymbol{u} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2} = \boldsymbol{w_1}^{f_1}\boldsymbol{w_2}$ & $\boldsymbol{v} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}\boldsymbol{u_1}^{e_2} = \boldsymbol{w_1}^{f_1}\boldsymbol{w_2}\boldsymbol{w_1}^{f_2}$

$$e_1 = f_1 = 1 \quad \Rightarrow \quad \boldsymbol{v} = \boldsymbol{u}\boldsymbol{u_1} = \boldsymbol{u}\boldsymbol{w_1} \quad \Rightarrow \quad \boldsymbol{u} = \boldsymbol{v}$$

WLOG assume that $f_1 > e_1 = 1$

$\boldsymbol{u} = \boldsymbol{u_1}\boldsymbol{u_2} = \boldsymbol{w_1}^{f_1}\boldsymbol{w_2}$ & $\boldsymbol{v} = \boldsymbol{u_1}\boldsymbol{u_2}\boldsymbol{u_1} = \boldsymbol{w_1}^{f_1}\boldsymbol{w_2}\boldsymbol{w_1}^{f_2} \Rightarrow$
$\boldsymbol{u_1} = \boldsymbol{w_1}^{f_2}$

$\boldsymbol{u_1}$ primitive forces $f_2 = 1$ and $\boldsymbol{u_1} = \boldsymbol{w_1}$

$\boldsymbol{u_1}\boldsymbol{u_2} = \boldsymbol{w_1}^{f_1}\boldsymbol{w_2} = \boldsymbol{u_1}^{f_1}\boldsymbol{w_2}$, implies that $f_1 = 1$
a contradiction

show that $\boldsymbol{v}$ is primitive

suppose the contrary: $\boldsymbol{v} = \boldsymbol{w}^k, k \geq 2$

$|\boldsymbol{w}| \leq \frac{|\boldsymbol{v}|}{2} \leq |\boldsymbol{u_1}^{e_1}| + |\boldsymbol{u_2}|$

$\boldsymbol{w}^{2k} = \boldsymbol{v}^2 = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2}\boldsymbol{u_1}^{e_1+e_2}\boldsymbol{u_2}\boldsymbol{u_1}^{e_2}$

$\boldsymbol{w}^{2k}$ and $\boldsymbol{u_1}^{e_1+e_2}\boldsymbol{u_2}$ have a common factor $\boldsymbol{u_1}^{e_1+e_2}\boldsymbol{u_2}$ of length $(|\boldsymbol{u_1}^{e_1}|+|\boldsymbol{u_2}|)+|\boldsymbol{u_1}^{e_2}| \geq |\boldsymbol{w}|+|\boldsymbol{u_1}|$

apply Common Factor Lemma to conclude that $\boldsymbol{w} \sim \boldsymbol{u_1}$, thus $\boldsymbol{w} = \boldsymbol{u_1}$

primitive string $\boldsymbol{u_1} = \boldsymbol{u_2}\overline{\boldsymbol{u}}_2$ aligns with $\boldsymbol{u_2}\boldsymbol{u_1}$, and so $\overline{\boldsymbol{u}}_2$ is a prefix of $\boldsymbol{u_1}$, in contradiction to Synchronization Principle

let $e_2 \geq 2$ show that $\boldsymbol{u}$ is primitive
suppose the contrary: $\boldsymbol{u} = \boldsymbol{w}^k$, $k \geq 2$

Hence $|\boldsymbol{w}| \leq \frac{|\boldsymbol{u}|}{2} = \frac{(|\boldsymbol{u_1}^{e_1}|+|\boldsymbol{u_2}|)}{2} < |\boldsymbol{u_1}^{e_1-1}|+|\boldsymbol{u_2}|$

$e_2 \geq 1$ and $e_2 \geq 2 \quad \Rightarrow \quad e_1+e_2 \geq 3$

$\boldsymbol{u_1}^{e_1}\boldsymbol{u_2} \lhd \boldsymbol{u}^2 = \boldsymbol{w}^{2k}$

so $\boldsymbol{w}^{2k}$ and $\boldsymbol{u_1}^{e_1+e_2}$ have a common factor $\boldsymbol{u_1}^{e_1}\boldsymbol{u_2}$

since $|\boldsymbol{u_1}^{e_1}\boldsymbol{u_2}| \geq |\boldsymbol{v}|+|\boldsymbol{u_1}|$, applying Common Factor Lemma,
$\boldsymbol{u_1} = \boldsymbol{w}$

this in turn implies $\boldsymbol{u} = \boldsymbol{u_1}^{e_1}\boldsymbol{u_2} = \boldsymbol{u_1}^k$, impossible since
$0 < |\boldsymbol{u_2}| < |\boldsymbol{u_1}|$

observations:

$|\boldsymbol{u_2}| > 0$ if any one of the following conditions holds:

(a) $\boldsymbol{v}$ is primitive

(b) $\boldsymbol{u}$ is primitive

(c) $\boldsymbol{u}^2$ is rightmost in $\boldsymbol{v}^2$

moreover:

(d) $|\boldsymbol{u_2}| > 0 \iff \boldsymbol{v}$ is primitive

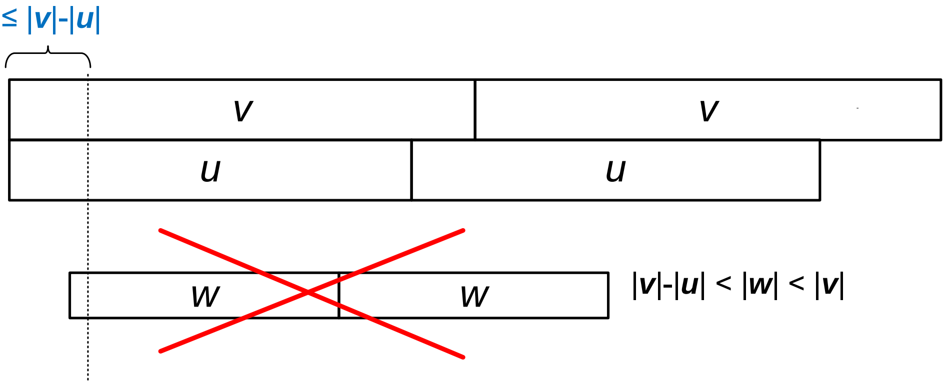
Applications

we concluded the paper with a comment and a sketch of how the canonical factorization could be applied to New Periodicity Lemma:

Lemma (2006, Fan, Puglisi, Smyth, and Turpin)

Let $\boldsymbol{x} = \mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})$, where we require that $\boldsymbol{u}^2$ be regular and that $\boldsymbol{v}$ be primitive. There is no square $\boldsymbol{w}^2$ starting at position i, $1 \leq i < |\boldsymbol{v}| - |\boldsymbol{u}|$ with $|\boldsymbol{v}| - |\boldsymbol{u}| < |\boldsymbol{w}| < |\boldsymbol{v}|$ except possibly $|\boldsymbol{w}| = |\boldsymbol{u}|$.

v^2 primitive, u^2 regular



we can report that since the final submission to PSC2014, we were able to prove using the canonical factorization an extended NPL:

Theorem

Consider a double square $\mathrm{DS}(\boldsymbol{u}, \boldsymbol{v})$ *and let* $\boldsymbol{u'}$ *be a suffix of* $\boldsymbol{u}$ *so that* $\boldsymbol{v} = \boldsymbol{uu'}$*. Let* $\boldsymbol{w}^2$ *be any square that is a factor of* $\boldsymbol{v}^2$*. Then exactly one of the following mutually exclusive cases holds:*

(a) $\boldsymbol{w} = \boldsymbol{v}$, *or*

(b) $|\boldsymbol{w}| < |\boldsymbol{u}|$, *or*

(c) $|\boldsymbol{u}| \leq |\boldsymbol{w}| < |\boldsymbol{v}|$ *and the primitive root of* $\boldsymbol{w}$ *is a conjugate of the primitive root of* $\boldsymbol{u'}$*.*

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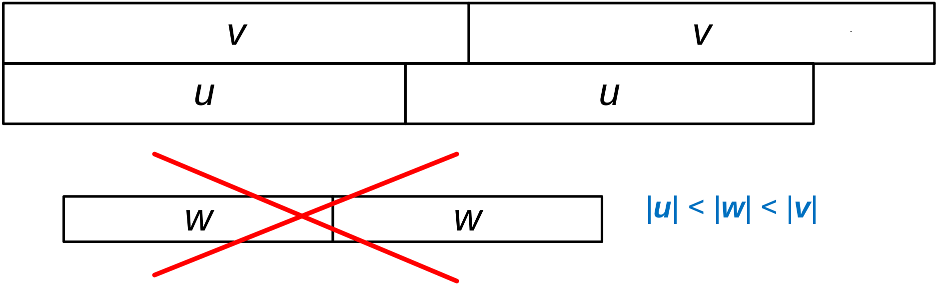

$|u| < |w| < |v|$

Lemma (*Crochemore-Rytter* (1995), *Fraenkel-Simpson* (1998))

Let $\boldsymbol{u}^2 \lhd \boldsymbol{v}^2 \lhd \boldsymbol{w}^2$ and let $\boldsymbol{u}$ be primitive, then $|\boldsymbol{u}|+|\boldsymbol{v}| \leq |\boldsymbol{w}|$.

we can also report that since the final submission to PSC2014, we were able to prove using the canonical factorization a generalization of the above lemma:

Theorem (*Bai*, *Deza*, and *F.*)

Let $\boldsymbol{u}^2 \lhd \boldsymbol{v}^2 \lhd \boldsymbol{w}^2$. Then either

(*a*) $|\boldsymbol{u}|+|\boldsymbol{v}| \leq |\boldsymbol{w}|$

or *(inclusive or)*

(*b*) *$\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ have the same primitive root*

THANK YOU

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