How many double squares can a string contain?

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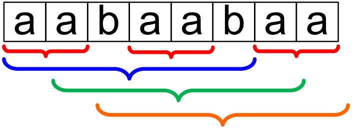
Outline

Motivation and background

We are dealing with finite strings over finite alphabets. There is no particular requirement about the order of the alphabet.

What is the *maximum number of distinct squares problem* ?

We are counting types of squares rather than their occurrences.

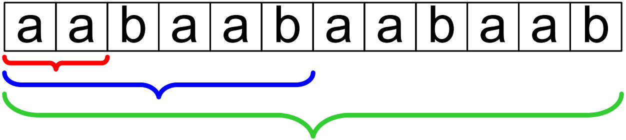


has 6 occurrences of squares, but only 4 distinct squares, *aa*, *aabaab*, *abaaba*, and *baabaa*.

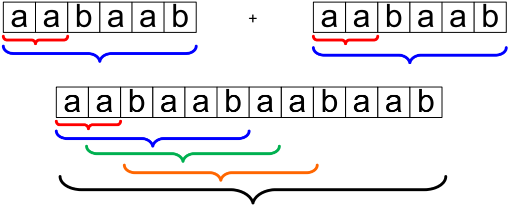
A trivial bound: the number of all occurrences of primitively rooted squares in a string of length n is bounded by $O(n \log n)$ (*Crochemore 1978*) and the number of distinct non-primitively rooted squares is $O(n)$ (*Kubica et al. 2013*)

Could it be $O(n)$? And if so, what would be the constant?

Why this is not simple? In a string of length n, $O(\log n)$ squares can start at the same position!



It is easy to compute it for short strings, so why induction cannot be used?



Concatenation does both “destroys" existing types through multiple-occurrences and “creates" new types. Of course, same holds true for the reverse process - partitioning of strings.

Theorem (*Fraenkel-Simpson*, 1998)

There are at most $2n$ distinct squares in a string of length n.

Count only the rightmost *occurrences. Fraenkel-Simpson showed that if there are three rightmost squares uu, vv, and ww starting at the same position so that $|u| < |v| < |w|$, then ww contains a farther copy of uu, based on Crochemore-Rytter (1995) Lemma showing that in such a case, $|w| \geq |u| + |v|$.*

Fraenkel-Simpson hypothesized that the number of distinct squares should be bounded by n, i.e.

$$\sigma(n) \leq n$$

where $\sigma(n) = \max \{ \ s(x) \ : \ x \textit{ is a string of length } n \ \}$.

Fraenkel-Simpson gave an infinite sequence of strings $\{x_n\}_{n=1}^{\infty}$ so that $|x_n| \nearrow \infty$ and

$$\frac{s(x_n)}{|x_n|} \nearrow 1$$

where $s(x) = \textit{number of distinct squares in } x$.

- In 2005 *Ilie* provided a simpler proof of *Fraenkel-Simpson*'s Theorem and in 2007 presented an asymptotic upper bound of $2n - \theta(\log n)$.

- In 2011 *Deza-F.* proposed a d-step approach to the problem and conjectured that $\sigma_d(n) \leq n - d$, where $\sigma_d(n) = \max \{ s(x) \ : \ x \textit{ is a string of length } n \textit{ with } d \textit{ distinct symbols} \}$.

Basic notions and tools

Definition

non-trivial power of a string x is a concatenation of m copies of x denotes as x^m; x^2 is a *square*, x^3 a *cube*.

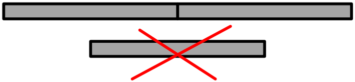
A string x is *primitive* if $x \neq y^n$ for any y and any $n \geq 2$.

primitive root of x is the smallest primitive y so that $x = y^n$.

x and y are *conjugates* if $x = uv$ and $y = vu$ for some u, v.

Lemma (*Synchronization principle*)

Given a primitive string x, a proper suffix y of x, a proper prefix z of x, and $m \geq 0$, there are exactly m occurrences of x in $yx^m z$.



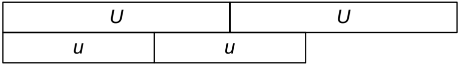
Lemma (*Common factor lemma*)

For any strings x and y, if a non-trivial power of x and a non-trivial power of y have a common factor of length $|x|+|y|$, then the primitive roots of x and y are conjugates. In particular, if x and y are primitive, then x and y are conjugates.

Double squares

- *Fraenkel-Simpson*: only two rightmost squares can start at the same position. Thus, only one rightmost square or two rightmost squares may start at any position.
- *Lam* (2009 – *unpublished*) tried bounding the number of *double squares* and hence bound the number of distinct squares. His approach is based on a taxonomy of all possible configurations of two double squares yielding a bound of $\frac{94}{48}n \approx 1.98n$.

A configuration of two squares



has been investigated in many different contexts:

- *Smyth et. al.*: with intention to find a position for amortization argument for runs conjecture.
- in computational framework by *Deza-F.-Jiang*: such configurations are used in *Liu*'s Ph.D. thesis to speed up computation of $\sigma_d(n)$.
- *Lam*: two rightmost squares have a unique structure.

Lemma

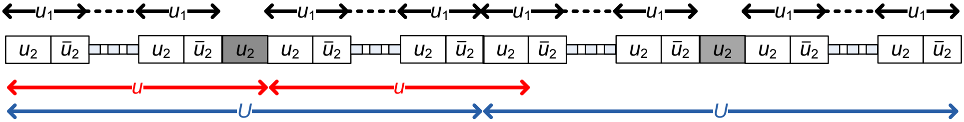
Let uu and UU be two squares in a string x starting at the same position with $|u| < |U|$ such that either

(b) *both uu and UU are rightmost occurrences, or*

(a) $|U| < |uu|$ *and either uu or UU is primitively rooted.*

Then $|u| < |U| < |uu| < |UU|$ and there is a unique primitive string u_1, a unique proper prefix u_2 of u_1, and unique integers e_1 and e_2 satisfying $1 \leq e_2 \leq e_1$ such that $\boldsymbol{u = u_1{}^{e_1} u_2}$ and $\boldsymbol{U = u_1{}^{e_1} u_2 u_1{}^{e_2}}$; i.e. uu and UU form a double square.

$$u_1^{\mathcal{U}(1)}\ u_2\ u_1^{\mathcal{U}(2)}\ u_1^{\mathcal{U}(1)}\ u_2\ u_1^{\mathcal{U}(2)}$$



Thus, only strings of length at least 10 may contain a double square: $|UU| = 2((\mathcal{U}(1)+\mathcal{U}(2))|u_1|+|u_2|) \geq 2((1+1)2+1) = 10$.

Cyclic shift (rotation) to the right is controlled by

$$lcp(u_1, \overline{u}_1)$$

while cyclic shift to the left is controlled by

$$lcs(u_1, \overline{u}_1)$$

lcp = largest common prefix
lcs = largest common suffix

$$u_1 = aaabaa,\ u_2 = aaab,\ \overline{u}_2 = aa,\ \mathcal{U}(1) = \mathcal{U}(2) = 2$$

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$u_1 = aaabaa$, $u_2 = aaab$, $\overline{u}_2 = aa$, $\mathcal{U}(1) = 2$, and $\mathcal{U}(2) = 1$.

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Definition

For a double square $\mathcal{U}$, $\overline{v}vv\overline{v}$ where $|\overline{v}| = |\overline{u}_2|$ and $|v| = |u_2|$ is an *inversion factor*

$$\mathcal{U} = u_1{}^{\mathcal{U}(1)} u_2 u_1{}^{\mathcal{U}(2)+\mathcal{U}(1)} u_2 u_1{}^{\mathcal{U}(2)} =$$

$$u_1{}^{(\mathcal{U}(1)-1)} u_2 \overline{u}_2 u_2 u_2 \overline{u}_2 u_1{}^{\mathcal{U}(2)+\mathcal{U}(1)-2} u_2 \overline{u}_2 u_2 u_2 \overline{u}_2 u_1{}^{(\mathcal{U}(2)-1)}$$

N_1 N_2

natural inversion factors

A cyclic shift of an inversion factor is an inversion factor, also controlled by $lcp(u_1, \overline{u}_1)$ and $lcs(u_1, \overline{u}_1)$.

L_1 L_2

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R_1 R_2

All inversion factors are cyclic shifts of the natural ones:

Lemma (*Inversion factor lemma*)

Given a double square $\mathcal{U}$, there is an inversion factor of $\mathcal{U}$ within the string UU starting at position $i \iff i \in [L_1, R_1] \cup [L_2, R_2]$.

Inversion factor lemma for distinct squares

Theorem (*Fraenkel-Simpson, Ilie*)

At most two rightmost squares can start at the same position.

Let us assume that 3 rightmost squares uu, UU, vv start at the same position.
By item (c) of Inversion factor lemma, uu and UU form a double square $\mathcal{U}$: $u = u_1{}^{\mathcal{U}(1)}u_2$ and $U = u_1{}^{\mathcal{U}(1)}u_2u_1{}^{\mathcal{U}(2)}$.
Since the first v contains an inversion factor, the second v must also contain an inversion factor.

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If the inversion factor in the second v *were from* $[L_2, R_2]$, *then* $|v| = |U|$, *a contradiction.*
Hence v *must not contain an inversion factor from* $[L_2, R_2]$ *and so* $u_1{}^{\mathcal{U}(1)}u_2u_1{}^{\mathcal{U}(1)+\mathcal{U}(2)-1}u_2$ *must be a prefix of* v.
Therefore vv *contains another copy of* $u_1{}^{\mathcal{U}(1)}u_2u_1{}^{\mathcal{U}(1)}u_2 = uu$, *a contradiction.*

Fundamental Lemma:

Lemma

Let x be a string starting with a double square $\mathcal{U}$. Let $\mathcal{V}$ be a double square with $\mathfrak{s}(\mathcal{U}) < \mathfrak{s}(\mathcal{V})$, then either

(a) $\mathfrak{s}(\mathcal{V}) < R_1(\mathcal{U})$, *in which case either*

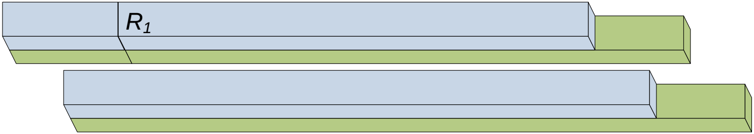
- (a_1) $\mathcal{V}$ *is an* α*-mate of* $\mathcal{U}$ *(cyclic shift), or*
- (a_2) $\mathcal{V}$ *is a* β*-mate of* $\mathcal{U}$ *(cyclic shift of U to V), or*
- (a_3) $\mathcal{V}$ *is a* γ*-mate of* $\mathcal{U}$ *(cyclic shift of U to v), or*
- (a_4) $\mathcal{V}$ *is a* δ*-mate of* $\mathcal{U}$ *(big tail),*

or

(b) $R_1(\mathcal{U}) \leq \mathfrak{s}(\mathcal{V})$, *then*

- (b_1) $\mathcal{V}$ *is a* ε*-mate of* $\mathcal{U}$ *(big gap).*

α-mate (cyclic shift):

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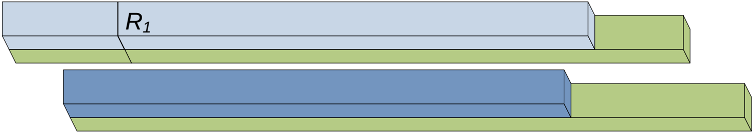
β-mate (cyclic shift of U to V)



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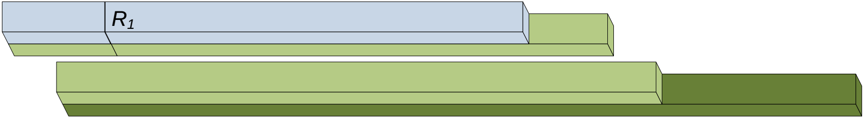
γ-mate (cyclic shift of U to v)



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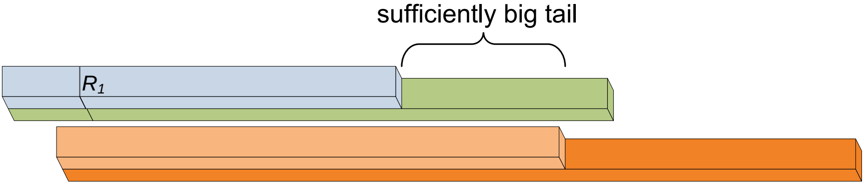
δ-mate (big tail)

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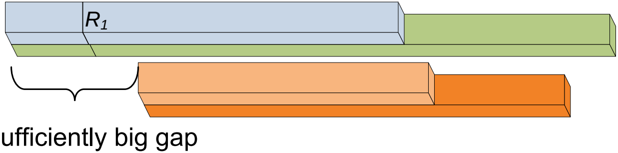
ε-mate (big gap)

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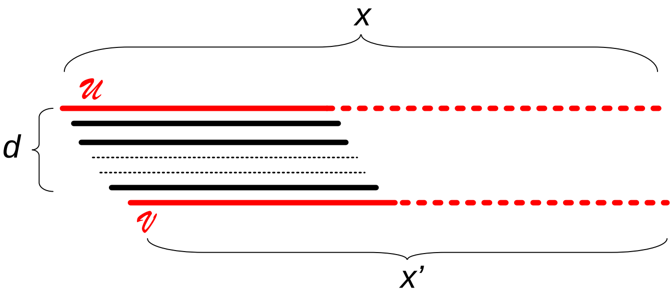
An upper bound for the number of double squares

We show by induction a bound $\delta(x) \leq \frac{5}{6}|x| - \frac{1}{3}|u|$, where uu is the shorter square of the leftmost double square of x.

The fundamental lemma basically says that either the gap $G(\mathcal{U},\mathcal{V})$ is "big" or the tail $T(\mathcal{U},\mathcal{V})$ is "big" (for δ-mate and ε-mate), or it is case of α-mate, β-mate, or γ-mate.



Lemma (Gap-Tail lemma)

$\delta(x') \leq \frac{5}{6}|x'| - \frac{1}{3}|v|$ *implies*

$\delta(x) \leq \frac{5}{6}|x| - \frac{1}{3}|u| + d - \frac{1}{2}|G(\mathcal{U},\mathcal{V})| - \frac{1}{3}|T(\mathcal{U},\mathcal{V})|$

We deal with α-mates, β-mates, and γ-mates separately.

It is possible as they form families, either a pure α-family, or $\alpha{+}\beta$-family, or $\alpha{+}\beta{+}\gamma$-family.

$\mathcal{U}$-family consists only of α-mates

Illustration of α-family with $\mathcal{U}(1) = \mathcal{U}(2)$

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Illustration of α-family with $\mathcal{U}(1) > \mathcal{U}(2)$

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It is easy to estimate the size of α-family, as it is controlled by $lcp(u_1, \overline{u}_1)$ and $lcp(y, u_2)$ where y is x without UU: the size $\leq |u_1|$.

- Either there are no other double squares, and then it can be shown directly that the bound holds, or
- There is a $\mathcal{V}$ underneath, and we can use induction using the Gap-Tail lemma. $\mathcal{V}$ must be either γ-mate, or δ-mate, or ε-mate, and the Gap-Tail lemma can be applied to propagate the bound.

$\mathcal{U}$-family consists of α-mates and β-mates

Illustration of $\alpha+\beta$-family

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It is more complicated to estimate the size of a $\alpha+\beta$-family:

$$\leq \begin{cases} \lceil \frac{\mathcal{U}(1)-\mathcal{U}(2)}{2} \rceil |u_1| & \text{if } \mathcal{U}(2) = 1 \\ \frac{\mathcal{U}(1)-\mathcal{U}(2)}{2} |u_1| & \text{if } \mathcal{U}(2) > 1 \end{cases}$$

- Either there are no other double squares, and then it can be shown directly that the bound holds, or
- There is a $\mathcal{V}$ underneath, and we can use induction using the Gap-Tail lemma. $\mathcal{V}$ must be either δ-mate, or ε-mate, and the Gap-Tail lemma can be applied to propagate the bound. (*Special care needed for ε-mate case and super-ε-mate must be put in play !*)

$\mathcal{U}$-family consists of α-mates, β-mates, and γ-mates

Illustration of $\alpha+\beta+\gamma$-family

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         [                  ][        )(       ]                   )
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It is quite complex to estimate the size of a $\alpha{+}\beta{+}\gamma$-family:

$$\leq \frac{2}{3}(\mathcal{U}(1) + 1)|u_1|$$

- Either there are no other double squares, and then it can be shown directly that the bound holds, or
- There is a $\mathcal{V}$ underneath, and we can use induction using the Gap-Tail lemma. $\mathcal{V}$ must be either δ-mate, or ε-mate, and the Gap-Tail lemma can be applied to propagate the bound.

Main theorems

Theorem

The number of double squares in a string of length n is bounded by $\lfloor 5n/6 \rfloor$.

Corollary

The number of distinct squares in a string of length n is bounded by $\lfloor 11n/6 \rfloor$.

- We presented a universal upper bound of $\frac{11n}{6}$ for the maximum number of distinct squares in a string of length n
- A bound of $\frac{5n}{6}$ for the maximum number of double squares
- It improves the universal bound of $2n$ by Fraenkel-Simpson
- It improves the asymptotic bound of $2n - \Theta(n)$ by Ilie
- The combinatorics of double squares is interesting on its own and possibly can be used for some other problems

THANK YOU

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