Two-Pattern Strings — Computing Repetitions & Near-Repetitions

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Abstract

In a recent paper we introduced infinite two-pattern strings on the alphabet $\{a, b\}$ as a generalization of Sturmian strings, and we posed three questions about them:

- Given a finite string $\boldsymbol{x}$, can we in linear time $O(|\boldsymbol{x}|)$ recognize whether or not $\boldsymbol{x}$ is a prefix/substring of some infinite two-pattern string?
- If recognized as two-pattern, can all the repetitions in $\boldsymbol{x}$ be computed in linear time?
- Given an integer ℓ, how many of these "two-pattern" strings $\boldsymbol{x}$ of length ℓ are there?

In the previous paper we were able to answer the first of these questions in the affirmative, at least for "complete" two-pattern strings $\boldsymbol{x}$. Here we show that, once a complete two-pattern string $\boldsymbol{x}$ has

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been recognized, its repetitions can all be computed in linear time using an iterative algorithm that in addition computes all the "near-repetitions" in $\boldsymbol{x}$. The third question is dealt with in a subsequent paper.

1 Introduction

In a recent paper [FLS03] the notion of two-pattern binary strings as a generalization of Sturmian strings was introduced. This paper follows on immediately from [FLS03], which we recommend that the reader consult. Nevertheless, to provide a measure of self-containment, we review the main ideas here.

Suppose an integer $\lambda \geq 1$ is given (the ***scope***), together with two nonempty strings $\boldsymbol{p}$ and $\boldsymbol{q}$ on $\{a, b\}$ such that $|\boldsymbol{p}| \leq \lambda, |\boldsymbol{q}| \leq \lambda$. We call $\boldsymbol{p}$ and $\boldsymbol{q}$ ***patterns of scope*** λ, and we require that they be ***suitable*** (see below for definitions — roughly speaking, $\boldsymbol{p}$ and $\boldsymbol{q}$ are constrained to be dissimilar enough that they can be efficiently distinguished from each other). For any pair of positive integers i and j, $i \neq j$, consider a morphism σ that maps single letters into ***blocks***:

$$\sigma: \ a \to \boldsymbol{p}^i\boldsymbol{q}, \quad b \to \boldsymbol{p}^j\boldsymbol{q}. \tag{1}$$

We call σ an ***expansion of scope*** λ and denote it by the 4-tuple $[\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ (or just $[\boldsymbol{p}, \boldsymbol{q}, i, j]$ if the scope is clear from the context).

Of course an expansion can be defined on any string $\boldsymbol{x}$ on $\{a, b\}$ by

$$\sigma(\boldsymbol{x}) = \sigma\big(\boldsymbol{x}[1]\big)\sigma\big(\boldsymbol{x}[2]\big)\cdots,$$

and the composition of two expansions is equally well-defined:

$$(\sigma_2 \circ \sigma_1)(\boldsymbol{x}) = \sigma_2\big(\sigma_1(\boldsymbol{x})\big).$$

Suppose that for some positive integer k, a sequence

$$S_k = \sigma_1, \sigma_2, \ldots, \sigma_k$$

of expansions of scope λ is given, where

$$\sigma_r = [\boldsymbol{p_r}, \boldsymbol{q_r}, i_r, j_r]_\lambda \tag{2}$$

for $r = 1, 2, \dots, k$. Then the string

$$\boldsymbol{x} = S_k(a) = (\sigma_k \circ \cdots \circ \sigma_2 \circ \sigma_1)(a) \tag{3}$$

is called a ***complete two-pattern string of scope*** λ. Here $\boldsymbol{x}$ is of course a finite string. A ***complete infinite two-pattern string of scope*** λ is a string $\boldsymbol{x}$ such that for *every* $k > 0$ there exists a sequence S_k of expansions (2) such that $S_k(a)$ is a prefix of $\boldsymbol{x}$. Then an ***infinite two-pattern string of scope*** λ is just any suffix of such a string $\boldsymbol{x}$.

In the special case $\lambda = 1$, $\boldsymbol{p}$ and $\boldsymbol{q}$ must both be single letters, and the suitability condition requires that $\boldsymbol{p} = a$, $\boldsymbol{q} = b$ (or of course *vice versa*). With the further restriction that $j_r = i_r \pm 1$ in (2), every Sturmian string is an infinite two-pattern string of scope 1 and every block-complete Sturmian string [FKS00] is a complete infinite two-pattern string of scope 1 [FLS03].

In this paper, as in [FLS03], we concern ourselves exclusively with computations on *finite complete* two-pattern strings, that we therefore refer to for short simply as ***two-pattern strings*** when no ambiguity results. In [FKS00] we showed how to recognize finite substrings of Sturmian strings in time proportional to their length, a result extended in [FLS03] to complete two-pattern strings. Because the patterns in two-pattern strings are much less constrained than those in Sturmian strings, the possibility arises that a two-pattern string $\boldsymbol{x}$ might result from two distinct expansions

$$\boldsymbol{x} = \sigma_1(\boldsymbol{y_1}), \quad \boldsymbol{x} = \sigma_2(\boldsymbol{y_2}),$$

where $\boldsymbol{y_1}$ is a two-pattern string but $\boldsymbol{y_2}$ is not. [FLS03] essentially shows that this circumstance is impossible, hence that an appropriate expansion can be found without backtracking, hence that recognition requires only time linear in $|\boldsymbol{x}|$.

In this paper we suppose that a two-pattern string $\boldsymbol{x}$ of scope λ has been recognized, and that a corresponding sequence of k expansions (2), $r = 1, 2, \dots, k$, has been identified that produces it. Our task here is to compute all the runs in $\boldsymbol{x}$ in linear time. As discussed in [FLS03], a recent algorithm [KK00] can compute the runs of *any* string in linear time, but nevertheless questions remain about the possibility of discovering other more direct and therefore more efficient approaches. Thus, while the algorithm presented in this paper is more restricted than that of [KK00], we believe that it may contribute to our theoretical understanding of periodicity, as well as to the design of future repetitions algorithms. Essentially the results presented here are generalizations of those given in [FKS00] for Sturmian strings and in [IMS97] for Fibonacci strings.

In Section 2 we state the main result of this paper and provide an overview of the algorithm. Section 3 provides further explanation about the formation of repetitions (runs) in two-pattern strings as a result of expansions (2). Then Section 4 provides a high-level description of the algorithm that is reinforced in Section 5 by a specification of the mechanisms by which each run and near-repetition is derived. Finally, Section 6 provides concluding remarks and links to detailed proofs.

We conclude this section by giving the promised definition of suitability:

Definition 1 *An ordered pair* $(\boldsymbol{p}, \boldsymbol{q})$ *of nonempty binary strings is said to be* **suitable** *if and only if*

- $\boldsymbol{p}$ *is* **primitive** (*that is,* $\boldsymbol{p}$ *has no nonempty border*)*;*
- $\boldsymbol{p}$ *is not a suffix of* $\boldsymbol{q}$*;*
- $\boldsymbol{q}$ *is neither a prefix nor a suffix of* $\boldsymbol{p}$*;*
- $\boldsymbol{q}$ *is not* $\boldsymbol{p}$*-regular.*

Definition 2 *Given binary strings* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*,* $\boldsymbol{q}$ *is said to be* $\boldsymbol{p}$**-regular** *if and only if* $\boldsymbol{q} = \boldsymbol{upvu}$ *for some choice of* (*possibly empty*) *substrings* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*.*

Actually, the second definition is a simplified one used in [FLS03], where it was mentioned that the algorithm would actually work for a more restrictive definition of $\boldsymbol{p}$-regularity.

The more restrictive definition, according to Definition 1, permits a greater number of suitable patterns to be used. The proofs of the lemmas and theorems of this paper all use the more restrictive definition, however, similarly as in [FLS03], the precise nature of the definition has no direct bearing on the workings and nature of the repetition algorithm.

The more restrictive definition as given in [FLS03] contains typos, and so we repeat it here in corrected form:

Definition 3 *Given binary strings* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*,* $\boldsymbol{q}$ *is said to be* $\boldsymbol{p}$**-regular** *if and only if there exist (possibly empty) strings* $\boldsymbol{u}, \boldsymbol{v}$ *together with nonnegative integers* $n_1, n_2, \ldots, n_k$*,* $k \geq 1$*,* $r \geq 1$*, such that*

- *the integers n_i assume at most two distinct values — that is,*

$$\big|\{n_i : \ i \in 1..k\}\big| \leq 2;$$

- $\boldsymbol{q} = (\boldsymbol{u}\boldsymbol{p}^r\boldsymbol{v}\boldsymbol{p}^{n_1})(\boldsymbol{u}\boldsymbol{p}^r\boldsymbol{v}\boldsymbol{p}^{n_2})\cdots(\boldsymbol{u}\boldsymbol{p}^r\boldsymbol{v}\boldsymbol{p}^{n_k})\boldsymbol{u}$ *for some* $\boldsymbol{u}$, $\boldsymbol{v}$, $r \geq 0$, *where* $\boldsymbol{v} = \boldsymbol{\varepsilon}$ *(the* ***empty string****) if* $r = 0$.

Also for clarity and economy of presentation, we confine ourselves in this paper, without loss of generality, to the case $i < j$; that is, to expansions in which a always maps into the short block $\boldsymbol{p}^i\boldsymbol{q}$, b into the long block $\boldsymbol{p}^j\boldsymbol{q}$.

2 Overview

It is well known [C81] that a string $\boldsymbol{x}[1..n]$ can contain $O(n \log n)$ distinct ***repetitions*** $\boldsymbol{x}[s..f] = \boldsymbol{u}^e$, where $1 \leq s < f \leq n$, $\boldsymbol{u}$ is the ***generator*** (and *not* a repetition), $|\boldsymbol{u}|$ the ***period***, and $e \geq 2$ the ***exponent***. Thus a repetition in $\boldsymbol{x}$ can be specified by a triple

$$(s, g, e),$$

where $g = |\boldsymbol{u}|$ is the minimum period and $(s, g, e+1)$ is not a repetition. A ***run*** (***maximal periodicity***) in $\boldsymbol{x}$ is a nonempty substring $\boldsymbol{x}[s..f] = \boldsymbol{u}^e\boldsymbol{u'}$ of minimum period $|\boldsymbol{u}| > |\boldsymbol{u'}|$, $e \geq 2$, that is ***nonextendible*** (neither $\boldsymbol{x}[s{-}1..f]$ nor $\boldsymbol{x}[s..f{+}1]$ is a substring of period $|\boldsymbol{u}|$). We call $\boldsymbol{u'}$ the ***right extension*** of the run. Thus a run in $\boldsymbol{x}$ can be specified by a 4-tuple

$$(s, g, e, t),$$

where $t = |\boldsymbol{u'}| \in 0..g-1$ is the length of the right extension that we call the ***tail***. The run was first defined and used in [M89]; it was shown in [KK00] that the number of runs in $\boldsymbol{x}$ is $O(n)$. As explained in [M89], computing all the runs in $\boldsymbol{x}$ implicitly yields all the repetitions.

Suppose that a sequence of expansions (2) has been found that operates on $\boldsymbol{x_0} = a$, yielding successively

$$\boldsymbol{x_r} = \sigma_r(\boldsymbol{x_{r-1}}), \quad r = 1, 2, \ldots, k,$$

where $\boldsymbol{x} = \boldsymbol{x_k}$ is a two-pattern string of scope λ. Of course if a run $\boldsymbol{u}^t\boldsymbol{u'}$ exists in $\boldsymbol{x_{r-1}}$, then its expansion $\sigma_r(\boldsymbol{u}^t\boldsymbol{u'})$ is a substring of a run of minimum period $|\sigma_r(\boldsymbol{u})|$ in $\boldsymbol{x_r}$. Thus every run in $\boldsymbol{x_{r-1}}$ expands into a corresponding run in $\boldsymbol{x_r}$.

However there may in addition be runs in $\boldsymbol{x_r}$ that do not result from runs in $\boldsymbol{x_{r-1}}$. To take a very simple example, every mapping under σ_r of a single letter into $\boldsymbol{p}^h\boldsymbol{q}$, $h \geq 2$, must yield a run of minimum period $|\boldsymbol{p}|$ (since by the definition of regularity, $\boldsymbol{p}$ must be primitive). As for Sturmian strings [FKS00], it turns out that, except for "short" runs (defined below), *every* run in $\boldsymbol{x_r}$ can be identified as *either* an expansion of a run in $\boldsymbol{x_{r-1}}$ *or* derived from a run in $\boldsymbol{x_{r-1}}$ *or* from one of only three other ***configurations*** in $\boldsymbol{x_{r-1}}$:

$$a\boldsymbol{u}b\boldsymbol{u},\ b\boldsymbol{u}a\boldsymbol{u},\ b\boldsymbol{u}aa\boldsymbol{u}b, \tag{4}$$

where $\boldsymbol{u}$ can be any substring (including $\boldsymbol{\varepsilon}$, a or b) of $\boldsymbol{x_{r-1}}$. It is natural to call these configurations ***near-repetitions***: reversing the first letter ($a \to b$, $b \to a$) of the first two configurations, and reversing both the first and last letters of the last configuration transforms it into squares. For obvious reasons we call the configuration $a\boldsymbol{u}bb\boldsymbol{u}a$ also a near-repetition and will include it in the set of configurations computed by our algorithm though it is not really needed for the computation of runs.

Definition 4 *A run $\boldsymbol{u}^e\boldsymbol{u'}$ or a near-repetition (4) in a two-pattern string of scope λ is said to be **short** if $|\boldsymbol{u}| \leq 3\lambda$; otherwise, **long**.*

We are now able to state the fundamental result of this paper:

Theorem 1 *Every long run or long near-repetition in an expanded two-pattern string $\boldsymbol{x_r} = \sigma_r(\boldsymbol{x_{r-1}})$, $r = 1, 2, \ldots, k$, can be computed in $O(1)$ steps from exactly one of the runs or near-repetitions in $\boldsymbol{x_{r-1}}$.*

Proof See Sections 3–5. $\square$

Based on this result, our algorithmic strategy for the computation of runs in a two-pattern string (3) is simple. For each $r = 1, 2, \ldots, k$, we

(1) scan $\boldsymbol{x_r}$ to compute the short configurations (runs and near-repetitions) by brute force;

(2) compute the long runs in $\boldsymbol{x_r}$ from the configurations already computed in $\boldsymbol{x_{r-1}}$;

$\mathcal{L} \leftarrow \emptyset$
for $g \leftarrow 1$ **to** 3λ **do**
  $s_0 \leftarrow 1$
  while $s_0 \leq n-2g+1$ **do**
    $s \leftarrow s_0$
    while $\boldsymbol{x}[s] = \boldsymbol{x}[s+g]$ **do**
      $s \leftarrow s+1$
    if $s-s_0 \geq g$ **then**
      $\mathcal{L} \overset{+}{\leftarrow} \big(s_0, g, \lfloor (s+g-1)/g \rfloor, (s+g-1) \bmod g\big)$
    $s_0 \leftarrow s+1$

Figure 1: Brute Force for Short Runs

(3) compute the long near-repetitions in $\boldsymbol{x_r}$ from the configurations of $\boldsymbol{x_{r-1}}$.

Thus, after k steps, the runs in $\boldsymbol{x} = \boldsymbol{x_k}$ are computed, as required.

Figure 1 shows the brute force algorithm that computes a list $\mathcal{L}$ of 4-tuples that identify all the short runs of period at most 3λ in $\boldsymbol{x}[1..n]$. It is not hard to show that this algorithm requires exactly $3\lambda(n-3\lambda)$ letter comparisons, hence $\Theta(n)$ time when λ is a constant. The brute force calculations for short near-repetitions are similar.

Since by Theorem 1 every long configuration (run or near-repetition) can be computed in $O(1)$ time, and since the total number of configurations is linear in $|\boldsymbol{x_r}|$ [KK00], it follows that the r^{th} step of this iteration requires $\Theta(|\boldsymbol{x_r}|)$ time for the calculation of long configurations. Since by (1)

$$|\boldsymbol{x_r}| \geq 2|\boldsymbol{x_{r-1}}|,$$

we conclude that the total time requirement for computing all short and long configurations in $\boldsymbol{x}$ is $\Theta(|\boldsymbol{x}|)$.

3 Runs & Their Expansions

In this section we first provide a complete explanation of the processing required to compute a run in an expansion $\sigma(\boldsymbol{x})$ from an existing run in $\boldsymbol{x}$. The explanation is nontechnical and makes use of an extended example to clarify the situations that can arise.

Of course the computation of runs in $\sigma(\boldsymbol{x})$ from runs in $\boldsymbol{x}$ is just one of many possible cases: runs from near-repetitions of type $a\boldsymbol{u}b\boldsymbol{u}$, or near-repetitions of type $a\boldsymbol{u}bb\boldsymbol{u}a$ from runs, and so on. At the end of this section we again use examples in order to provide insight into these cases. But the main point is this: the pattern of processing is in all cases the same, the variations are in detail only.

Observe that since $i < j$, in every expansion a is transformed into a ***short block***, b into a ***long block***. Thus expansions may affect the starting position s of an expanded run, its period g, and its tail t, according to the number of occurrences of a and b in $\boldsymbol{p}$ and $\boldsymbol{q}$. To make the calculations associated with runs easier, we represent them as follows:

$$(s, g, e, t) = \big((s_a, s_b), (g_a, g_b), e, (t_a, t_b)\big),$$

where

- s_μ is the number of occurrences of letter μ *preceding* the starting position of the run;
- g_μ is the number of occurrences of letter μ in the generator;
- t_μ is the number of occurrences of letter μ in the run's right extension.

Consider for instance a string

$$\begin{array}{ccccccccc} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} \\ \boldsymbol{x} = & a & a & a & b & a & b & a & a \end{array} \tag{5}$$

In our notation the run $(3, 2, 2, 1)$ will be designated $\big((2, 0), (1, 1), 2, (1, 0)\big)$. Note that simply summing the elements in each pair (adding 1 in the case of the starting position) enables us to recapture the usual representation. Now suppose that $\boldsymbol{x}$ is expanded by $\sigma = [ab, bbb, 2, 3]_3$, yielding $\boldsymbol{y} = \sigma(\boldsymbol{x})$:

$$\begin{array}{cccccccc} {\scriptstyle 1234567} & {\scriptstyle 8911111} & {\scriptstyle 1111122} & {\scriptstyle 222222223} & {\scriptstyle 3333333} & {\scriptstyle 334444444} & {\scriptstyle 4445555} & {\scriptstyle 5555556} \\ & {\scriptstyle \;\;01234} & {\scriptstyle 5678901} & {\scriptstyle 234567890} & {\scriptstyle 1234567} & {\scriptstyle 890123456} & {\scriptstyle 7890123} & {\scriptstyle 4567890} \\ ababbbb & ababbbb & ababbbb & abababbbb & ababbbb & abababbbb & ababbbb & ababbbb \end{array} \tag{6}$$

Let p_μ (respectively, q_μ) denote the number of occurrences of μ in $\boldsymbol{p}$ (respectively, $\boldsymbol{q}$), where $\mu = a$ or b. Each a in $\boldsymbol{x}$ contributes $ip_a + q_a$ new a's in the expanded string $\boldsymbol{y}$, while each b in $\boldsymbol{x}$ contributes $jp_a + q_a$ new a's.

Similarly, each a in $\boldsymbol{x}$ contributes $ip_b + q_b$ new b's in $\boldsymbol{y}$, and each b $jp_b + q_b$ new b's. Thus in order to compute the effect of σ on each pair (h_a, h_b) listed in the run for $\boldsymbol{x}$, we need only compute a transformation τ as follows:

$$\begin{aligned}
\tau(h_a, h_b) &= \big(h_a(ip_a + q_a) + h_b(jp_a + q_a), h_a(ip_b + q_b) + h_b(jp_b + q_b)\big) \\
&= \big((h_a i + h_b j)p_a + (h_a + h_b)q_a, (h_a i + h_b j)p_b + (h_a + h_b)q_b\big).
\end{aligned}$$

In our example, $p_a = 1$, $p_b = 1$, $q_a = 0$, $q_b = 3$, $i = 2$, and $j = 3$, so that

$$\tau(2, 0) = \big((2{\cdot}2 + 0{\cdot}3){\cdot}1 + (2 + 0){\cdot}0, (2{\cdot}2 + 0{\cdot}3){\cdot}1 + (2 + 0){\cdot}3\big) = (4, 10),$$

while similar calculations yield $\tau(1, 1) = (5, 11)$, $\tau(1, 0) = (2, 5)$. Hence the expanded string (6) contains the expanded run $\rho_0 = \big((4, 10), (5, 11), 2, (2, 5)\big)$. Almost. Examination of (6) reveals that ρ_0 is not really a run since the leading square can be extended to the left by 8 positions — more precisely, by $(2, 6)$.

In general, we must recognize three situations:

1. The original run starts at position 1 of the original string. Then the leading square of the expanded run is not left-extendible.

2. The run starts at position 2. Then the leading square of the expanded run can be left-extended by $i|\boldsymbol{p}|$ positions; that is, by (ip_a, ip_b). The starting position must be decrement accordingly.

3. The run starts at a position ≥ 3. Then the expanded run can be left-extended by (ip_a, ip_b) as in case 2, but in addition by $gcs(\boldsymbol{p}, \boldsymbol{q}) = |GCS(\boldsymbol{p}, \boldsymbol{q})|$, the length of the greatest common suffix of $\boldsymbol{p}$ and $\boldsymbol{q}$. Let $gcs_a(\boldsymbol{p}, \boldsymbol{q})$ denote the number of a's in $GCS(\boldsymbol{p}, \boldsymbol{q})$, $gcs_b(\boldsymbol{p}, \boldsymbol{q})$ the number of b's. The total left extension of the run therefore amounts to $\big(ip_a + gcs_a(\boldsymbol{p}, \boldsymbol{q}), ip_b + gcs_b(\boldsymbol{p}, \boldsymbol{q})\big)$. Again, the starting position must be decrement by this amount.

In our example case 3 applies. $GCS(\boldsymbol{p}, \boldsymbol{q}) = b$, and so $gcs_a(\boldsymbol{p}, \boldsymbol{q}) = 0$ and $gcs_b(\boldsymbol{p}, \boldsymbol{q}) = 1$. Hence the total left extension will be $(2{\cdot}1{+}0{+}0, 2{\cdot}1{+}3{+}1) = (2, 6)$. Therefore we must update the starting position to $(4, 10) - (2, 6) = (2, 4)$, yielding the expanded run $\rho_1 = \big((2, 4), (5, 11), 2, (2, 5)\big)$. This looks much better, but it is still not correct: it can easily be checked that the right extension should have been $(4, 7)$ rather than $(2, 5)$.

In general, we must recognize two situations:

1. The right extension of the original run extends all the way to the end of the string; that is, there is no extra letter beyond the end of the extension. Then there is no additional right extension to the one computed directly.

2. Otherwise there is an additional right extension of $i|\boldsymbol{p}|$ positions plus $gcp(\boldsymbol{p}, \boldsymbol{q}) = |GCP(\boldsymbol{p}, \boldsymbol{q})|$ positions, where $GCP(\boldsymbol{p}, \boldsymbol{q})$ is the greatest common prefix of $\boldsymbol{p}$ and $\boldsymbol{q}$. More precisely, there is an additional right-extension $\big(ip_a + gcp_a(\boldsymbol{p}, \boldsymbol{q}), ip_b + gcp_b(\boldsymbol{p}, \boldsymbol{q})\big)$.

In our example, case 2 applies. Thus there is $(2{\cdot}1 + 0, 2{\cdot}1 + 0) = (2, 2)$ additional right extension. Hence we must modify the run to include the additional right extension: $\big((2,4), (5,11), 2, (4,7)\big)$. This almost looks correct except for one problem: a careful examination of (6) reveals that in fact the run contains a cube, so we should have $e = 3$. The explanation is simple: by expansion, the run gained $(2, 6)$ in left extension and $(4, 7)$ in right extension, altogether $(6, 13)$, enough gain to make another repeat of the generator that requires $(5, 11)$. Hence we increment the exponent and reduce the right extension:

$$\big((2,4), (5,11), 2, (6,13)\big) = \big((2,4), (5,11), 3, (1,2)\big).$$

Finally we have the proper expanded run! Observe the calculation requires only a restricted number of elementary operations on the four elements of the run together with knowledge of $gcs(\boldsymbol{p}, \boldsymbol{q})$ and $gcp(\boldsymbol{p}, \boldsymbol{q})$ (of course pre-computed once only).

Based on this example, we make the following claim:

a proper expansion of a run in $\boldsymbol{x}$ to a run in $\sigma(\boldsymbol{x})$ is computable in $O(1)$ time (7)

However, as noted above, runs in $\sigma(\boldsymbol{x})$ do not arise solely as expansions of runs in $\boldsymbol{x}$. For instance, in our example, each a in (5) gives rise to a substring $ababbb$ of (6) that contains two runs, $\underline{abab}bbbb$ and $aba\underline{bbbb}$. These new runs are of course short and easy to determine using brute force.

But there are less obvious runs that arise during an expansion. Take for instance an expansion $\sigma = [aabb, ab, 2, 3]_4$, and consider a near-repetition ab in the string $\boldsymbol{x}$ to be expanded: ab will expand to

$$aabb\ aab\underline{b\ ab\ a}abb\ aabb\ aabb\ ab$$

that contains a run with generator $\boldsymbol{u} = ba$. So we must somehow track near-repetitions of type ab in $\boldsymbol{x}$ in order to keep track of all runs as they arise in $\sigma(\boldsymbol{x})$.

One more example. Consider a near-repetition $aabaa$ in a string $\boldsymbol{x}$ to be expanded by $\sigma = [ab, bbb, 2, 3]_3$: $aabaa$ expands to

$$ababbbb\ ababbbb\ abababbbb\ ababbbb\ ababbbb$$

that contains the run $(ababbbb\ ababbbb\ ab)(ababbbb\ ababbbb\ ab)ab$. So we must somehow track near-repetitions of type $aabaa$ in $\boldsymbol{x}$ in order to keep track of all runs as they arise in $\sigma(\boldsymbol{x})$. Note that, unlike the previous example, the near-repetition ab here does not give rise to any new run.

At this point the reader may believe that there simply is no possibility of tracking all the possible ways that runs can arise in $\sigma(\boldsymbol{x})$. But it is actually straightforward, though broken down into many special cases that result from the combinations of configurations we need to track.

4 The Algorithm

In this section we return to the overview of the algorithm given in Section 2, interpreted now in the light of the examples and analysis of Section 3.

As indicated in Section 2, our algorithm is a simple iteration that, for each $r = 1, 2, \ldots, k$, computes the configurations (runs and near-repetitions) in $\boldsymbol{x_r} = \sigma_r(\boldsymbol{x_{r-1}})$ based on those already computed for $\boldsymbol{x_{r-1}}$. The iteration begins with $\boldsymbol{x_0} = a$.

list	type of configuration
$\mathcal{L}_1$	runs
$\mathcal{L}_2$	$a\boldsymbol{u}b\boldsymbol{u}$
$\mathcal{L}_3$	$b\boldsymbol{u}a\boldsymbol{u}$
$\mathcal{L}_4$	$b\boldsymbol{u}aa\boldsymbol{u}b$
$\mathcal{L}_5$	$a\boldsymbol{u}bb\boldsymbol{u}a$

Table 1: Lists & Configurations

For each r, we maintain five lists $\mathcal{L}_m(\boldsymbol{x_r})$, $m = 1, 2, \ldots, 5$, corresponding to the five types of configurations, as shown in Table 1. When $\boldsymbol{x_r}$ has been completely processed, then, $\mathcal{L}_1(\boldsymbol{x_r})$ lists all the runs in $\boldsymbol{x_r}$, $\mathcal{L}_2(\boldsymbol{x_r})$ all the near-repetitions of type $a\boldsymbol{u}b\boldsymbol{u}$, and so on. For $r = 0$, all the lists $\mathcal{L}_m(\boldsymbol{x_0})$ are of course empty. Note that while a 4-tuple is required to specify each run, a near-repetition can be specified using only a pair (s, g), where $g = |\boldsymbol{u}|$. This is because the form of each near-repetition is known in advance — there is no exponent e and no right extension $\boldsymbol{u'}$.

		$\boldsymbol{x_{r-1}}$			
		$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
	$\mathcal{L}_1$	(7), (9)	(10)	(11)	(12)
	$\mathcal{L}_2$	(13)	(14)	(15)	(16)
$\boldsymbol{x_r}$	$\mathcal{L}_3$	(17)	(18)	(19)	(20)
	$\mathcal{L}_4$	(21)	(22)	(23)	(24)
	$\mathcal{L}_5$	(25)	(26)	(27)	(28)

Table 2: Dependency of $\mathcal{L}_m(\boldsymbol{x_r})$ on $\mathcal{L}_{m'}(\boldsymbol{x_{r-1}})$

For each $r = 1, 2, \ldots, k$, we begin by computing the short configurations of each type m, using algorithms similar to the one specified in Figure 1, and placing them in their corresponding lists $\mathcal{L}_m(\boldsymbol{x_r})$. Then the lists are updated from the lists $\mathcal{L}_{m'}(\boldsymbol{x_{r-1}})$ for $\boldsymbol{x_{r-1}}$, according to procedures π_m as follows:

$$\mathcal{L}_m(\boldsymbol{x_r}) \stackrel{+}{\leftarrow} \pi_m\big(\mathcal{L}_1(\boldsymbol{x_{r-1}}), \mathcal{L}_2(\boldsymbol{x_{r-1}}), \mathcal{L}_3(\boldsymbol{x_{r-1}}), \mathcal{L}_4(\boldsymbol{x_{r-1}})\big), \tag{8}$$

$m = 1, 2, \ldots, 5$. In other words, the long configurations in each list $\mathcal{L}_m(\boldsymbol{x_r})$ are computed from the configurations (long and short) in the four lists $\mathcal{L}_{m'}(\boldsymbol{x_{r-1}})$, $m' = 1, 2, 3, 4$. The relationship between the lists for $\boldsymbol{x_r}$ and those for $\boldsymbol{x_{r-1}}$ is shown explicitly in Table 2, where the reference numbers in each position refer to the calculations specified in the text ((7) in Section 3 and (9)–(28) in Section 5).

With the information provided by (8) and Table 2, we can now give a structured overview of the processing that computes the runs in a two-pattern string $\boldsymbol{x}$. The algorithm is given in Figure 2, where we suppose that the scope λ and k expansions $\sigma_r = [\boldsymbol{p_r}, \boldsymbol{q_r}, i_r, j_r]_\lambda$, $r = 1, 2, \ldots, k$, are given. The algorithm stores only two sets of list for each r, those corresponding to $\boldsymbol{x_{r-1}}$ and $\boldsymbol{x_r}$.

In order to establish both the correctness and complexity of the all-runs algorithm, we need essentially to establish Theorem 1. We have already seen

$\delta \leftarrow 0;\ \boldsymbol{x}^{(\delta)} \leftarrow a$
for $m \leftarrow 1$ **to** 5 **do**
  $\mathcal{L}_m^{(\delta)} \leftarrow \emptyset$
for $r \leftarrow 1$ **to** k **do**
  $\boldsymbol{x}^{(1-\delta)} \leftarrow \sigma_r\big(\boldsymbol{x}^{(\delta)}\big)$
  for $m \leftarrow 1$ **to** 5 **do**
    $\mathcal{L}_m^{(1-\delta)} \leftarrow$ short configs ($|\boldsymbol{u}| \leq 3\lambda$) of type m in $\boldsymbol{x}^{(1-\delta)}$
  for $m \leftarrow 1$ **to** 5 **do**
    $\mathcal{L}_m^{(1-\delta)} \stackrel{+}{\leftarrow} \pi_m\big(\mathcal{L}_1^{(\delta)}, \mathcal{L}_2^{(\delta)}, \mathcal{L}_3^{(\delta)}, \mathcal{L}_4^{(\delta)}\big)$
  $\delta \leftarrow 1-\delta$
output $\mathcal{L}_1^{(\delta)}$

Figure 2: Computing All the Runs in $S_k(a)$

(Figure 1) that the short configurations can be computed in time linear in string length, and it is straightforward to verify that the calculations (7) and (9)–(28) can all be performed in constant time given a precomputation of gcs and gcp values. Thus, as observed in Section 2, the overall time requirement of the algorithm is $\Theta(|\boldsymbol{x}|)$.

It remains then to be shown that

- The calculations (7) and (9)–(28) are correct; that is, that the runs specified are in fact those that arise as a result of an expansion. These proofs are available in the web supplement identified for each calculation in Section 5.

- The calculations are complete; that is, that no runs other than those specified can occur in an expansion $\sigma(\boldsymbol{x})$. As discussed in Section 6, the completeness proof is also available on the web.

5 Deriving the Runs and Near-Repetitions

In this section we give the conditions under which a configuration (run or near-repetition) in an expanded string $\boldsymbol{y} = \sigma(\boldsymbol{x})$ can be derived from a configuration in $\boldsymbol{x}$, and we specify in each case the form of the expanded configuration. We work with a general expansion $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]$; for the sake of brevity, we use $\overline{\boldsymbol{u}}$ to denote the expansion $\sigma(\boldsymbol{u})$ of $\boldsymbol{u}$.

Deriving run from run (not run expansion!). (9)

1. The run in $\boldsymbol{x}$ has generator a, $\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$.
   For every $2 \leq r < i$ and every square aa in the run: $\sigma(aa) = \boldsymbol{p}^{i-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q} = \boldsymbol{p}^{i-r-1}\boldsymbol{p}\boldsymbol{p}^{r}\boldsymbol{q}\boldsymbol{p}^{r}\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q} = \boldsymbol{p}^{i-r-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{i-r-1}\boldsymbol{q}$.
   The derived run has exponent 2 and the square $\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}_1, \hat{\boldsymbol{p}}_2)$ and has maximal right-extension of size $gcp(\boldsymbol{p}_2, \hat{\boldsymbol{p}}_1)$.
   (*See Corollary 1, case 3 for aa, in web supplement for proof.*)
   (Note: *the conditions imply that* $\boldsymbol{p}_1 \neq \hat{\boldsymbol{p}}_2$ *and* $\boldsymbol{p}_2 \neq \hat{\boldsymbol{p}}_2$ *— otherwise* $\boldsymbol{p} = \boldsymbol{q}$, *and* $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$ *— otherwise* $\boldsymbol{p}$ *has a non-trivial border* $\boldsymbol{p}_1$, *and so the squares produced by this derivation are distinct from the expansion of aa*)

2. The run in $\boldsymbol{x}$ has generator b, $\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$.
   For every $2 \leq r < j$ and every square bb in the run: $\sigma(bb) = \boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{j-r-1}\boldsymbol{q} = \boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^{r}\boldsymbol{q}\boldsymbol{p}^{r}\boldsymbol{p}\boldsymbol{p}^{j-r-1}\boldsymbol{q} = \boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
   The derived run has exponent 2 and the square $\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}_1, \hat{\boldsymbol{p}}_2)$ and has maximal right-extension of size $gcp(\boldsymbol{p}_2, \hat{\boldsymbol{p}}_1)$.
   (*See Corollary 1, case 3 for bb, in web supplement for proof.*)
   (Note: *the conditions imply that* $\boldsymbol{p}_1 \neq \hat{\boldsymbol{p}}_2$ *and* $\boldsymbol{p}_2 \neq \hat{\boldsymbol{p}}_2$ *— otherwise* $\boldsymbol{p} = \boldsymbol{q}$, *and* $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$ *— otherwise* $\boldsymbol{p}$ *has a non-trivial border* $\boldsymbol{p}_1$, *and so the squares produced by this derivation are distinct from the expansion of bb*)

3. The run in $\boldsymbol{x}$ has a string a as a generator, $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $i = 2r+1$ for some $r \geq 2$.
   $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1\boldsymbol{q}$.
   The derived run has exponent 2 and the square $\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.
   (*See Corollary 1, case 6, for aa in the web supplement for proof.*)

(Note: *the conditions imply that* $\boldsymbol{q}_1 \neq \hat{\boldsymbol{q}}_2$ *and* $\boldsymbol{q}_2 \neq \hat{\boldsymbol{q}}_2$ *— otherwise* $\boldsymbol{p} = \boldsymbol{q}$, *and* $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$ *— otherwise* $\boldsymbol{p}$ *has a non-trivial border* $\boldsymbol{q}_1$, *and so the squares produced by this derivation are distinct from the expansion of aa*)

4. The run in $\boldsymbol{x}$ has a string b as a generator, $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $j = 2r{+}1$ for some $r \geq 2$.
$\sigma(aa) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.
(*See Corollary 1, case 6 for bb, in the web supplement for proof.*)
(Note: *the conditions imply that* $\boldsymbol{q}_1 \neq \hat{\boldsymbol{q}}_2$ *and* $\boldsymbol{q}_2 \neq \hat{\boldsymbol{q}}_2$ *— otherwise* $\boldsymbol{p} = \boldsymbol{q}$, *and* $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$ *— otherwise* $\boldsymbol{p}$ *has a non-trivial border* $\boldsymbol{q}_1$, *and so the squares produced by this derivation are distinct from the expansion of bb*)

Deriving run from near-repetition $a\boldsymbol{u}b\boldsymbol{u}$. (10)

1. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ in $\boldsymbol{x}$ is followed by an a, $j \leq 2i$. $\sigma(a\boldsymbol{u}b\boldsymbol{u}a) = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\boldsymbol{p}^{2i-j}\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]$ has maximal left-extension of size 0 (if $a\boldsymbol{u}b\boldsymbol{u}a$ is an initial segment of $\boldsymbol{x}$) or of size $gcs(\boldsymbol{p}, \boldsymbol{q})$ and has maximal right-extension of size $(i|\boldsymbol{p}| + gcp(\boldsymbol{p}, \boldsymbol{q}))$.
(*See Corollary 1, case 2, in web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$.
For every $2 \leq r < i$: $\sigma(ab) = \boldsymbol{p}^{i-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{j-r-1}\boldsymbol{q} = \boldsymbol{p}^{i-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{j-r-1}\boldsymbol{q} = \boldsymbol{p}^{i-r-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}_1, \hat{\boldsymbol{p}}_2)$ and has maximal right-extension of size $gcp(\boldsymbol{p}_2, \hat{\boldsymbol{p}}_1)$.
(*See Corollary 1, case 3 for ab, in web supplement for proof.*)

3. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $j = 2r{+}1$ for some $r \geq 2$.

$\sigma(ab) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.
(*See Corollary 1, case 6 for ab, in the web supplement for proof.*)

4. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by an a and is not an initial segment of $\boldsymbol{x}$. $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $j = 2i{+}1$.
$\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1 =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1$.
The derived run has exponent 2 and the square $\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.
(*See Corollary 1, case 7, in the web supplement for proof.*)

Deriving run from near-repetition $b\boldsymbol{u}a\boldsymbol{u}$. (11)

1. $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} = \boldsymbol{p}^{j-i}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]$.
The derived run has exponent 2 and the square $\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}, \boldsymbol{q})$ and maximal right-extension of size 0 (if $b\boldsymbol{u}a\boldsymbol{u}$ is an end segment of $\boldsymbol{x}$) or of size $(i|\boldsymbol{p}| + gcp(\boldsymbol{p}, \boldsymbol{q}))$.
(*See Corollary 1, case 5, in the web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$.
For every $2 \leq r < j$: $\sigma(ba) = \boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q} =$
$\boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q} = \boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}_1, \hat{\boldsymbol{p}}_2)$ and has maximal right-extension of size $gcp(\boldsymbol{p}_2, \hat{\boldsymbol{p}}_1)$.
(*See Corollary 1, case 3 for ba, in web supplement for proof.*)

3. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $i = 2r{+}1$ for some $r \geq 2$.
$\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1\boldsymbol{q}$.
The derived run has exponent 2 and the square $\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.
(*See Corollary 1, case 6 for ba, in the web supplement for proof.*)

Deriving run from near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$. (12)

1. $\boldsymbol{q}=\boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1\neq\varepsilon$, $\boldsymbol{p}_2\neq\varepsilon$, $\boldsymbol{p}=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1=\hat{\boldsymbol{p}}_2\boldsymbol{p}_2$.
   $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b)=\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}=$
   $\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\boldsymbol{p}_2\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.
   The derived run has exponent 2 and the square $\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]$ has maximal left-extension of size $gcs(\boldsymbol{p}_1,\hat{\boldsymbol{p}}_2)$ and maximal right-extension of size $gcp(\boldsymbol{p}_2,\hat{\boldsymbol{p}}_1)$.
   (*See Corollary 1, case 4, in the web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ from run. (13)

1. The run in $\boldsymbol{x}$ has a string a as a generator, $\boldsymbol{q}=\boldsymbol{p}_1b\boldsymbol{p}_2$, $\boldsymbol{p}=\hat{\boldsymbol{p}}_2a\boldsymbol{p}_2=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
   For every $2\leq r<i$ and every square aa in the run: $\sigma(aa)=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=$
   $\boldsymbol{p}^{i-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q}=\boldsymbol{p}^{i-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q}=$
   $\boldsymbol{p}^{i-r-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{i-r-1}\boldsymbol{q}$.
   (*See Theorem 2 , case 1 for aa, in web supplement for proof.*)

2. The run in $\boldsymbol{x}$ has a string b as a generator, $\boldsymbol{q}=\boldsymbol{p}_1b\boldsymbol{p}_2$, $\boldsymbol{p}=\hat{\boldsymbol{p}}_2a\boldsymbol{p}_2=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
   For every $2\leq r<j$ and every square aa in the run: $\sigma(bb)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}=$
   $\boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{j-r-1}\boldsymbol{q}=\boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{j-r-1}\boldsymbol{q}=$
   $\boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
   (*See Theorem 2, case 1 for bb, in web supplement for proof.*)

3. The run in $\boldsymbol{x}$ has generator a, $\boldsymbol{p}=\boldsymbol{q}_1b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2a\boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2\leq r\leq i/2{-}1$ and every square aa in the run: $\sigma(aa)=$ $\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q}=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q}=\boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 2, case 5 for aa, in web supplement for proof.*)

4. The run in $\boldsymbol{x}$ has generator b, $\boldsymbol{p}=\boldsymbol{q}_1b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2a\boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2\leq r\leq j/2{-}1$ and every square bb in the run: $\sigma(bb)=$ $\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q}=\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 2, case 5 for bb, in web supplement for proof.*)

5. The run in $\boldsymbol{x}$ has generator a, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 \hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i = 2r{+}1$ for some $2 \leq r$.
   For any square aa in the run: $\sigma(aa) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] b \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big]\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 2, case 9 for aa, in web supplement for proof.*)

6. The run in $\boldsymbol{x}$ has generator b, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 \hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2r{+}1$ for some $2 \leq r$.
   For any square bb in the run: $\sigma(bb) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} = \boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] b \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big]\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 2, case 9 for bb, in web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ from near-repetition $a\boldsymbol{u}b\boldsymbol{u}$. (14)

1. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1 b \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 \hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
   For every $2 \leq r < j$: $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^{i-r-1} \boldsymbol{p}^{r+1} \boldsymbol{q} \boldsymbol{p}^{r+1} \boldsymbol{p}^{j-r-1} \boldsymbol{q} = \boldsymbol{p}^{i-r-1} \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^{j-r-1} \boldsymbol{q} = \boldsymbol{p}^{i-r-1} \hat{\boldsymbol{p}}_2 \big[a \big[\boldsymbol{p}_2 \boldsymbol{p}^r \boldsymbol{p}_1\big] b \big[\boldsymbol{p}_2 \boldsymbol{p}^r \boldsymbol{p}_1\big]\big] \hat{\boldsymbol{p}}_1 \boldsymbol{p}^{j-r-1} \boldsymbol{q}$.
   (*See Theorem 2, case 1 for ab, in web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2 \leq r \leq j/2{-}1$: $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r+2} \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] b \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big]\big] b \boldsymbol{q}_2 \boldsymbol{q}$.
   (*See Theorem 2, case 5 for ab, in web supplement for proof.*)

3. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a b and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i < j{+}1/2$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}b) = \cdot\cdot \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^j \boldsymbol{q} = \cdot\cdot \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} = \cdot\cdot \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{q}_1\big] b \big[\boldsymbol{q}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{q}_1\big]\big] b \boldsymbol{q}_2 \boldsymbol{p}^i \boldsymbol{q}$.
   (*See Theorem 2, case 6, in web supplement for proof.*)

4. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i \geq j{+}1/2$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} = \cdot\cdot \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^{2i-j-1} \boldsymbol{q} = \cdot\cdot \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{q}_1\big] b \big[\boldsymbol{q}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^{j-i-1} \boldsymbol{q}_1\big]\big] b \boldsymbol{q}_2 \boldsymbol{p}^{2i-j-1} \boldsymbol{q}$.
   (*See Theorem 2, case 7, in web supplement for proof.*)

5. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $j = 2i$.
$\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{q} =$
$\cdot\cdot \hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{q}$.
(*See Theorem 2, case 8, in web supplement for proof.*)

6. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2r+1$ for some $2 \le r$.
$\sigma(ab) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} =$
$\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
(*See Theorem 2, case 9 for ab, in web supplement for proof.*)

7. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by an a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 b \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2i+1$.
$\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
$\cdot\cdot \hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
(*See Theorem 2, case 10, in web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ from near-repetition $b\boldsymbol{u}a\boldsymbol{u}$. (15)

1. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1 b \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
For every $2 \le r < i$: $\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} =$
$\boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q} = \boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q} =$
$\boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{i-r-1}\boldsymbol{q}$.
(*See Theorem 2, case 1 for ba, in web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1 b \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
$\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]\big]b\boldsymbol{p}_2$.
(*See Theorem 2, case 3, in web supplement for proof.*)

3. $\boldsymbol{u} \ne \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1 b \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
Since $\boldsymbol{u} \ne \varepsilon$, $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{q}$ for some $\boldsymbol{u}_1$. Thus $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} =$
$\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} =$
$\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\big]\big]b\boldsymbol{p}_2$.
(*See Theorem 2, case 4, in web supplement for proof.*)

4. $\boldsymbol{u}=\varepsilon$, $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
For every $2 \le r \le i/2{-}1$: $\sigma(ba)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q}=$
$\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{q}$.
(*See Theorem 2, case 5 for ba, in web supplement for proof.*)

5. $\boldsymbol{u}=\varepsilon$, $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2=\boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i=2r{+}1$ for some $2 \le r$.
$\sigma(ba)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}=$
$\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
(*See Theorem 2, case 9 for ba, in web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ from near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$. (16)

1. $\boldsymbol{q}=\boldsymbol{p}_1 b\boldsymbol{p}_2$, $\boldsymbol{p}=\hat{\boldsymbol{p}}_2 a\boldsymbol{p}_2=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
$\sigma(b\boldsymbol{u}aa\boldsymbol{u}b)=\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}=\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q}=$
$\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.
(*See Theorem 2, case 2, in web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}a\boldsymbol{u}$ from run. (17)

1. The run in $\boldsymbol{x}$ has a string a as a generator, $\boldsymbol{q}=\boldsymbol{p}_1 a\boldsymbol{p}_2$, $\boldsymbol{p}=\hat{\boldsymbol{p}}_2 b\boldsymbol{p}_2=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
For every $2 \le r < i$ and every square aa in the run: $\sigma(aa)=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=$
$\boldsymbol{p}^{i-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q}=\boldsymbol{p}^{i-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q}=$
$\boldsymbol{p}^{i-r-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{i-r-1}\boldsymbol{q}$.
(*See Theorem 3, case 1 for aa, in web supplement for proof.*)

2. The run in $\boldsymbol{x}$ has a string b as a generator, $\boldsymbol{q}=\boldsymbol{p}_1 a\boldsymbol{p}_2$, $\boldsymbol{p}=\hat{\boldsymbol{p}}_2 b\boldsymbol{p}_2=\boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
For every $2 \le r < j$ and every square aa in the run: $\sigma(bb)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}=$
$\boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{j-r-1}\boldsymbol{q}=\boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{j-r-1}\boldsymbol{q}=$
$\boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
(*See Theorem 3, case 1 for bb, in web supplement for proof.*)

3. The run in $\boldsymbol{x}$ has generator a, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2 \le r \le i/2-1$ and every square aa in the run: $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q} = \boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 3, case 5 for aa, in web supplement for proof.*)

4. The run in $\boldsymbol{x}$ has generator b, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2 \le r \le j/2-1$ and every square bb in the run: $\sigma(bb) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q} = \boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 3, case 5 for bb, in web supplement for proof.*)

5. The run in $\boldsymbol{x}$ has generator a, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i = 2r+1$ for some $2 \le r$.
   For any square aa in the run: $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} =$
   $\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} = \boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 3, case 9 for aa, in web supplement for proof.*)

6. The run in $\boldsymbol{x}$ has generator b, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2r+1$ for some $2 \le r$.
   For any square bb in the run: $\sigma(bb) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} =$
   $\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} = \boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 3, case 9 for bb, in web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}a\boldsymbol{u}$ from near-repetition $a\boldsymbol{u}b\boldsymbol{u}$. (18)

1. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
   For every $2 \le r < j$: $\sigma(ab) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} =$
   $\boldsymbol{p}^{i-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{j-r-1}\boldsymbol{q} = \boldsymbol{p}^{i-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{j-r-1}\boldsymbol{q} =$
   $\boldsymbol{p}^{i-r-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-r-1}\boldsymbol{q}$.
   (*See Theorem 3, case 1 for ab, in web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2 \le r \le j/2-1$: $\sigma(ab) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+2}\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{q} =$
   $\boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 3, case 5 for ab, in web supplement for proof.*)

3. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a b and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i < j{+}1/2$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}b) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q} =$
   $\cdot\cdot \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}$.
   (*See Theorem 3, case 6, in web supplement for proof.*)

4. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i \geq j{+}1/2$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^{2i-j-1}\boldsymbol{q} =$
   $\cdot\cdot \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{p}^{2i-j-1}\boldsymbol{q}$.
   (*See Theorem 3, case 7, in web supplement for proof.*)

5. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by a a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $j = 2i$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{q} =$
   $\cdot\cdot \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{q}$.
   (*See Theorem 3, case 8, in web supplement for proof.*)

6. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2r{+}1$ for some $2 \leq r$.
   $\sigma(ab) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} =$
   $\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 3, case 9 for ab, in web supplement for proof.*)

7. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ is followed by an a and it is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2i{+}1$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
   $\cdot\cdot \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 3, case 10, in web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}a\boldsymbol{u}$ from near-repetition $b\boldsymbol{u}a\boldsymbol{u}$. (19)

1. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{p}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$.
   For every $2 \leq r < i$: $\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} =$
   $\boldsymbol{p}^{j-r-1}\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}\boldsymbol{p}^{i-r-1}\boldsymbol{q} = \boldsymbol{p}^{j-r-1}\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^{i-r-1}\boldsymbol{q} =$
   $\boldsymbol{p}^{j-r-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{i-r-1}\boldsymbol{q}$.
   (*See Theorem 3, case 1 for ba, in web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
   $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p} i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^{j-i-1} \hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q}_1 \big] a \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q}_1 \big] \big] a \boldsymbol{p}_2$.
   (See Theorem 3, case 3, in web supplement for proof.)

3. $\boldsymbol{u} \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
   Since $\boldsymbol{u} \neq \varepsilon$, $\boldsymbol{u} = \boldsymbol{u}_1 \boldsymbol{q}$ for some $\boldsymbol{u}_1$. Thus $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} = \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u_1}} \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}}_1 \boldsymbol{q} = \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u_1}} \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}}_1 \boldsymbol{q} =$
   $\boldsymbol{p}^{j-i-1} \hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u_1}} \boldsymbol{q}_1 \big] a \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}}_1 \big] \big] a \boldsymbol{p}_2$.
   (See Theorem 3, case 4, in web supplement for proof.)

4. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$.
   For every $2 \leq r \leq i/2-1$: $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r+2} \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{q} =$
   $\boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1 \big] a \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1 \big] \big] a \boldsymbol{q}_2 \boldsymbol{q}$.
   (See Theorem 3, case 5 for ba, in web supplement for proof.)

5. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1 \hat{\boldsymbol{q}}_1$, for some $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i = 2r+1$ for some $2 \leq r$.
   $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} =$
   $\boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1 \big] a \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1 \big] \big] \hat{\boldsymbol{q}}_1$.
   (See Theorem 3, case 9 for ba, in web supplement for proof.)

Deriving near-repetition $b\boldsymbol{u}a\boldsymbol{u}$ from near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$. (20)

1. $\boldsymbol{q} = \boldsymbol{p}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 \hat{\boldsymbol{p}}_1$, for some $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$.
   $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) = \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p} \boldsymbol{p}^{j-i-1} \boldsymbol{q} =$
   $\boldsymbol{p}^{j-i-1} \hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p}_1 \big] a \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p}_1 \big] \big] \hat{\boldsymbol{p}}_1 \boldsymbol{p}^{j-i-1} \boldsymbol{q}$.
   (See Theorem 3, case 2, in web supplement for proof.)

Deriving near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$ from run. (21)

1. The run has generator a, $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i \geq 3$.
   For every square aa of the run: $\sigma(aa) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p} \boldsymbol{p}^{i-1} \boldsymbol{q} \boldsymbol{p}^{i-1} \boldsymbol{p} \boldsymbol{q} =$
   $\hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p}^{i-1} \boldsymbol{p}_1 \big] aa \big[\boldsymbol{p}_2 \boldsymbol{p}^{i-1} \boldsymbol{p}_1 \big] b \big] \hat{\boldsymbol{p}}_1 \boldsymbol{q}$.
   (See Theorem 4, case 1 for aa, in the web supplement for proof.)

2. The run has generator b, $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $j \geq 3$.
   For every square bb of the run: $\sigma(bb) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}\boldsymbol{p}^{j-1} \boldsymbol{q} \boldsymbol{p}^{j-1} \boldsymbol{p}\boldsymbol{q} = \hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p}^{j-1} \boldsymbol{p}_1\big] aa \big[\boldsymbol{p}_2 \boldsymbol{p}^{j-1} \boldsymbol{p}_1\big] b\big] \hat{\boldsymbol{p}}_1 \boldsymbol{q}$.
   (*See Theorem 4, case 1 for bb, in the web supplement for proof.*)

3. The run has generator a, $\boldsymbol{p} = a$, $\boldsymbol{q} = b\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $i = 2r$, $r \geq 3$.
   $\sigma(aa) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r} \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{p}^{r-1}\big] aa \big[\boldsymbol{p}^{r-1}\big] b\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 3 for aa, in the web supplement for proof.*)

4. The run has generator b, $\boldsymbol{p} = a$, $\boldsymbol{q} = b\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $j = 2r$, $r \geq 3$.
   $\sigma(bb) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r} \boldsymbol{q} = \boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{p}^{r-1}\big] aa \big[\boldsymbol{p}^{r-1}\big] b\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 3 for bb, in the web supplement for proof.*)

5. The run has generator a, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa \boldsymbol{q}_2$, $i = 2r{+}1$, $r \geq 3$.
   For any square aa in the run: $\sigma(aa) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{q}_2 \boldsymbol{p}^r\big] \boldsymbol{q}_1\big] aa \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] b\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 5 for aa, in the web supplement for proof.*)

6. The run has generator b, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa \boldsymbol{q}_2$, $j = 2r{+}1$, $r \geq 3$.
   For any square aa in the run: $\sigma(bb) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} = \boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{q}_2 \boldsymbol{p}^r\big] \boldsymbol{q}_1\big] aa \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] b\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 5 for bb, in the web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$ from near-repetition $a\boldsymbol{u}b\boldsymbol{u}$. (22)

1. $\boldsymbol{u} = \varepsilon$, $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i \geq 3$.
   $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}\boldsymbol{p}^{i-1} \boldsymbol{q} \boldsymbol{p}^{i-1} \boldsymbol{p}\boldsymbol{p}^{j-i-1} \boldsymbol{q} =$
   $\hat{\boldsymbol{p}}_2 \big[b \big[\boldsymbol{p}_2 \boldsymbol{p}^{i-1} \boldsymbol{p}_1\big] aa \big[\boldsymbol{p}_2 \boldsymbol{p}^{i-1} \boldsymbol{p}_1\big] b\big] \hat{\boldsymbol{p}}_1 \boldsymbol{p}^{j-i-1} \boldsymbol{q}$.
   (*See Theorem 4, case 1 for ab, in the web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = a$, $\boldsymbol{q} = b\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $j = 2r$, $r \geq 3$.
   $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r} \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2 \big[b \big[\boldsymbol{p}^{r-1}\big] aa \big[\boldsymbol{p}^{r-1}\big] b\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 3 for ab, in the web supplement for proof.*)

3. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ in $\boldsymbol{x}$ is followed by an a and is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p}=a$, $\boldsymbol{q}=b\hat{\boldsymbol{q}}_1=\hat{\boldsymbol{q}}_2 b$, $j=2i{+}2$. $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a)=$ $\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}=\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]\boldsymbol{p}\boldsymbol{p}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]b\big]\hat{\boldsymbol{q}}_1=$ $\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]aa\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]b\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 4, in the web supplement for proof.*)

4. $\boldsymbol{u}=\varepsilon$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 b\boldsymbol{q}_2=\boldsymbol{q}_1 b\hat{\boldsymbol{q}}_1$, $\boldsymbol{p}=\boldsymbol{q}_1 aa\boldsymbol{q}_2$, $j=2r{+}1$, $r\geq 3$.
   $\sigma(ab)=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}=\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}=\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}=$
   $\boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]aa\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 5 for ab, in the web supplement for proof.*)

5. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ in $\boldsymbol{x}$ is followed by an a and is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 b\boldsymbol{q}_2=\boldsymbol{q}_1 b\boldsymbol{q}_1$, $\boldsymbol{p}=\boldsymbol{q}_1 aa\boldsymbol{q}_2$, $j=2i{+}1$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a)=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}=$
   $\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]aa\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 6, in the web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$ from near-repetition $b\boldsymbol{u}a\boldsymbol{u}$. (23)

1. $\boldsymbol{u}=\varepsilon$, $\hat{\boldsymbol{p}}_2 b\boldsymbol{p}_2=\boldsymbol{p}_1 b\hat{\boldsymbol{p}}_1$, $\boldsymbol{q}=\boldsymbol{p}_1 aa\boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i\geq 3$.
   $\sigma(ba)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^{j-1}\boldsymbol{q}\boldsymbol{p}^{i-1}\boldsymbol{p}\boldsymbol{q}=$
   $\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]aa\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]b\big]\hat{\boldsymbol{p}}_1\boldsymbol{q}$.
   (*See Theorem 4, case 1 for ba, in the web supplement for proof.*)

2. $\boldsymbol{u}=\varepsilon$, $\boldsymbol{p}=a$, $\boldsymbol{q}=b\hat{\boldsymbol{q}}_1=\hat{\boldsymbol{q}}_2 b$, $i=2r$, $r\geq 3$.
   $\sigma(ba)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q}=\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^{r-1}\big]aa\big[\boldsymbol{p}^{r-1}\big]b\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 3 for ba, in the web supplement for proof.*)

3. $\boldsymbol{u}=\varepsilon$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 b\boldsymbol{q}_2=\boldsymbol{q}_1 b\hat{\boldsymbol{q}}_1$, $\boldsymbol{p}=\boldsymbol{q}_1 aa\boldsymbol{q}_2$, $i=2r{+}1$, $r\geq 3$.
   $\sigma(ba)=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}=\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}=\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}=$
   $\boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]aa\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 4, case 5 for ba, in the web supplement for proof.*)

Deriving near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$ from near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$. (24)

1. $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$.
   $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]aa\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]b\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.
   (*See Theorem 4, case 2, in the web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}bb\boldsymbol{u}a$ from run. (25)

1. The run has generator a, $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i \geq 3$.
   For every square aa of the run: $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}\boldsymbol{p}^{i-1}\boldsymbol{q}\boldsymbol{p}^{i-1}\boldsymbol{p}\boldsymbol{q} = \hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1\boldsymbol{q}$.
   (*See Theorem 5, case 1 for aa, in the web supplement for proof.*)

2. The run has generator b, $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $j \geq 3$.
   For every square bb of the run: $\sigma(bb) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}\boldsymbol{p}^{j-1}\boldsymbol{q}\boldsymbol{p}^{j-1}\boldsymbol{p}\boldsymbol{q} = \hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^{j-1}\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^{j-1}\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1\boldsymbol{q}$.
   (*See Theorem 5, case 1 for bb, in the web supplement for proof.*)

3. The run has generator a, $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $i = 2r$, $r \geq 3$.
   $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q} = \boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^{r-1}\big]bb\big[\boldsymbol{p}^{r-1}\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 3 for aa, in the web supplement for proof.*)

4. The run has generator b, $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $j = 2r$, $r \geq 3$.
   $\sigma(bb) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q} = \boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^{r-1}\big]bb\big[\boldsymbol{p}^{r-1}\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 3 for bb, in the web supplement for proof.*)

5. The run has generator a, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $i = 2r+1$, $r \geq 3$.
   For any square aa in the run: $\sigma(aa) = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} = \boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 5 for aa, in the web supplement for proof.*)

6. The run has generator b, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $j = 2r+1$, $r \geq 3$.
   For any square aa in the run: $\sigma(bb) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} = \boldsymbol{p}^j\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 5 for bb, in the web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}bb\boldsymbol{u}a$ from near-repetition $a\boldsymbol{u}b\boldsymbol{u}$. (26)

1. $\boldsymbol{u} = \varepsilon$, $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i \geq 3$.
   $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}\boldsymbol{p}^{i-1}\boldsymbol{q}\boldsymbol{p}^{i-1}\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
   $\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.
   (*See Theorem 5, case 1 for ab, in the web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $j = 2r$, $r \geq 3$.
   $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r} \boldsymbol{q} = \boldsymbol{p}^i \hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^{r-1}\big]bb\big[\boldsymbol{p}^{r-1}\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 3 for ab, in the web supplement for proof.*)

3. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ in $\boldsymbol{x}$ is followed by an a and is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $j = 2i+2$. $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) =$
   $\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]\boldsymbol{p}\boldsymbol{p}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]a\big]\hat{\boldsymbol{q}}_1 =$
   $\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]bb\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 4, in the web supplement for proof.*)

4. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $j = 2r+1$, $r \geq 3$.
   $\sigma(ab) = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} =$
   $\boldsymbol{p}^i\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 5 for ab, in the web supplement for proof.*)

5. The near-repetition $a\boldsymbol{u}b\boldsymbol{u}$ in $\boldsymbol{x}$ is followed by an a and is not an initial segment of $\boldsymbol{x}$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \boldsymbol{q}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $j = 2i+1$.
   $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
   $\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 6, in the web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}bb\boldsymbol{u}a$ from near-repetition $b\boldsymbol{u}a\boldsymbol{u}$. (27)

1. $\boldsymbol{u} = \varepsilon$, $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$, $i \geq 3$.
   $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^{j-1}\boldsymbol{q}\boldsymbol{p}^{i-1}\boldsymbol{p}\boldsymbol{q} =$
   $\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^{i-1}\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1\boldsymbol{q}$.
   (*See Theorem 5, case 1 for ba, in the web supplement for proof.*)

2. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $i = 2r$, $r \geq 3$.
   $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r} \boldsymbol{q} = \boldsymbol{p}^j \hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^{r-1}\big]bb\big[\boldsymbol{p}^{r-1}\big]a\big]\hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 3 for ba, in the web supplement for proof.*)

3. $\boldsymbol{u} = \varepsilon$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $i = 2r{+}1$, $r \geq 3$.
   $\sigma(ba) = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} = \boldsymbol{p}^j \boldsymbol{q} \boldsymbol{p}^{2r+1} \boldsymbol{q} = \boldsymbol{q} \boldsymbol{p}^r \boldsymbol{p} \boldsymbol{p}^r \boldsymbol{q} =$
   $\boldsymbol{p}^j \hat{\boldsymbol{q}}_2 \big[a \big[\boldsymbol{q}_2 \boldsymbol{p}^r\big] \boldsymbol{q}_1\big] bb \big[\boldsymbol{q}_2 \boldsymbol{p}^r \boldsymbol{q}_1\big] a\big] \hat{\boldsymbol{q}}_1$.
   (*See Theorem 5, case 5 for ba, in the web supplement for proof.*)

Deriving near-repetition $a\boldsymbol{u}bb\boldsymbol{u}a$ from near-repetition $b\boldsymbol{u}aa\boldsymbol{u}b$. (28)

1. $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$.
   $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) = \boldsymbol{p}^j \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^j \boldsymbol{q} = \boldsymbol{p}^{j-i-1} \boldsymbol{p} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{q} \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p} \boldsymbol{p}^{j-i-1} \boldsymbol{q} =$
   $\boldsymbol{p}^{j-i-1} \hat{\boldsymbol{p}}_2 \big[a \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p}_1\big] bb \big[\boldsymbol{p}_2 \boldsymbol{p}^i \boldsymbol{q} \overline{\boldsymbol{u}} \boldsymbol{p}^i \boldsymbol{p}_1\big] a\big] \hat{\boldsymbol{p}}_1 \boldsymbol{p}^{j-i-1} \boldsymbol{q}$.
   (*See Theorem 5, case 2, in the web supplement for proof.*)

6 Concluding Remarks

It is straightforward to see that the derivations described in (7)–(28) are sound as they derive the correct runs and near-repetitions. However, it remains to be shown that they are also complete; that is, that all long runs and near-repetitions arise in the ways described in (7)–(28) and in no other way. Otherwise, our algorithm might miss some of the runs or near-repetitions.

Note also that the fact that in an expansion $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]$, the pair $(\boldsymbol{p}, \boldsymbol{q})$ must be suitable, has not played any role in the discussions of the repetition algorithm. The truth is that it does play the most important role in the actual proofs of the completeness of the derivations (7)–(28).

The proofs of the completeness of the derivations (7)–(28) (and thus of Theorem 1) are mathematically uninteresting, tedious and lengthy, as they are based on "brute force" discussions of all possible ways a square or a near repetition can be laid out in an expanded string. For that reason, and since they do not facilitate understanding of how the algorithm works, we omit them from the paper. However, the interested reader can find them in all their gory details and length at the web site of the first author:

`www.cas.mcmaster.ca/∼franek/web-publications.html`

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This appendix contains the proofs only. For the definition of two-pattern strings and the motivation of the theorems presented here, please, see the report.

First we need to introduce or recall some terminology:

For a string $\boldsymbol{u}[1..k]$, its **segment** is $\boldsymbol{u}[m..n]$ for some $1 \leq m \leq n \leq k$. $\boldsymbol{u}^*$ will denote the **set of all its segments**. A segment $\boldsymbol{u}[1..n]$ is called an **initial segment**, while a segment $\boldsymbol{u}[m..k]$ is called an **end segment**. We say that segment $\boldsymbol{u}[m_1..n_1]$ **precedes** segment $\boldsymbol{u}[m_2..n_2]$ (or equivalently that segment $\boldsymbol{u}[m_2..n_2]$ **follows** segment $\boldsymbol{u}[m_1..n_1]$) if $n_1 < m_2$.

A square $\boldsymbol{u}[s..s{+}k{-}1]\boldsymbol{u}[s{+}k..s{+}2k{-}1]$ can be **left-extended** by m positions if $\boldsymbol{u}[s{-}m{+}r] = \boldsymbol{u}[s{-}m{+}r{+}k]$ for any $0 \leq r < k$ (and so $\boldsymbol{u}[s{-}m..s{-}m{+}k{-}1]\boldsymbol{u}[s{-}m{+}k..s{-}m{+}2k{-}1]$ is a square).
A square $\boldsymbol{u}[s..s{+}k{-}1]\boldsymbol{u}[s{+}k..s{+}2k{-}1]$ can be **right-extended** by m positions if $\boldsymbol{u}[s{+}m{+}r] = \boldsymbol{u}[s{+}m{+}r{+}k]$ for any $0 \leq r < k$ (and so $\boldsymbol{u}[s{+}m..s{+}m{+}k{-}1]\boldsymbol{u}[s{+}m{+}k..s{+}m{+}2k{-}1]$ is a square).

A square $\boldsymbol{u}[s..s{+}k{-}1]\boldsymbol{u}[s{+}k..s{+}2k{-}1]$ is **irreducible** if $\boldsymbol{u}[s..s{+}k{-}1]$ is not a repetition. A string is called **primitive** if it has no non-empty border.

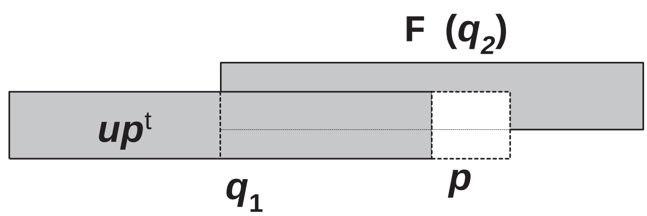
If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two-pattern strings and $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]$ an expansion and $\boldsymbol{y} = \sigma(\boldsymbol{x})$, then $\boldsymbol{y}$ is a concatenation of blocks $\boldsymbol{p}^i\boldsymbol{q}$ and $\boldsymbol{p}^j\boldsymbol{q}$. These occurrences of copies of $\boldsymbol{p}$ and $\boldsymbol{q}$ are called **restrained**.

$GCS(\boldsymbol{u}, \boldsymbol{v})$ denotes the greatest common suffix of $\boldsymbol{u}$ and $\boldsymbol{v}$, while $GCP(\boldsymbol{u}, \boldsymbol{v})$ denotes the greatest common prefix of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Lemma 1 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ an expansion, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $m < n$, $k \geq 1$ and let $\mathcal{F} : \boldsymbol{y}[m..n]^* \rightarrow \boldsymbol{y}[m{+}k..n{+}k]^*$ be a bijection defined by $\mathcal{F}(\boldsymbol{y}[s_1..s_2]) = \boldsymbol{y}[s_1{+}k..s_2{+}k]$. Let $\boldsymbol{p}_1$, a restrained copy of $\boldsymbol{p}$ that is a segment of $\boldsymbol{y}[m..n]$, precede $\boldsymbol{q}_1$, a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}$ however not an end segment. Then it is not possible for $\mathcal{F}(\boldsymbol{p}_1)$ to be a segment of $\boldsymbol{q}_1$.*

Proof By the way of contradiction, let us assume that $\boldsymbol{p}_1$ is a restrained copy of $\boldsymbol{p}$ that is a segment of $\boldsymbol{y}[m..n]$ and that is mapped by $\mathcal{F}$ onto a segment of $\boldsymbol{q}_1$. $\boldsymbol{p}_1$ is followed by some (or none) restrained copies of $\boldsymbol{p}$, let us denote them $\boldsymbol{p}_2$, ... $\boldsymbol{p}_t$, followed by a restrained copy of $\boldsymbol{q}$, let us denote it $\boldsymbol{q}_2$.

If the $\mathcal{F}$ images of $\boldsymbol{p}_1$, ..., $\boldsymbol{p}_t$ were not all segments of $\boldsymbol{q}_1$, then one of the $\mathcal{F}$-images of $\boldsymbol{p}_1$, ... $\boldsymbol{p}_t$ would either be an end segment of $\boldsymbol{q}_1$ (giving $\boldsymbol{p}$ a suffix of $\boldsymbol{q}$, a contradiction), or it would intersect $\boldsymbol{q}_1$ and the restrained copy of $\boldsymbol{p}$ immediately following $\boldsymbol{q}_1$ (which contradicts the primitiveness of $\boldsymbol{p}$). Thus we must conclude that all the $\mathcal{F}$-images of $\boldsymbol{p}_1...\boldsymbol{p}_t$ are segments of $\boldsymbol{q}_1$. It follows that $\mathcal{F}(\boldsymbol{q}_2)$, intersects with $\boldsymbol{q}_1$ while it is not its segment. The initial segment of $\boldsymbol{q}_1$ that is not a segment of $\mathcal{F}(\boldsymbol{q}_2)$ can be expressed as $\boldsymbol{up}^t$ (see the diagram below), and so $\boldsymbol{q} = (\boldsymbol{up}^t)^r\boldsymbol{v}$, where $\boldsymbol{v}$ is a prefix of $\boldsymbol{up}^t$, for some $r \geq 1$.



- case $\boldsymbol{v} = \varepsilon$: is impossible, for $\boldsymbol{p}$ would be a suffix of $\boldsymbol{q}$.

- case $\boldsymbol{v}$ is a proper prefix of $\boldsymbol{u}$: then $\boldsymbol{u} = \boldsymbol{v}\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is a non-empty prefix of $\boldsymbol{p}$, as $\boldsymbol{q}_1$ is followed by $\hat{\boldsymbol{p}}$, a restrained copy of $\boldsymbol{p}$. Henceforth the end segment of $\boldsymbol{y}$ of which $\hat{\boldsymbol{p}}$ is an initial segment has $\hat{\boldsymbol{v}}\boldsymbol{p}$ as a prefix, which contradicts the primitiveness of $\boldsymbol{p}$.

- case $\boldsymbol{v} = \boldsymbol{u}$: then $\boldsymbol{q}$ is $\boldsymbol{p}$-regular as $\boldsymbol{q} = (\boldsymbol{up}^t)^r\boldsymbol{u}$, a contradiction.

- case $\boldsymbol{v} = \boldsymbol{up}^s\hat{\boldsymbol{v}}$, where $0 \leq s \leq t$ and $\hat{v}$ is a prefix of $\boldsymbol{p}$: $\hat{\boldsymbol{v}} \neq \varepsilon$, for otherwise $\boldsymbol{p}$ would be a suffix of $\boldsymbol{q}$. It follows that $s < t$. $\boldsymbol{q}_1$ is followed by $\hat{\boldsymbol{p}}$, a restrained copy of $\boldsymbol{p}$. Consider the end segment $\mathcal{S}_1$ of $\boldsymbol{y}$ of which $\boldsymbol{up}^s\hat{\boldsymbol{v}}\hat{\boldsymbol{p}}$ (the end segment of $\boldsymbol{q}_1$ followed by $\hat{\boldsymbol{p}}$) is an initial segment. It has $\boldsymbol{up}^t\boldsymbol{up}^s\hat{\boldsymbol{v}}$ as a prefix (the end segment of $\mathcal{F}(\boldsymbol{q}_2)$). Let us move to the right past the prefix $\boldsymbol{up}^s$ in the segment $\mathcal{S}_1$. It is an end segment of $\boldsymbol{y}$, denote it $\mathcal{S}_2$. $\mathcal{S}_2$ has $\hat{\boldsymbol{v}}\boldsymbol{p}$ as a prefix and also $\boldsymbol{p}$ as a prefix, as $s < r$. This contradicts the primitiveness of $\boldsymbol{p}$.

All cases lead to contradictions, and so our initial assumption cannot hold. $\square$

Lemma 2 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ an expansion, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $m < n$, $k \geq 1$ and let $\mathcal{F} : \boldsymbol{y}[m..n]^* \to \boldsymbol{y}[m{-}k..n{-}k]^*$ be a bijection defined by $\mathcal{F}(\boldsymbol{y}[s_1..s_2]) = \boldsymbol{y}[s_1{-}k..s_2{-}k]$. Let $\boldsymbol{p}_1$, a restrained copy of $\boldsymbol{p}$ that is a segment of $\boldsymbol{y}[m..n]$, follow $\boldsymbol{q}_1$, a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}$. Then it is not possible for $\mathcal{F}(\boldsymbol{p}_1)$ to be a segment of $\boldsymbol{q}_1$.*

Proof Virtually identical to the proof of the previous lemma. $\square$

In the following theorem we will discuss In the following we are going to discuss all possible ways squares can arise. We say that $\boldsymbol{y}[s..s{+}2k{-}1]$ is a square if $\boldsymbol{y}[s{+}m] = \boldsymbol{y}[s{+}k{+}m]$ for any $0 \leq m < k$.

Theorem 2 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ an expansion, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $\mathcal{S} = \boldsymbol{y}[s..s{+}k{-}1]\boldsymbol{y}[s{+}k..s{+}2k{-}1]$ be a big irreducible square in $\boldsymbol{y}$ that cannot be left-extended. Then either*

1. *$\mathcal{S}$ is a square in the σ-expansion of a run in $\boldsymbol{x}$; or*

2. *$j \leq 2i$ and $\mathcal{S}$ is a left-extension of a square derived from an $a\boldsymbol{u}b\boldsymbol{u}a$ near repetition or itself derived from an $a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(a\boldsymbol{u}b\boldsymbol{u}a) = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} =$ $\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\boldsymbol{p}^{2i-j}\boldsymbol{q}$; or*

3. *$\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$ and $\mathcal{S}$ is derived from one of the near repetitions aa, ab, ba, and bb in $\boldsymbol{x}$ in the following way: $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ all have $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}$, $0 \leq r < i$ (for aa, ab, ba) and $0 \leq r < j$ (for bb), as a substring. $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1} = \boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p} =$ $\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1$; or*

4. *$\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \neq \varepsilon$, $\boldsymbol{p}_2 \neq \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$ and $\mathcal{S}$ is derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) =$ $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\boldsymbol{p}_2\boldsymbol{p}^{j-i-1}\boldsymbol{q}$; or*

5. *$\mathcal{S}$ is a left-extension of a square derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition or itself derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} = \boldsymbol{p}^{j-i}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]$; or*

6. *$\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $i = 2r{+}1$ (for aa, ab, ba) or $j = 2r{+}1$ (for bb) and $\mathcal{S}$ is derived from one of the near repetitions*

aa, ab, ba, and bb in $\boldsymbol{x}$ in the following way: $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ all contain $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$ as a substring. $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1$; or

7. *$\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $j = 2i{+}1$ and $\mathcal{S}$ is derived from a $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdots\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdots\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1 = \cdots\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1$.*

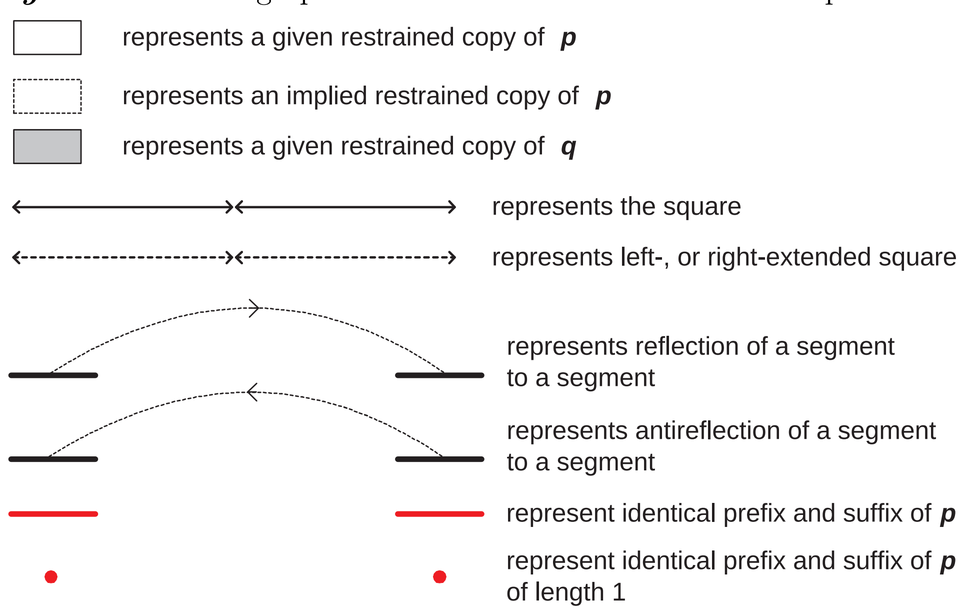
Proof For the square $\mathcal{S} = \boldsymbol{y}[s..s{+}k{-}1]\boldsymbol{y}[s{+}k..s{+}2k{-}1]$ in $\boldsymbol{y}$, the bijection $\mathcal{R}_{\mathcal{S}} : \boldsymbol{u}[s..s{+}k{-}1]^* \to \boldsymbol{u}[s{+}k..s{+}2k{-}1]^*$ defined by

$$\mathcal{R}_{\mathcal{S}}(\boldsymbol{u}[s{+}m..s{+}n]) = \boldsymbol{u}[s{+}k{+}m..s{+}k{+}n]$$

will be referred to as **reflection**, while its inverse as **antireflection**.

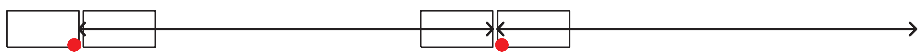
For the purpose of applying Lemma 1, the role of the bijection $\mathcal{F}$ will be played by the reflection, while for applying Lemma 2, the role of $\mathcal{F}$ will be played by the antireflection.

The proof is conducted by a "brute force" discussion of all possible ways the square $\mathcal{S}$ can be placed in $\boldsymbol{y}$ with respect to the restrained copies of $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\boldsymbol{y}$. We will use a graphical method to describe the various placements.

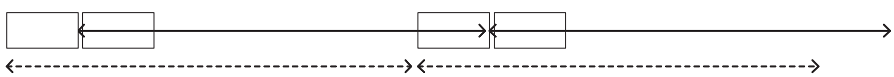


The cases are discussed based on where the points $\boldsymbol{y}[s]$ and $\boldsymbol{y}[s{+}k{-}1]$ are located. Recall that for the square $\mathcal{S}$ to be big, $k > 3\lambda$.

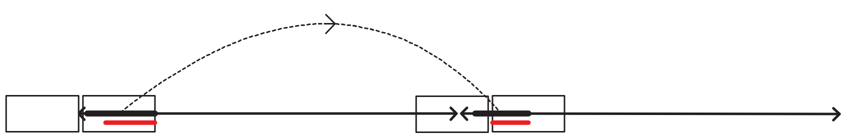
Case (1) – $\boldsymbol{pp}$-$\boldsymbol{pp}$ (i.e. the point $\boldsymbol{y}[s]$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$, and the point $\boldsymbol{y}[s+k-1]$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$):

1a 

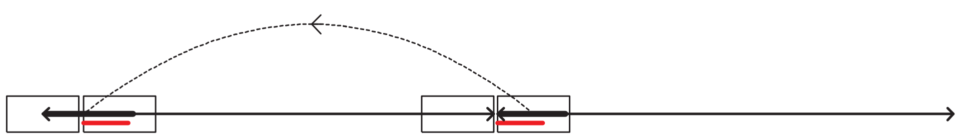
is possible only if $\boldsymbol{p} = a$. That reduces it to case 1h below.

1b 

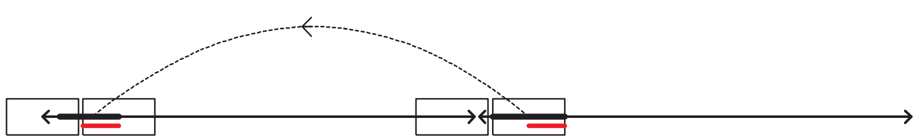
is not possible as it allows the square $\mathcal{S}$ to be left-extended.

1c 

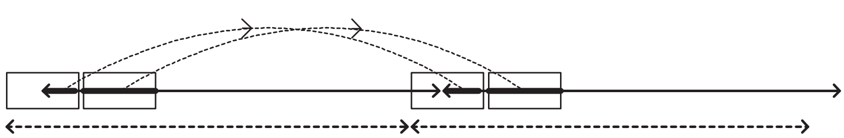
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1d 

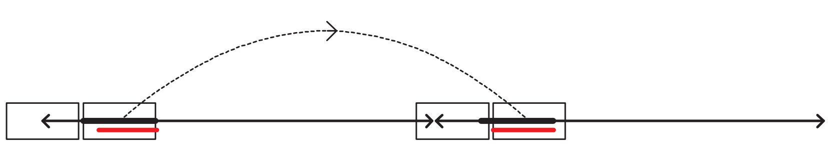
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1e 

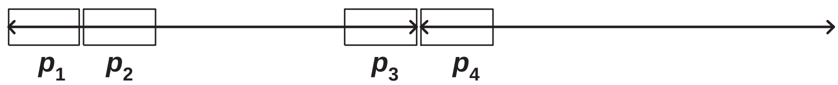
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1f 

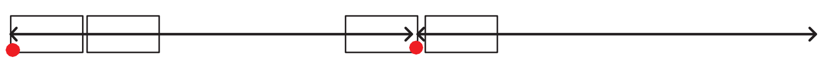
is not possible as it allows the square $\mathcal{S}$ to be left-extended.

1g 

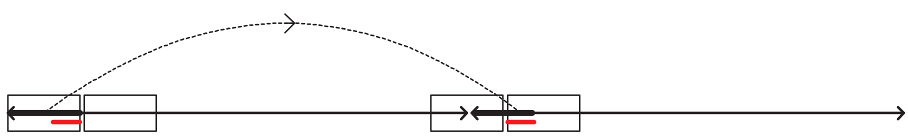
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1h


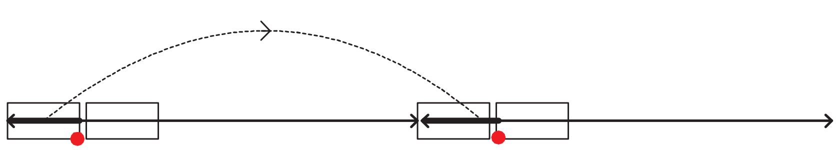
is possible. If $\boldsymbol{p}_1$ is preceded by a restrained copy of $\boldsymbol{p}$, then the square $\mathcal{S}$ can be left-extended, a contradiction. Hence $\boldsymbol{p}_1$ is preceded by a restrained copy of $\boldsymbol{q}$, or $\boldsymbol{p}_1$ is an initial segment of $\boldsymbol{y}$. Since $\mathcal{S}$ is irreducible, $\boldsymbol{y}[s..s{+}k{-}1] \neq \boldsymbol{p}^t$ for $t \geq 2$. Since it is big, it $\boldsymbol{y}[s..s{+}k{-}1] \neq \boldsymbol{p}$. Therefore there is $\boldsymbol{q}_2$, a restrained copy of $\boldsymbol{q}$, that is a segment of $\boldsymbol{y}[s..s{+}k{-}1]$. It follows that $\boldsymbol{p}^i\boldsymbol{q}$ is a prefix of $\boldsymbol{y}[s..s{+}k{-}1]$ and $\boldsymbol{p}^{j-i}$ is a suffix of $\boldsymbol{y}[s..s{+}k{-}1]$. Thus $\boldsymbol{y}[s..s{+}2k{-}1] = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ for some $\overline{\boldsymbol{u}}$. Thus $\boldsymbol{y}[s..s{+}k{-}1] = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ which is a substring of either $\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ (and so the square is derived from the expansion of the near repetition $a\boldsymbol{u}b\boldsymbol{u}b$ in $\boldsymbol{x}$, hence it is a square in the run in $\boldsymbol{y}$ that is the expansion of a run in $\boldsymbol{x}$ that contains the square $\boldsymbol{u}b\boldsymbol{u}b$) (so **case 1 of the theorem is satisfied**), or $\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ provided $j{-}i \leq i$ (and so the square is derived from the expansion of the near repetition $a\boldsymbol{u}b\boldsymbol{u}a$ in $\boldsymbol{x}$ and **case 2 of the theorem is satisfied**).

1i


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

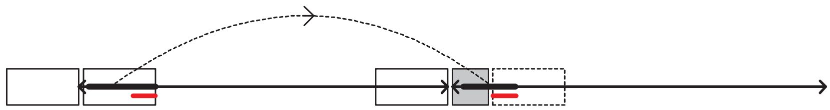
1j


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

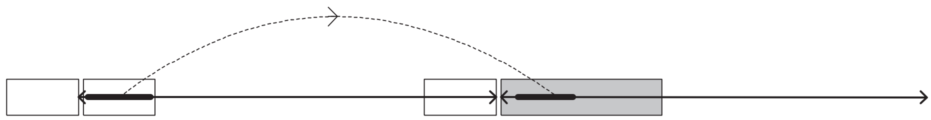
1k


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

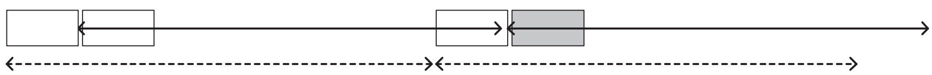
Case (2) – ***pp*-*pq***

2a 

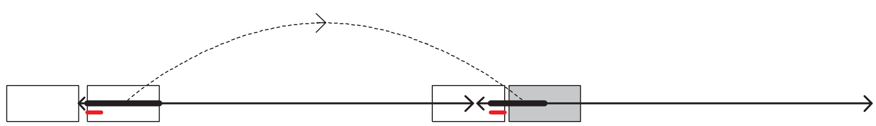
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2b 

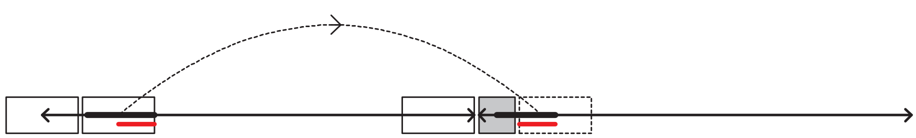
is not possible as it contradicts Lemma 1.

2c 

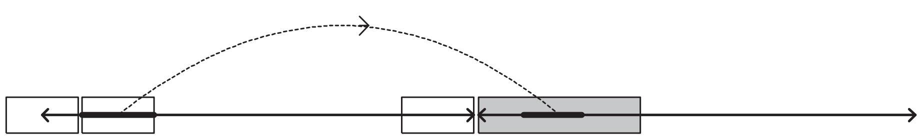
is not possible as it allows the square $\mathcal{S}$ to be left-extended (it would also imply that either $\boldsymbol{p}$ is a prefix $\boldsymbol{q}$ or $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$).

2d 

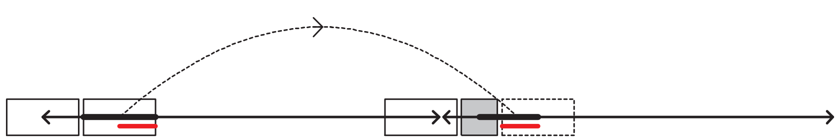
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2e 

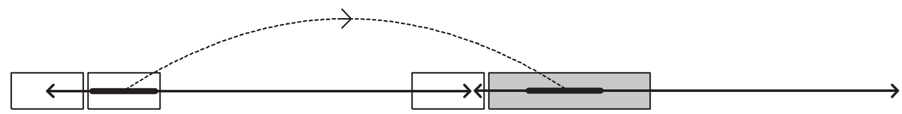
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2f 

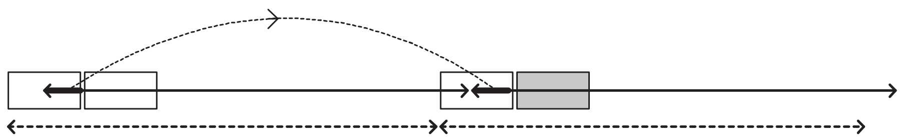
is not possible as it contradicts Lemma 1.

2g 

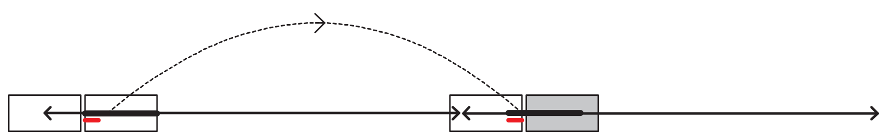
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2h 

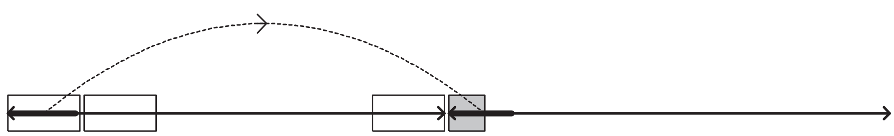
is not possible as it contradicts Lemma 1.

2i 

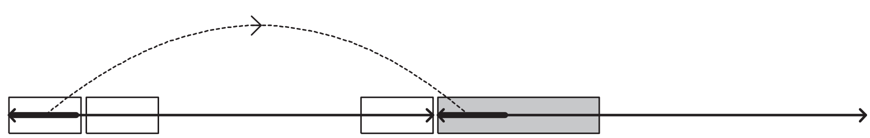
is not possible as it allows the square $\mathcal{S}$ to be left-extended (it would also imply that either $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$ or $\boldsymbol{q}$ a prefix of $\boldsymbol{p}$).

2j 

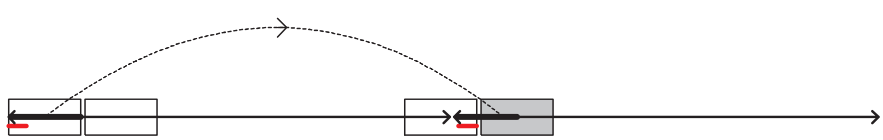
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2k 

is not possible as $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

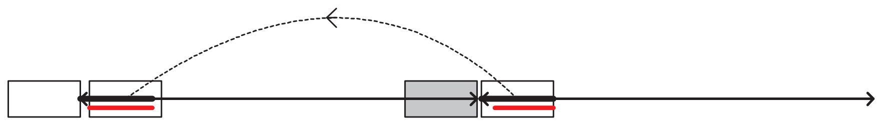
2l 

is not possible as $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

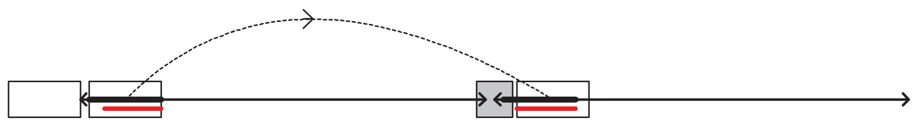
2m 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

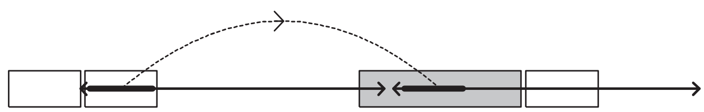
Case (3) – $\boldsymbol{pp}$-$\boldsymbol{qp}$

3a 

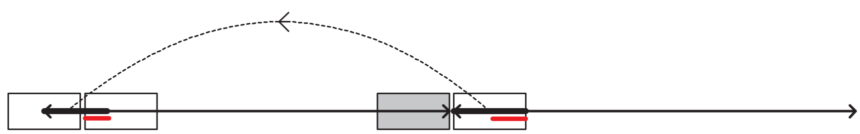
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3b 

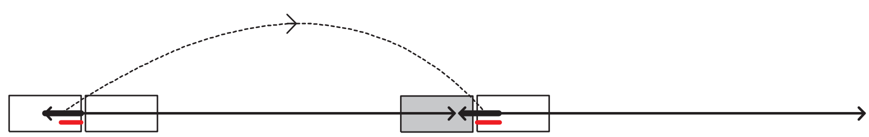
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3c 

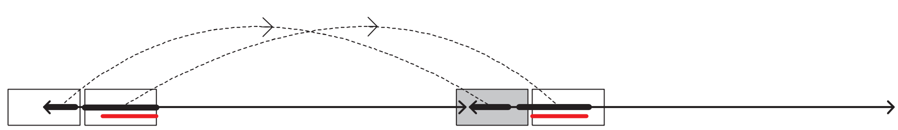
is not possible as it contradicts Lemma 1.

3d 

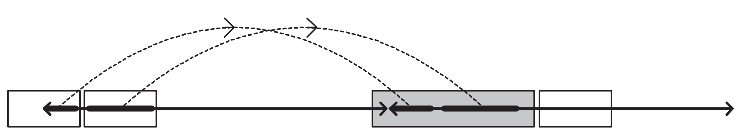
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3e 

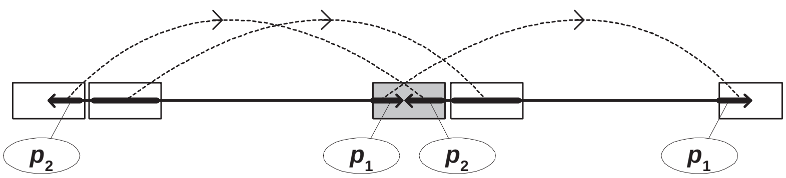
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3f 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3g 

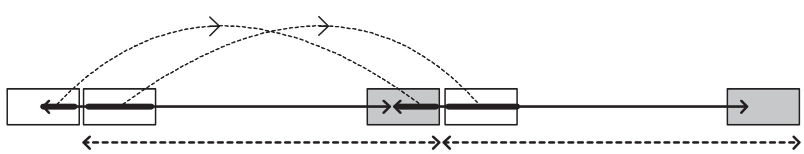
is not possible as it contradicts Lemma 1.

3h 

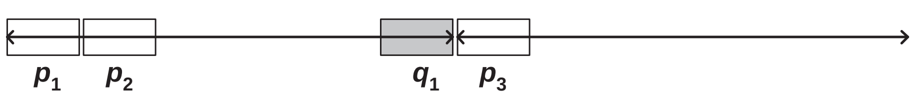
is possible if $\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, for some $\boldsymbol{p}_1 \neq \varepsilon$ that is a prefix of $\boldsymbol{p}$, and some $\boldsymbol{p}_2 \neq \varepsilon$ that is a suffix of $\boldsymbol{p}$. Since $\mathcal{S}$ is big, either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s..s{+}k{-}1]$, and so $\boldsymbol{y}[s..s{+}2k{-}1]$ is a substring of $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^r$ for some $r \geq 1$, and so $\mathcal{S}$ is derived from one of the near repetitions aa, ab, ba, and bb in $\boldsymbol{x}$ (and so **case 3 of the theorem is satisfied**); or

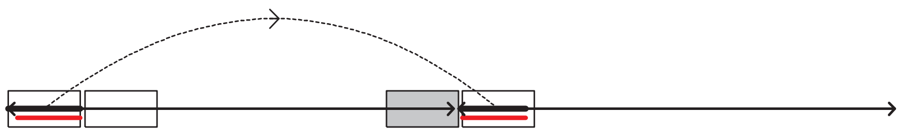
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s..s{+}1{-}k]$ and so $\boldsymbol{y}[s..s{+}2k{-}1]$ is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ that is the expansion of $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$, and thus $\mathcal{S}$ is derived from a near repetition $b\boldsymbol{u}aa\boldsymbol{u}b$ in $\boldsymbol{x}$ (and so **case 4 of the theorem is satisfied**).

3i 

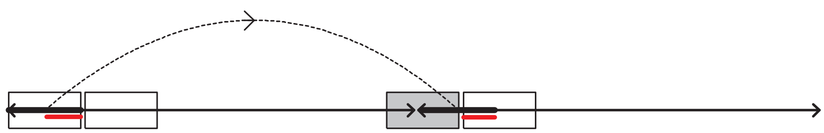
is possible. $\mathcal{S}$ is a left-extension of a square $\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that is the expansion of $b\boldsymbol{u}a\boldsymbol{u}$ near repetition in $\boldsymbol{x}$. Therefore $\mathcal{S}$ is a left-extension of a square derived from a near repetition $b\boldsymbol{u}a\boldsymbol{u}$ in $\boldsymbol{x}$ and so **case 5 of the theorem is satisfied**.

3j 

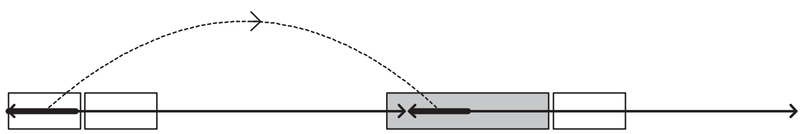
is possible. Since $\mathcal{S}$ cannot be left-extended, either $\boldsymbol{p}_1$ is an initial segment of $\boldsymbol{y}$, and so $\mathcal{S}$ is the expansion of a square in $\boldsymbol{x}$, or $\boldsymbol{p}_1$ is preceded by a restrained copy of $\boldsymbol{p}$, and so $\boldsymbol{y}[s..s{+}2k{-}1] = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that is the expansion of $b\boldsymbol{u}a\boldsymbol{u}$ near repetition in $\boldsymbol{x}$. Therefor $\mathcal{S}$ is derived from a near repetition $b\boldsymbol{u}a\boldsymbol{u}$ in $\boldsymbol{x}$ and so **case 5 of the theorem is satisfied**.

3k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

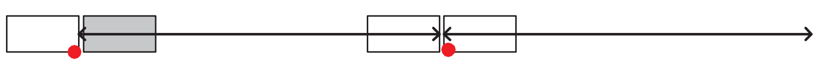
3l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

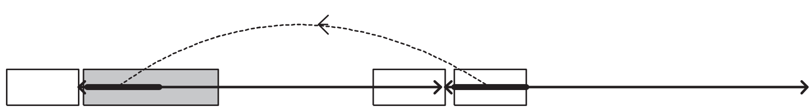
3m 

is not possible as it contradicts Lemma 1.

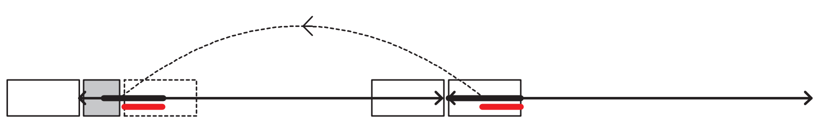
Case (4) – $\boldsymbol{pq}$-$\boldsymbol{pp}$

4a 

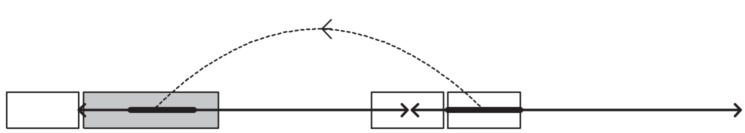
is possible only if $\boldsymbol{p} = a$. That reduces it to case 4k below.

4b 

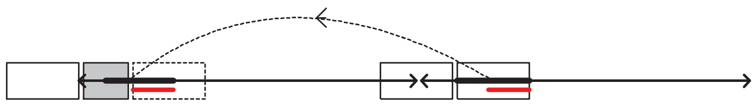
is not possible as it contradicts Lemma 2.

4c 

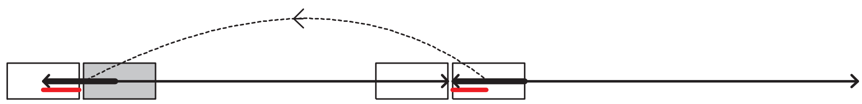
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4d 

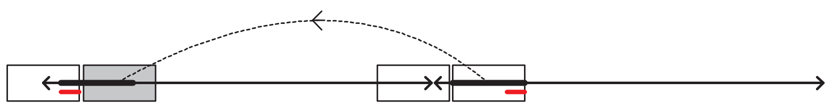
is not possible as it contradicts Lemma 2.

4e 

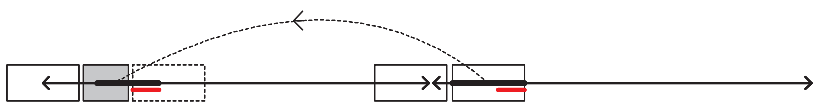
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4f 

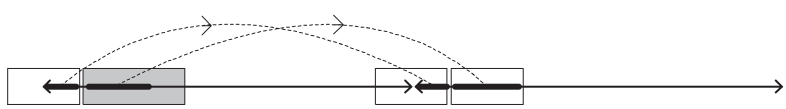
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4g 

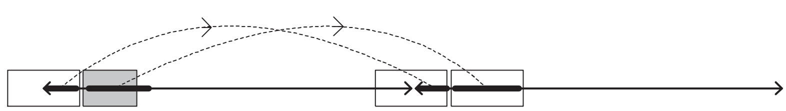
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4h 

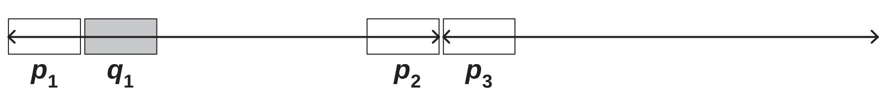
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4i 

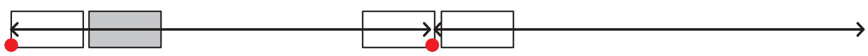
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

4j 

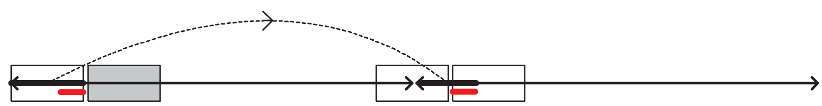
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

4k 

is possible. Since $\mathcal{S}$ cannot be left-extended, either $\boldsymbol{p}_1$ is an initial segment of $\boldsymbol{y}$, or $\boldsymbol{p}_1$ is preceded by a restrained copy of $\boldsymbol{q}$. Hence $i = 1$ and $\mathcal{S} = \boldsymbol{pq\bar{u}p}^{j-1}\boldsymbol{pq\bar{u}p}^{j-1}$ that is either a substring of $\boldsymbol{pq\bar{u}p}^j\boldsymbol{pq\bar{u}p}^j\boldsymbol{q}$ or $\boldsymbol{pq\bar{u}p}^j\boldsymbol{pq\bar{u}pq}$, if $j = 2$. In the former case, $\boldsymbol{pq\bar{u}p}^j\boldsymbol{pq\bar{u}p}^j\boldsymbol{q}$ is the expansion of $a\boldsymbol{u}b\boldsymbol{u}b$, and so $\mathcal{S}$ is a square in the run in $\boldsymbol{y}$ that is the expansion of a run in $\boldsymbol{x}$ that contains the square $\boldsymbol{u}b\boldsymbol{u}b$ (and so **case 1 of the theorem is satisfied**). In the latter case, $\mathcal{S}$ is derived from a near repetition $a\boldsymbol{u}b\boldsymbol{u}a$ (and so **case 2 of the theorem is satisfied**).

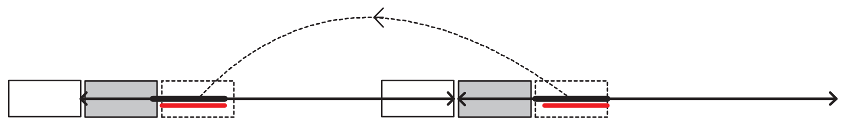
4l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

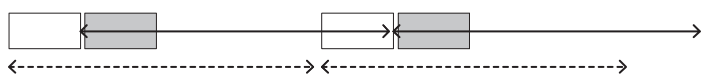
4m 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

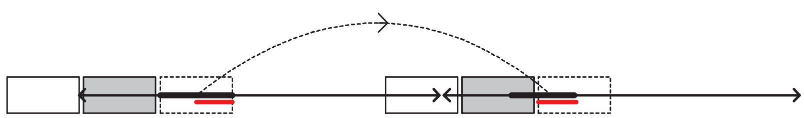
Case (5) – $\boldsymbol{pq}$-$\boldsymbol{pq}$

5a 

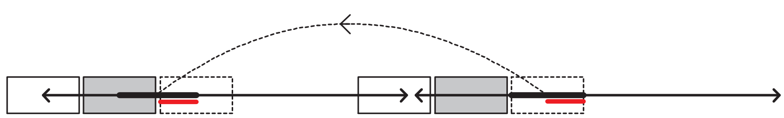
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5b 

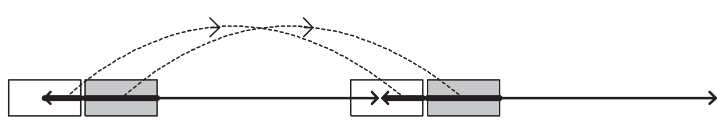
is not possible as it allows the square $\mathcal{S}$ to be left-extended.

5c 

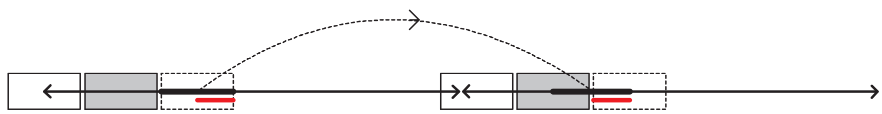
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5d 

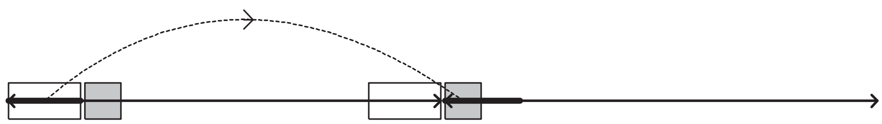
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5e 

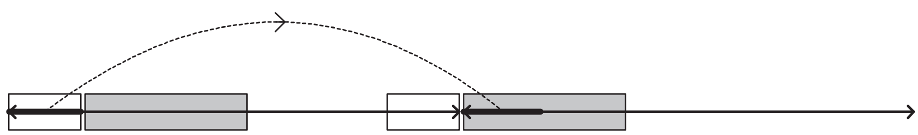
is not possible as it allows the square $\mathcal{S}$ to be left-extended.

5f 

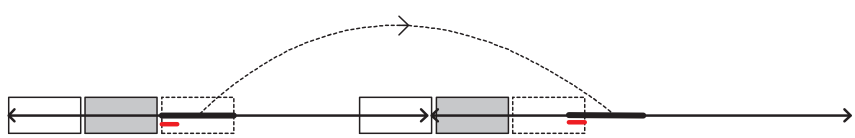
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5g 

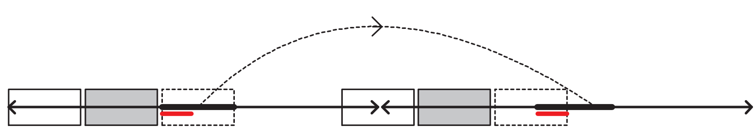
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

5h 

is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

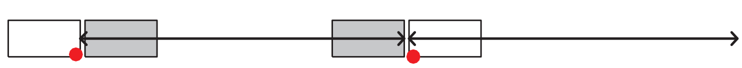
5i 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

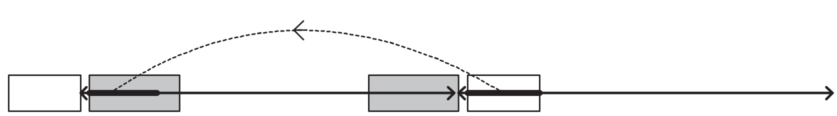
5j 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

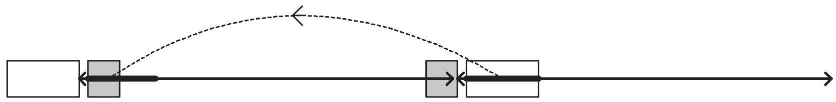
Case (6) – $\boldsymbol{pq}$-$\boldsymbol{qp}$

6a 

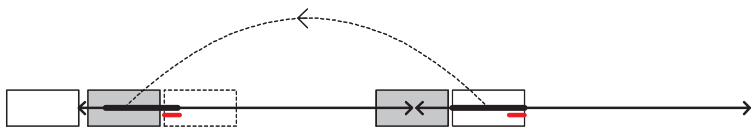
is possible only if $\boldsymbol{p} = a$ which reduces it to case 6k below.

6b 

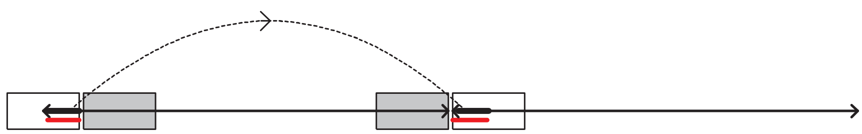
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6c 

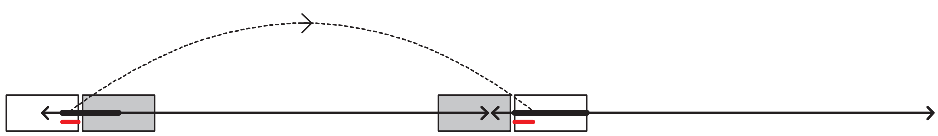
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6d 

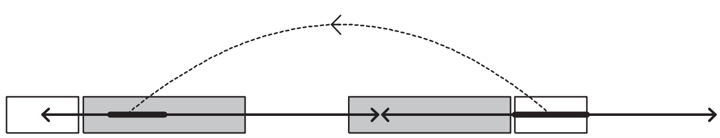
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6e 

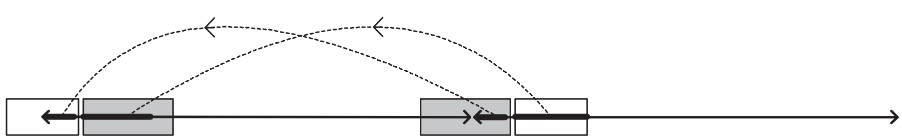
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6f 

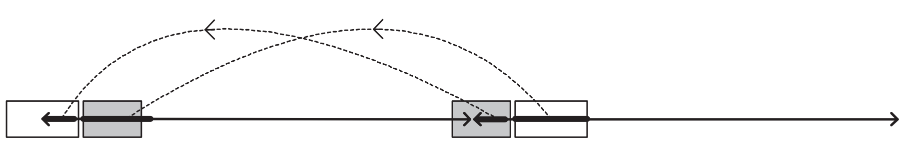
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6g 

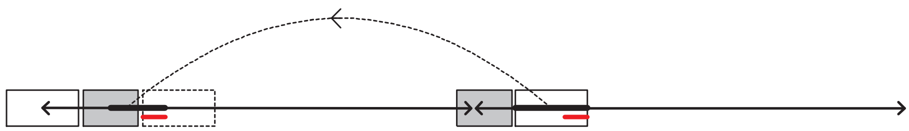
is not possible as it contradicts Lemma 2.

6h 

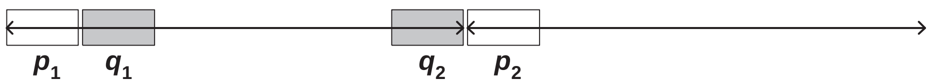
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6i 

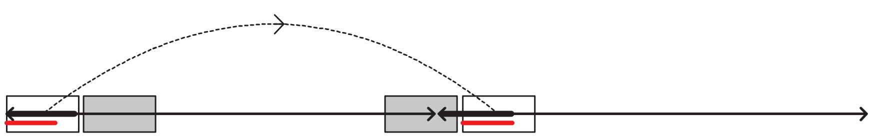
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6j 

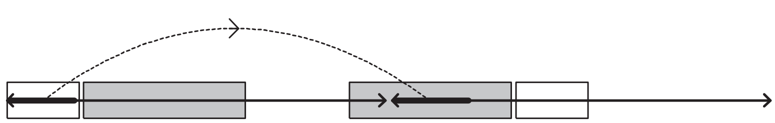
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6k 

is possible. Because $\mathcal{S}$ cannot be left-extended, $\boldsymbol{p}_1$ is preceded by a restrained copy of $\boldsymbol{p}$, or $\boldsymbol{p}_1$ is an initial segment of $\boldsymbol{y}$. In the former case, $\boldsymbol{y}[s..s{+}2k{-}1] = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$, the expansion of a near repetition $b\boldsymbol{u}a\boldsymbol{u}$ in $\boldsymbol{x}$ (and so **case 5 of the theorem is satisfied**). In the latter case, $i = 1$ and $\boldsymbol{y}[s..s{+}2k{-}1] = \boldsymbol{p}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}\boldsymbol{q}\overline{\boldsymbol{u}}$ that is the expansion of $a\boldsymbol{u}a\boldsymbol{u}$, and so $\mathcal{S}$ is a square in the expansion of a run that includes the square $a\boldsymbol{u}a\boldsymbol{u}$ in $\boldsymbol{x}$ (and so **case 1 of the theorem is satisfied**).

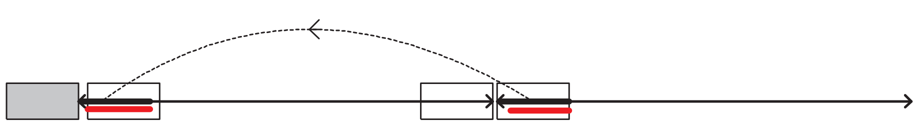
6l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6m 

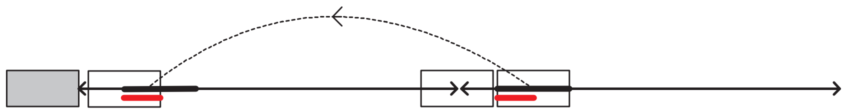
is not possible as it contradicts Lemma 1.

Case (7) – $\boldsymbol{qp}$-$\boldsymbol{pp}$

7a 

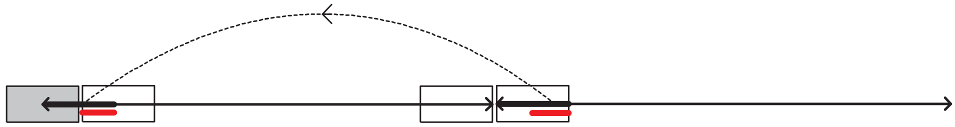
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7b



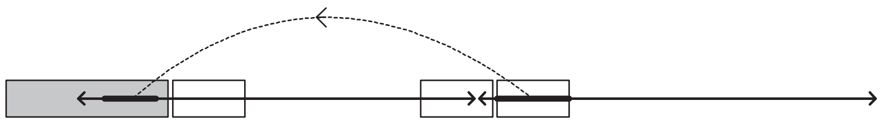
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7c



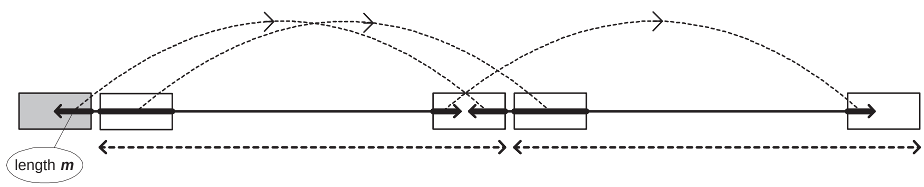
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7d



is not possible as it contradicts Lemma 2.

7e

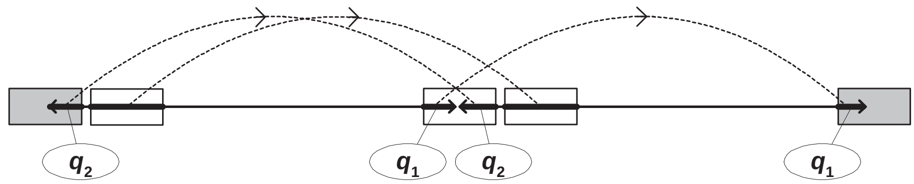


is possible. The square $\mathcal{S}$ is a left-extension of a square $\boldsymbol{y}[s{+}m..s{+}m{+}k{-}1]\boldsymbol{y}[s{+}m{+}k..s{+}m{+}2k{-}1]$. If there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}m..s{+}m{+}k{-}1]$, then $\boldsymbol{y}[s{+}m..s{+}m{+}k{-}1] = \boldsymbol{p}^r$ for some $r \geq 1$. Since $\mathcal{S}$ is irreducible, $r = 1$. But that contradicts the fact that $\mathcal{S}$ is big. Therefore there must be a restrained copy of $\boldsymbol{q}$ that is a segment $\boldsymbol{y}[s{+}m..s{+}m{+}k{-}1]$.
$\mathcal{S}$ is, thus, a left-extension of the square $\mathcal{S}_1 =$
$\boldsymbol{y}[s{+}m..s{+}m{+}k{-}1]\boldsymbol{y}[s{+}m{+}k..s{+}m{+}2k{-}1]$.
Since, $\boldsymbol{y}[s{+}m..s{+}m{+}2k{-}1] = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$, either

1. $j \leq 2i$ and $\boldsymbol{y}[s{+}m..s{+}m{+}2k{-}1]$ is a substring of $\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i$ and so $\mathcal{S}_1$ is derived from an $a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$, and so **case 2 of the theorem holds**; or

2. $\boldsymbol{y}[s{+}m..s{+}m{+}2k{-}1]$ is a substring of $\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j$ that is the expansion of $a\boldsymbol{u}b\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ and so **case 1 of the theorem is satisfied**.

7f

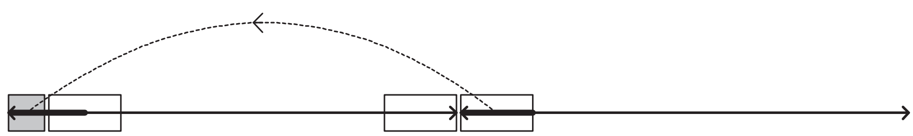


is possible if $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, for some $\boldsymbol{q}_1 \neq \varepsilon$ that is a prefix of $\boldsymbol{q}$ and some $\boldsymbol{q}_2 \neq \varepsilon$ that is a suffix of $\boldsymbol{q}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s..s+k-1]$ and then $\boldsymbol{y}[s..s+2k-1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{q}$, $r = i$ or $r = j$, r is odd, which is a substring of the expansion of one of the following near repetitions in $\boldsymbol{x}$: aa, ab, ba, and bb (and so **case 6 of the theorem is satisfied**); or

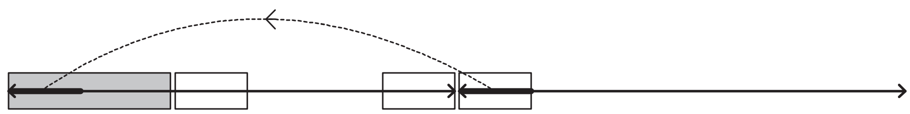
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s..s+k-1]$ and then $\boldsymbol{y}[s..s+2k-1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{q}$ that is a substring of the extension of $\cdot a\boldsymbol{u}b\boldsymbol{u}\cdot$, provided $j = 2i+1$ (and so **case 7 of the theorem is satisfied**).

7g



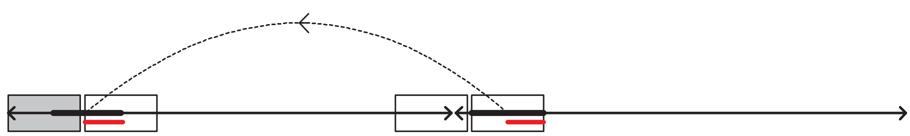
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

7h



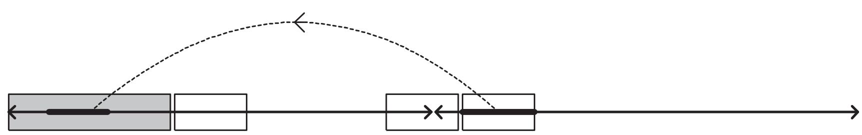
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

7i

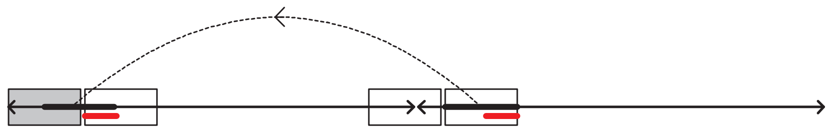


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

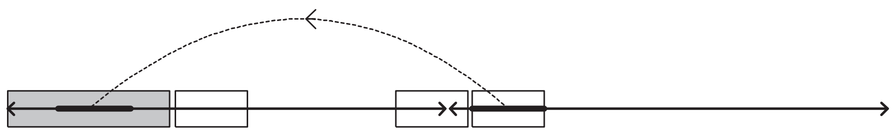
7j



is not possible as it contradicts Lemma 2.

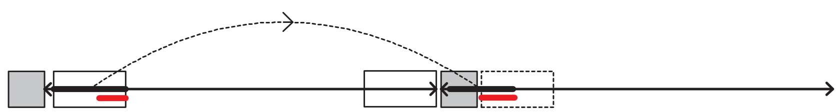
7k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

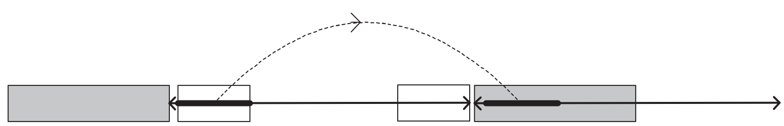
7l 

is not possible as it contradicts Lemma 2.

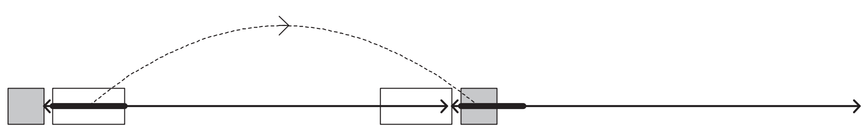
Case (8) – $\boldsymbol{qp}$-$\boldsymbol{pq}$

8a 

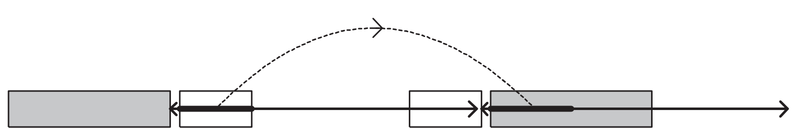
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8b 

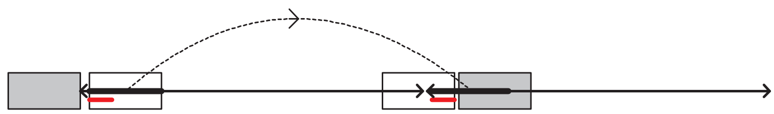
is not possible as it contradicts Lemma 1.

8c 

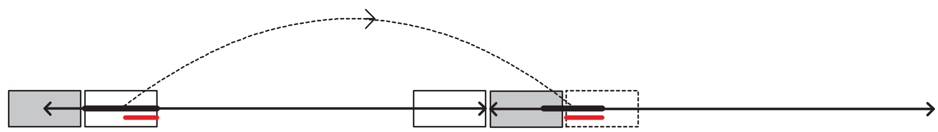
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

8d 

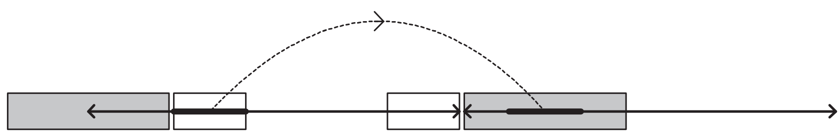
is not possible as it contradicts Lemma 1.

8e 

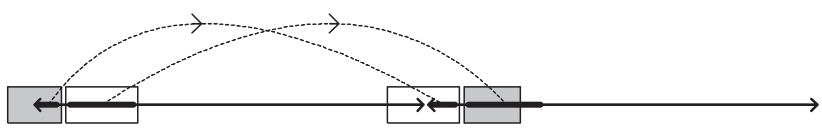
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8f 

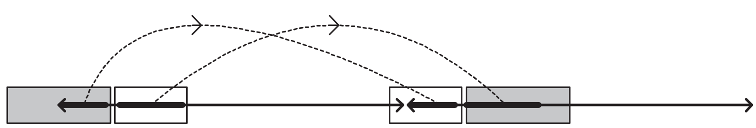
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8g 

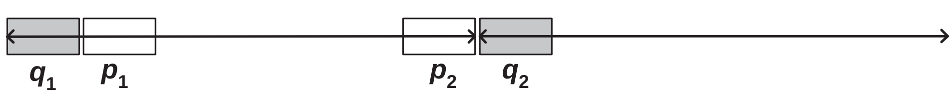
is not possible as it contradicts Lemma 1.

8h 

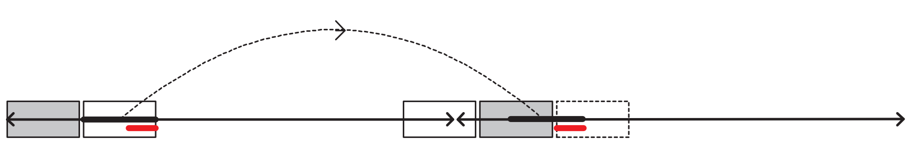
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

8i 

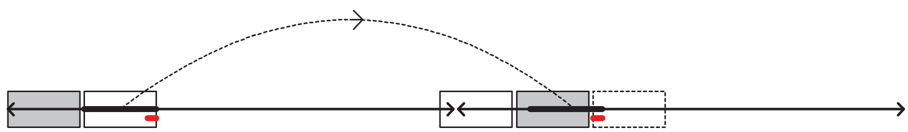
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

8j 

is not possible. For $\mathcal{S}$ cannot be left-extended and so either $\boldsymbol{q}_1$ is an initial segment of $\boldsymbol{y}[s..s+k-1]$, which is a contradiction, or $\boldsymbol{q}_1$ is preceded by a restrained copy of $\boldsymbol{q}$, also a contradiction.

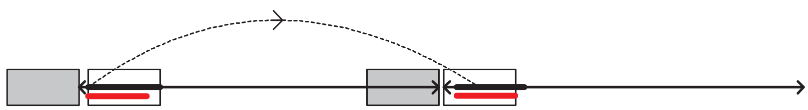
8k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

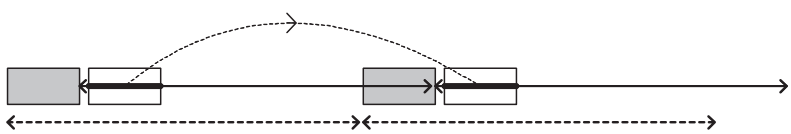
8l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

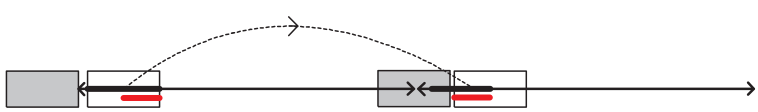
Case (9) – $\boldsymbol{qp}$-$\boldsymbol{qp}$

9a 

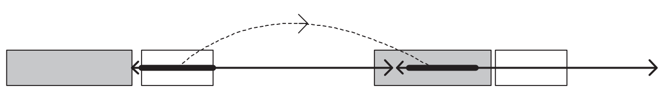
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9b 

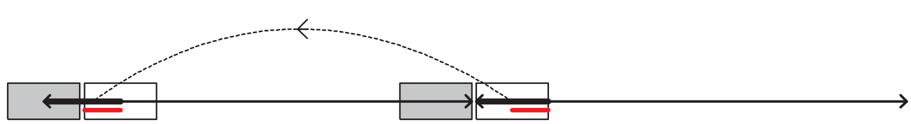
is not possible as it allows $\mathcal{S}$ to be left-extended.

9c 

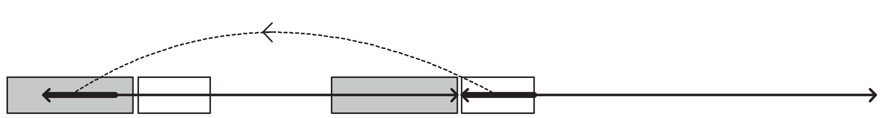
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9d 

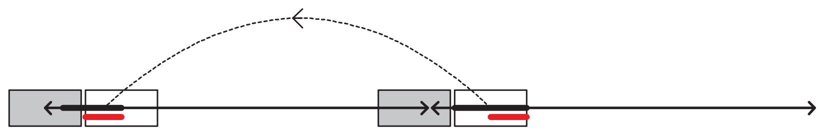
is not possible as it contradicts Lemma 1.

9e 

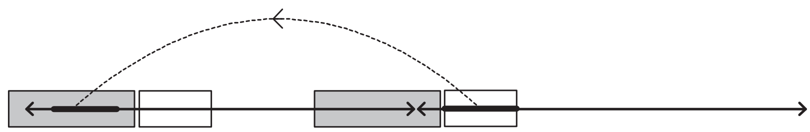
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9f 

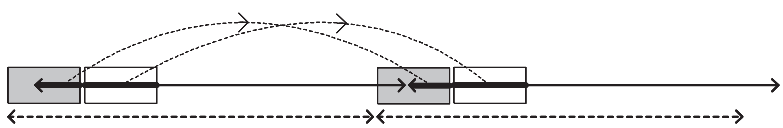
is not possible as it contradicts Lemma 2.

9g 

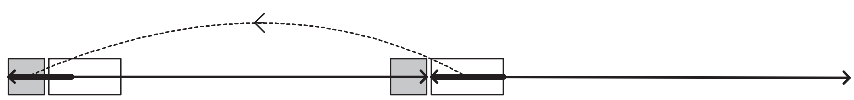
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9h 

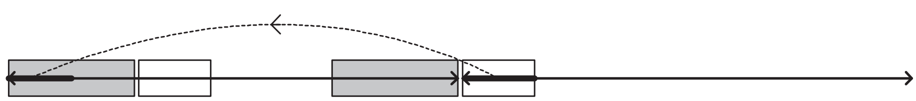
is not possible as it contradicts Lemma 2.

9i 

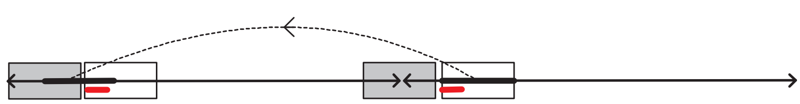
is not possible as it allows $\mathcal{S}$ to be left-extended.

9j 

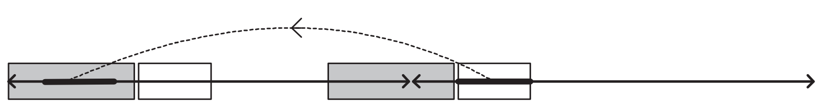
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

9k 

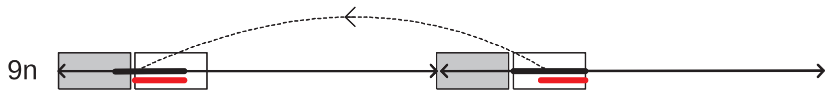
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

9l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9m 

is not possible as it contradicts Lemma 2.



is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

□

Corollary 1 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p},\boldsymbol{q},i,j]_\lambda$ an expansion, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $\mathcal{S}$ be a run in $\boldsymbol{y}$. Then either*

1. *$\mathcal{S}$ is an expansion of a run in $\boldsymbol{x}$; or*

2. *$j \le 2i$ and $\mathcal{S}$ is a run of power 2 derived from an $a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(a\boldsymbol{u}b\boldsymbol{u}a) = \boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} =$ $\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}\big]\boldsymbol{p}^{2i-j}\boldsymbol{q}$ with left-extension of size 0 (if $a\boldsymbol{u}b\boldsymbol{u}a$ is an initial segment of $\boldsymbol{x}$) or of size $GCS(\boldsymbol{p},\boldsymbol{q})$ and with right-extension of size $(i|\boldsymbol{p}| + GCP(\boldsymbol{p},\boldsymbol{q}))$; or*

3. *$\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \ne \varepsilon$, $\boldsymbol{p}_2 \ne \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$ and $\mathcal{S}$ is a run of power 2 derived from one of the near repetitions aa, ab, ba, and bb in $\boldsymbol{x}$ in the following way: $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ all have $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}$, $2 \le r < i$ (for aa, ab, ba) and $2 \le r < j$ (for bb), as a substring. $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1} = \boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p} = \hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\hat{\boldsymbol{p}}_1$ with left-extension of size $GCS(\boldsymbol{p}_1,\hat{\boldsymbol{p}}_2)$ and right-extension of size $GCP(\boldsymbol{p}_2,\hat{\boldsymbol{p}}_1)$; or*

4. *$\boldsymbol{q} = \boldsymbol{p}_1\boldsymbol{p}_2$, $\boldsymbol{p}_1 \ne \varepsilon$, $\boldsymbol{p}_2 \ne \varepsilon$, $\boldsymbol{p} = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{p}}_2\boldsymbol{p}_2$ and $\mathcal{S}$ is a run of power 2 derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} =$ $\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\boldsymbol{p}_2\boldsymbol{p}^{j-i-1}\boldsymbol{q}$ with left-extension of size $GCS(\boldsymbol{p}_1,\hat{\boldsymbol{p}}_2)$ and right-extension of size $GCP(\boldsymbol{p}_2,\hat{\boldsymbol{p}}_1)$; or*

5. *$\mathcal{S}$ is a run of power 2 derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition in $\boldsymbol{x}$ in the following way: $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} = \boldsymbol{p}^{j-i}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\big]$ with left-extension of size $GCS(\boldsymbol{p},\boldsymbol{q})$ and right-extension of size 0 (if $b\boldsymbol{u}a\boldsymbol{u}$ is an end segment of $\boldsymbol{x}$) or of size $(i|\boldsymbol{p}| + GCP(\boldsymbol{p},\boldsymbol{q}))$; or*

6. *$\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \ne \varepsilon$, $\boldsymbol{q}_2 \ne \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $i = 2r{+}1$ (for aa, ba) or $j = 2r{+}1$ (for ab, bb) for some $r \ge 2$, and $\mathcal{S}$ is a run of power 2 derived from one of the near repetitions aa, ab, ba, and bb in $\boldsymbol{x}$ in the following way: $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ all contain $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$ as a*

substring. $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1$ *with left-extension of size* $GCS(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ *and right-extension of size* $GCP(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$*; or*

7. $\boldsymbol{p} = \boldsymbol{q}_1\boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \varepsilon$, $\boldsymbol{q}_2 \neq \varepsilon$, $\boldsymbol{q} = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2\boldsymbol{q}_2$, $j = 2i{+}1$ *and* $\mathcal{S}$ *is a run of power 2 derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1 = \cdot\cdot\hat{\boldsymbol{q}}_2\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\big[\boldsymbol{q}_2\boldsymbol{p}^i\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}_1\big]\hat{\boldsymbol{q}}_1$ *with left-extension of size* $GCS(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2)$ *and right-extension of size* $GCP(\boldsymbol{q}_2, \hat{\boldsymbol{q}}_1)$.

Proof Just apply the previous theorem to the leading square of the run. $\square$

In the following we are going to discuss all possible ways near repetitions of type $a\boldsymbol{u}b\boldsymbol{u}$ can arise. We say that $\boldsymbol{y}[s..s{+}2k{+}1]$ is an $a\boldsymbol{u}b\boldsymbol{u}$ near repetition if $\boldsymbol{y}[s] = a$, $\boldsymbol{y}[s{+}k{+}1] = b$ and $\boldsymbol{y}[s{+}m] = \boldsymbol{y}[s{+}k{+}1{+}m]$ for any $1 \leq i \leq k$.

Theorem 3 *Let* $\boldsymbol{x}$, $\boldsymbol{y}$ *be two-pattern strings of scope* λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ *an extension, and* $\boldsymbol{y} = \sigma(\boldsymbol{x})$. *Let* $\mathcal{S} = \boldsymbol{y}[s..s{+}2k{+}1]$ *be a big* $a\boldsymbol{u}b\boldsymbol{u}$ *near repetition in* $\boldsymbol{y}$. *Then either*

1. $\boldsymbol{q} = \boldsymbol{p}_1 b\boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a\boldsymbol{p}_2 = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from aa, ab, ba or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, *and* $\sigma(bb)$ *all contain* $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}$ *as a substring* ($2 \leq r < i$ *for aa, ab, ba, and* $2 \leq r < j$ *for bb*). $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1} = \boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p} = \hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1$.

2. $\boldsymbol{q} = \boldsymbol{p}_1 b\boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a\boldsymbol{p}_2 = \boldsymbol{p}_1\hat{\boldsymbol{p}}_1$, *for some* $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a* $b\boldsymbol{u}aa\boldsymbol{u}b$ *near repetition in* $\boldsymbol{x}$ *the following way:* $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.

3. $\boldsymbol{q} = \boldsymbol{q}_1 b\boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a\boldsymbol{p}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a ba near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ *contains* $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ *as a substring.* $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]\big]b\boldsymbol{p}_2$.

4. $\boldsymbol{q} = \boldsymbol{q}_1 b\boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 a\boldsymbol{p}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a* $b\boldsymbol{u}a\boldsymbol{u}$ *near repetition* ($\boldsymbol{u} \neq \varepsilon$) *in* $\boldsymbol{x}$ *in the following way:* $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{q}$, $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u_1}}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u_1}}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u_1}}\boldsymbol{q}_1\big]b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\big]\big]b\boldsymbol{p}_2$.

5. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i\geq 6$ *(for aa, ba) or* $j\geq 6$ *(for ab, bb) and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ *all contain* $\boldsymbol{q}\boldsymbol{p}^{2r+2}$ *as a substring* ($2\leq r\leq \frac{i}{2}-1$ *for aa, ba,* $2\leq r\leq \frac{j}{2}-1$ *for ba, bb).*
$\boldsymbol{q}\boldsymbol{p}^{2r+2}=\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p}=\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2$.

6. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i<\frac{j+1}{2}$ *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}b$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}b)=$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}=$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}$.

7. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i\geq\frac{j+1}{2}$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a)=$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^{2i-j-1}\boldsymbol{q}=$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{p}^{2i-j-1}\boldsymbol{q}$.

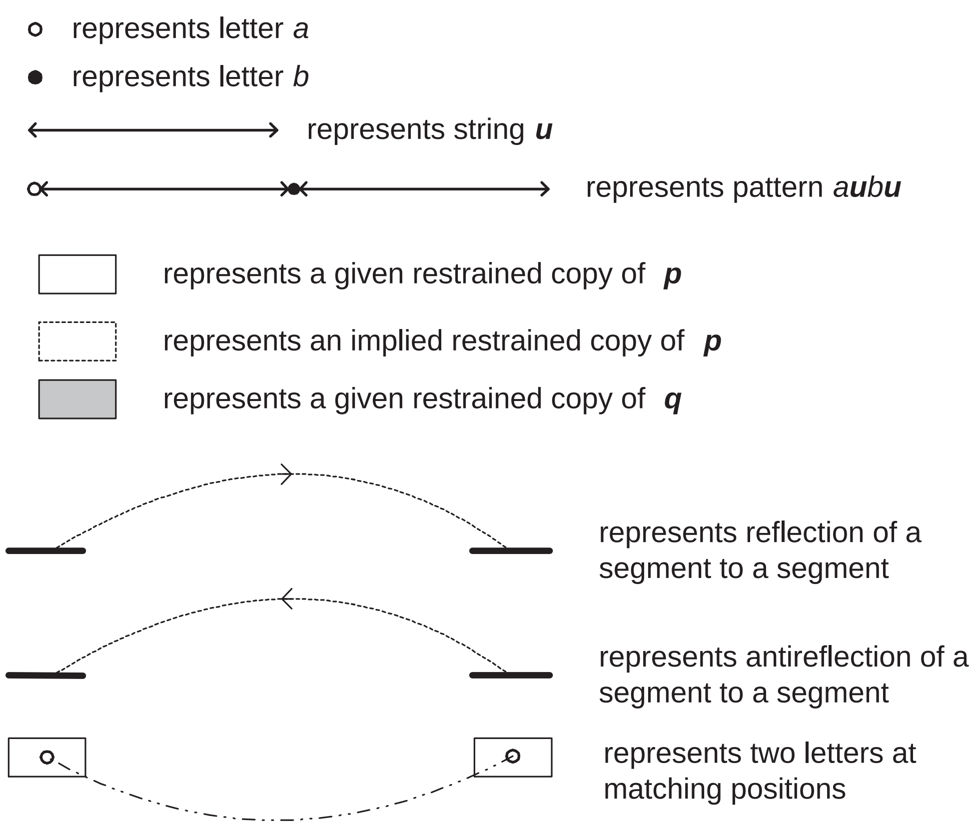
8. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $j=2i$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a)=$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{q}=$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]b\boldsymbol{q}_2\boldsymbol{q}$.

9. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2=\boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i=2r{+}1$ *(for aa, ba) or* $j=2r{+}1$ *(for ab, bb) for some* $2\leq r$, *and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *they all include* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$ *as a substring.* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}=$
$\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}=\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.

10. $\boldsymbol{p}=\boldsymbol{q}_1 b\boldsymbol{q}_2$, $\boldsymbol{q}=\hat{\boldsymbol{q}}_2 a\boldsymbol{q}_2=\boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j=2i{+}1$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:*
$\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a)=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}=\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}=$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.

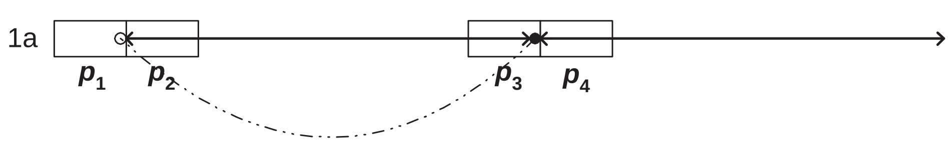
Proof

We shall conduct the proof in the same spirit as the proof of Theorem 2 by a "brute force" discussion of all possible ways the near repetition $a\boldsymbol{u}b\boldsymbol{u}$ can be placed in $\boldsymbol{y}$ with respect to restrained copies of $\boldsymbol{p}$ and $\boldsymbol{q}$. We shall employ a graphical language similar to the one used in the proof of Theorem 2. Also similarly we define the reflection $\mathcal{R}_{\mathcal{S}}:\boldsymbol{y}[s{+}1..s{+}k]^*\rightarrow\boldsymbol{y}[s{+}k{+}2..s{+}2k{+}1]^*$ and its inverse, the antireflection.

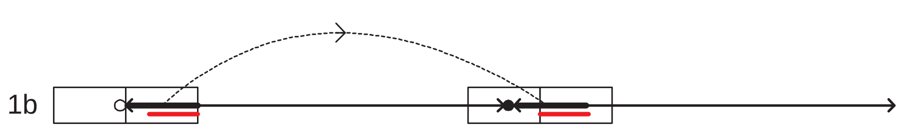


Recall that for the near repetition $\mathcal{S}$ to be big, $k > 3\lambda$.

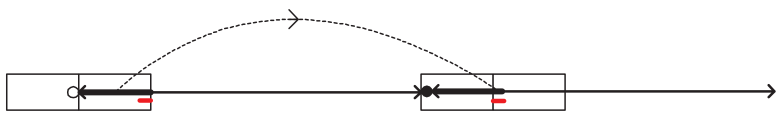
Case (1) – $\boldsymbol{pp}$-$\boldsymbol{pp}$ (i.e. the point $\boldsymbol{y}[s] = a$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$, and the point $\boldsymbol{y}[s+k+1] = b$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$):



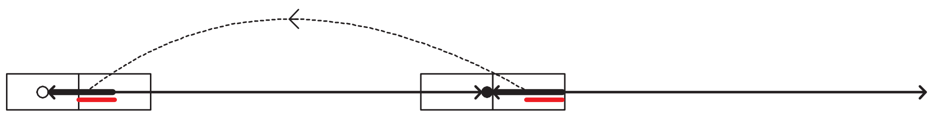
is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is a and the last letter of $\boldsymbol{p}_3$ is b.



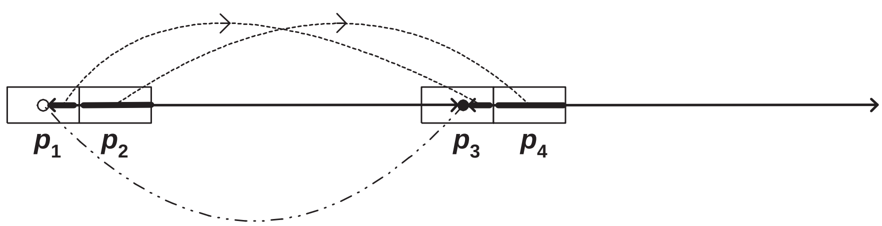
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1c 

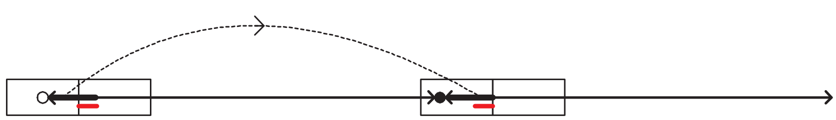
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1d 

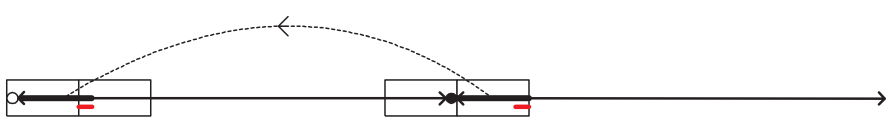
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1e 

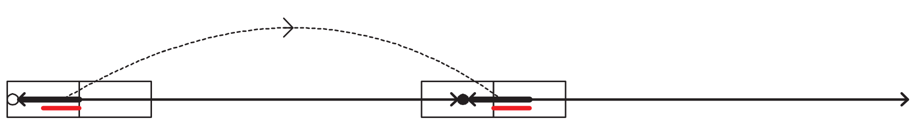
is not possible as it implies that the n-th letter of $\boldsymbol{p}_1$ is a and the n-th letter of $\boldsymbol{p}_3$ is b, where n is the position of $\boldsymbol{y}[s]$ in $\boldsymbol{p}_1$ and the position of $\boldsymbol{y}[s+k+1]$ in $\boldsymbol{p}_3$.

1f 

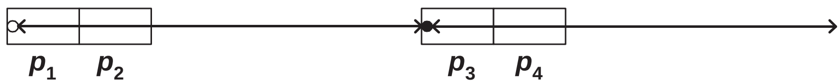
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1g 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

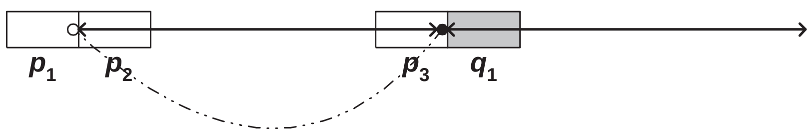
1h 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

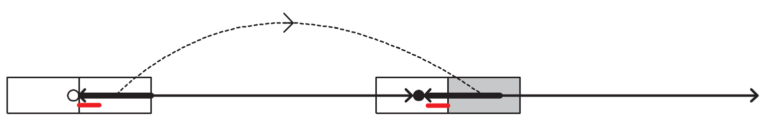
1i 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is a and the first letter of $\boldsymbol{p}_3$ is b.

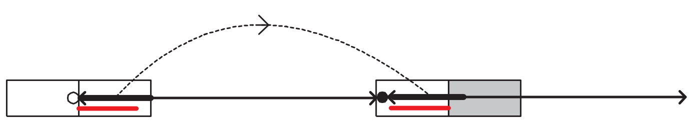
Case (2) – $\boldsymbol{pp}$-$\boldsymbol{pq}$

2a 

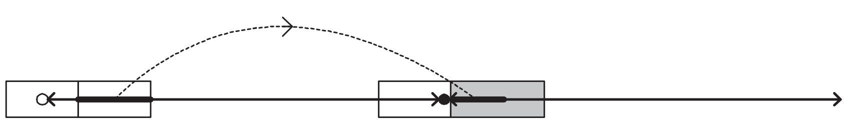
is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is a and the last letter of $\boldsymbol{p}_3$ is b.

2b 

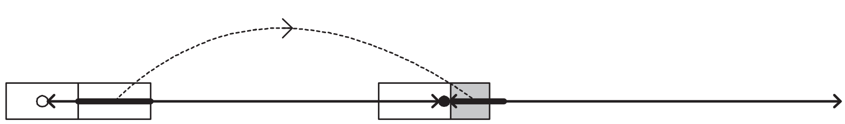
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2c 

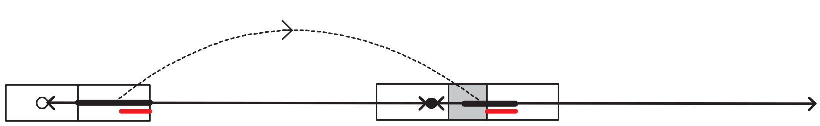
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2d 

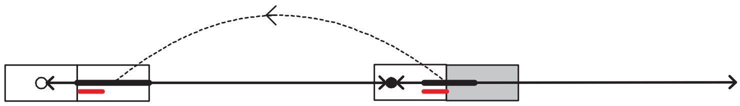
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

2e 

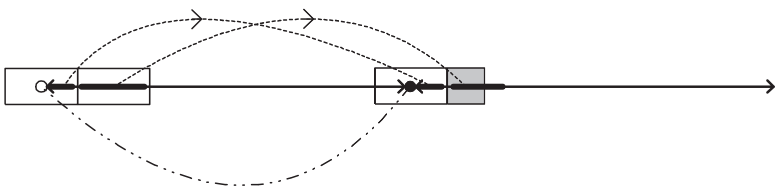
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

2f 

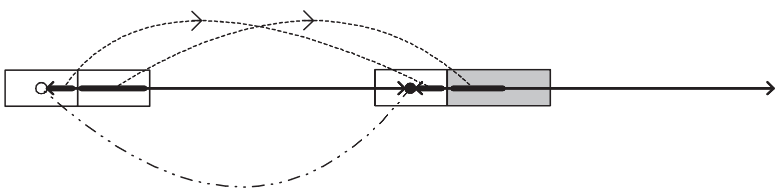
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2g


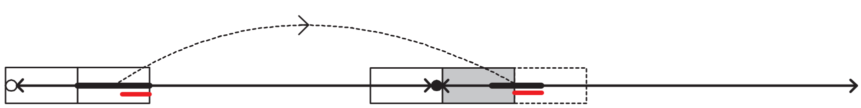
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2h


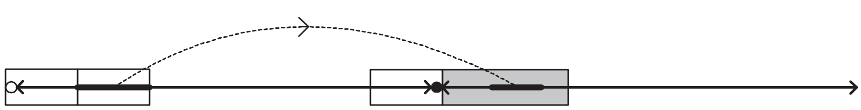
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$ (it also implies that the n-the letter of $\boldsymbol{p}$ is both a and b).

2i


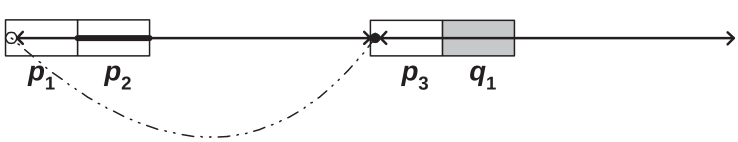
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$. (it also implies that the n-the letter of $\boldsymbol{p}$ is both a and b).

2j


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

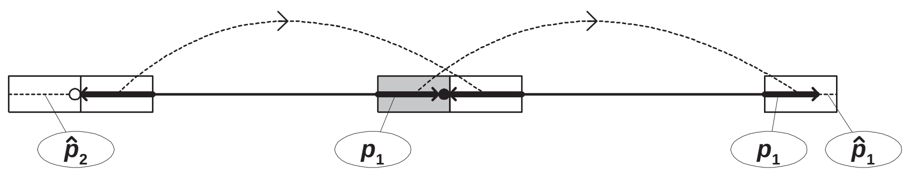
2k


is not possible as it contradicts Lemma 1.

2l


is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is a while the first letter of $\boldsymbol{p}_3$ is b.

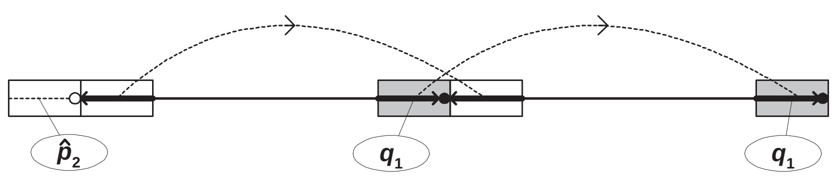
Case (3) – $\boldsymbol{pp}$-$\boldsymbol{qp}$

3a


is possible if $\boldsymbol{q} = \boldsymbol{p}_1 b$ for some $\boldsymbol{p}_1$ that is a prefix of $\boldsymbol{p}$ and a is a suffix of $\boldsymbol{p}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$ (and so it is derived from aa, ab, ba or bb near repetitions in $\boldsymbol{x}$ and thus **case 1 of the theorem holds for** $\boldsymbol{p}_2 = \varepsilon$); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{i+1}\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ (and so it is derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition and **case 2 of the theorem holds for** $\boldsymbol{p}_2 = \varepsilon$).

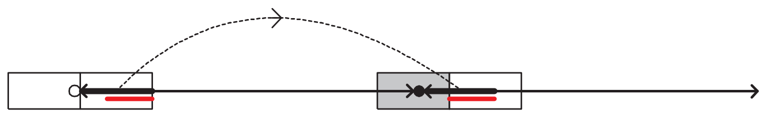
3b


is possible if b is a suffix of $\boldsymbol{q}$ and a is a suffix of $\boldsymbol{p}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ (and so it is derived from a ba near repetition in $\boldsymbol{x}$ and so **case 3 of the theorem holds for** $\boldsymbol{p}_2 = \varepsilon$); or

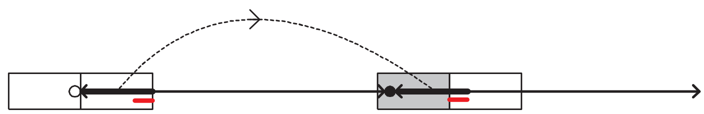
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ (and so it is derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition with $\boldsymbol{u} \neq \varepsilon$ and so **case 4 of the theorem holds for** $\boldsymbol{p}_2 = \varepsilon$).

3c



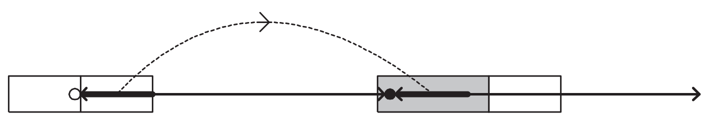
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3d



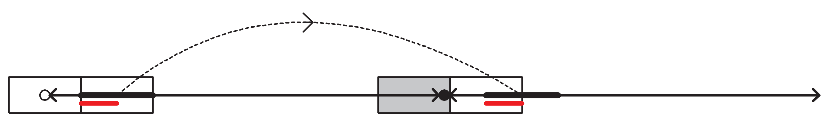
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3e



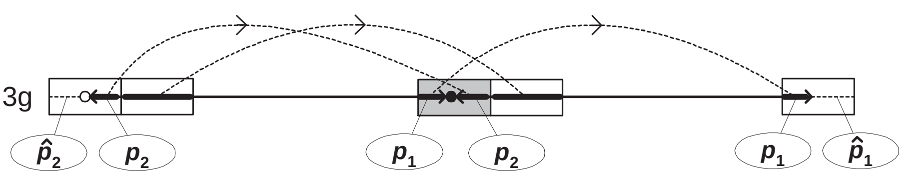
is not possible as it contradicts Lemma 1.

3f



is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3g

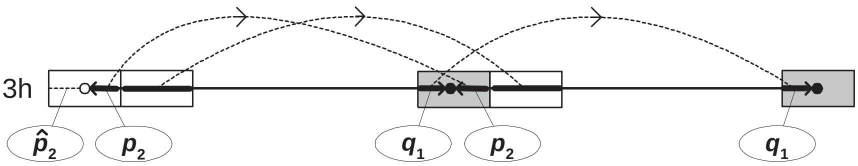


is possible if $\boldsymbol{q} = \boldsymbol{p}_1 b \boldsymbol{p}_2$ for some $\boldsymbol{p}_1$ that is a prefix of $\boldsymbol{p}$ and some $\boldsymbol{p}_2$ so that $a\boldsymbol{p}_2$ is a suffix of $\boldsymbol{p}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$ (and so it is derived from aa, ab, ba or bb near repetitions in $\boldsymbol{x}$ and so **case 1 of the theorem holds**); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^{i+1}\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ (and so it is derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition and so **case 2 of the theorem holds**).

3h

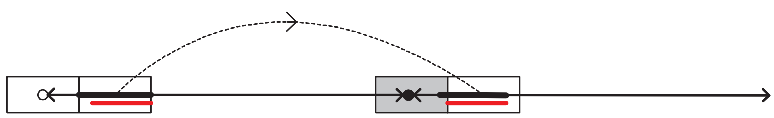


is possible if for some $\boldsymbol{p}_2$, $b\boldsymbol{p}_2$ is a suffix of $\boldsymbol{q}$ and $a\boldsymbol{p}_2$ a suffix of $\boldsymbol{p}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^{i}\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^{j}\boldsymbol{q}\boldsymbol{p}^{i}\boldsymbol{q}$ (and so it is derived from a ba near repetition in $\boldsymbol{x}$ and so **case 3 of the theorem holds**); or

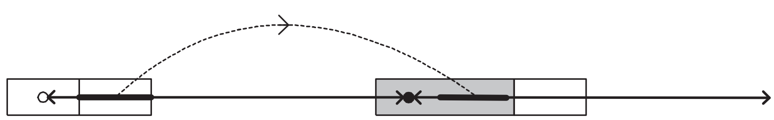
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{i}\boldsymbol{q}\overline{\boldsymbol{u}}$ that itself is a substring of $\boldsymbol{p}^{j}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{i}\boldsymbol{q}\overline{\boldsymbol{u}}$ (and so it is derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition with $\boldsymbol{u} \neq \varepsilon$ and so **case 4 of the theorem holds**).

3i



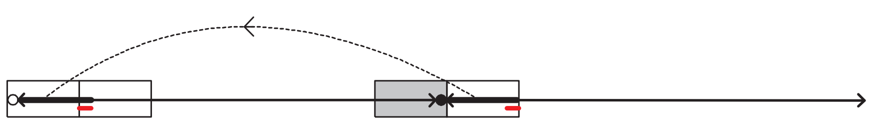
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3j



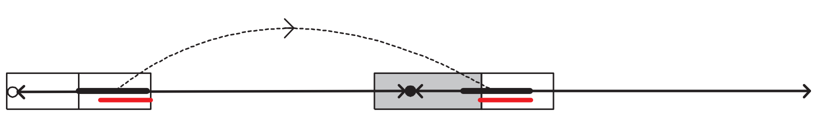
is not possible as it contradicts Lemma 1.

3k

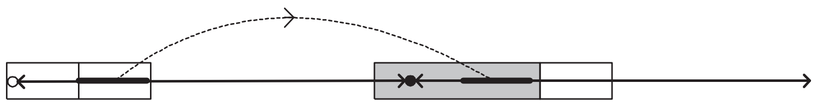


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

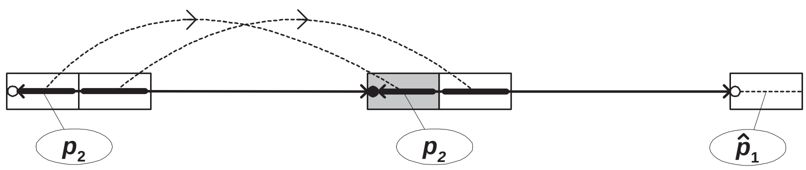
3l



is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3m 

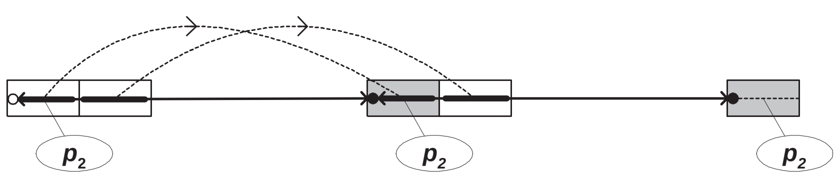
is not possible as it contradicts Lemma 1.

3n 

is possible if $\boldsymbol{q} = b\boldsymbol{p}_2$ and $\boldsymbol{p} = b\boldsymbol{p}_2$ for some $\boldsymbol{p}_2$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$ (and so it is derived from aa, ab, ba or bb near repetitions in $\boldsymbol{x}$ and so **case 1 of theorem holds for** $\boldsymbol{p}_1 = \hat{\boldsymbol{p}}_2 = \varepsilon$); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{i+1}\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ (and so it is derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition and so **case 2 of the theorem holds for** $\boldsymbol{p}_1 = \hat{\boldsymbol{p}}_2 = \varepsilon$).

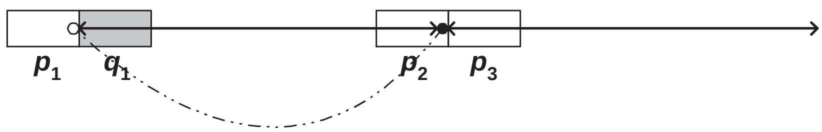
3o 

is possible if for some $\boldsymbol{p}_2$, $\boldsymbol{q} = a\boldsymbol{p}_2$ and $\boldsymbol{p} = b\boldsymbol{p}_2$. Either

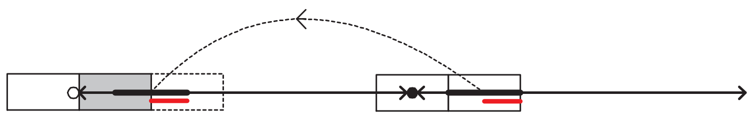
1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ (and so it is derived from a ba near repetition in $\boldsymbol{x}$ and so **case 3 of the theorem holds for** $\boldsymbol{q}_1 = \hat{\boldsymbol{p}}_2 = \varepsilon$); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ that itself is a substring of $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}$ (and so it is derived from a $b\boldsymbol{u}a\boldsymbol{u}$ near repetition with $\boldsymbol{u} \neq \varepsilon$ and so **case 4 of the theorem holds for** $\boldsymbol{q}_1 = \hat{\boldsymbol{p}}_2 = \varepsilon$).

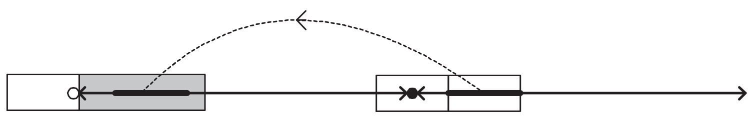
Case (4) – $\boldsymbol{pq}$-$\boldsymbol{pp}$

4a


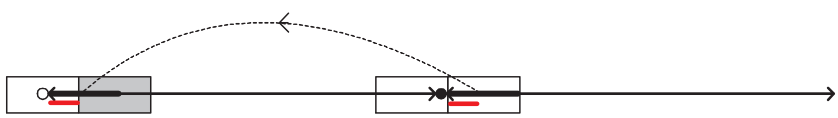
is not possible as it implies that a is the last letter of $\boldsymbol{p}_1$ and b is the last letter of $\boldsymbol{p}_2$.

4b


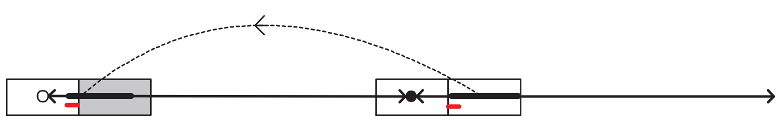
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4c


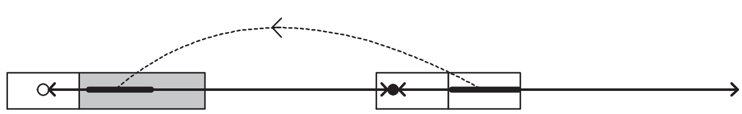
is not possible as it contradicts Lemma 2.

4d


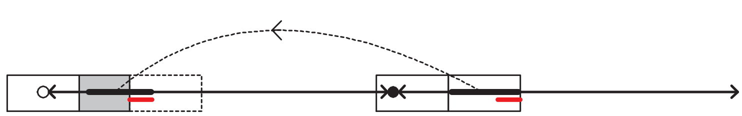
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4e


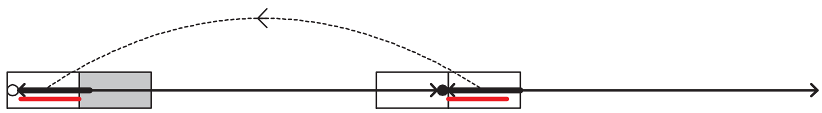
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4f


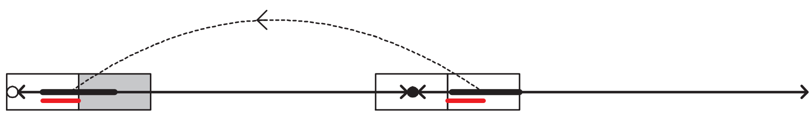
is not possible as it contradicts Lemma 2.

4g


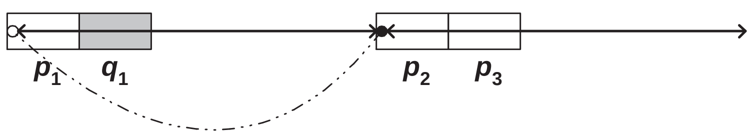
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4h 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

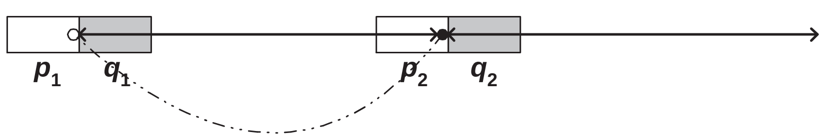
4i 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

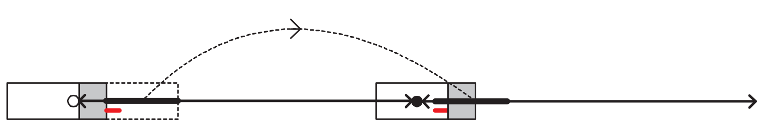
4j 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is a and the first letter of $\boldsymbol{p}_2$ is b.

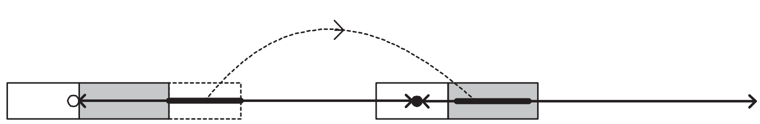
Case (5) – $\boldsymbol{pq}$-$\boldsymbol{pq}$

5a 

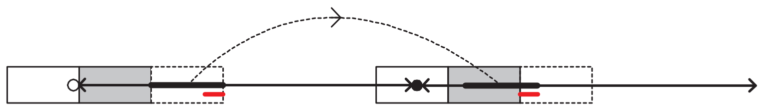
is not possible as it implies that a is the last letter of $\boldsymbol{p}_1$ and b is the last letter of $\boldsymbol{p}_3$.

5b 

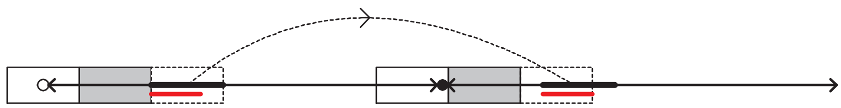
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5c 

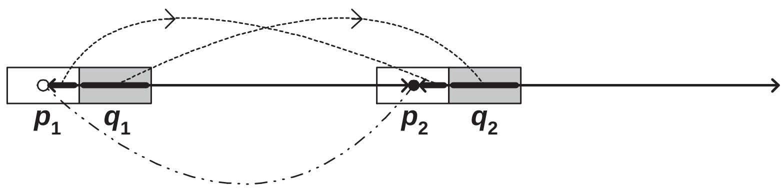
is not possible as it contradicts Lemma 1.

5d 

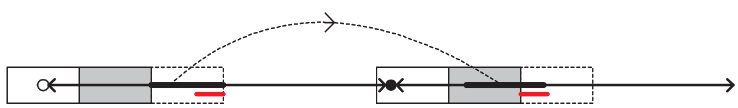
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5e 

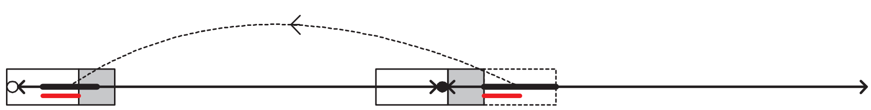
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5f 

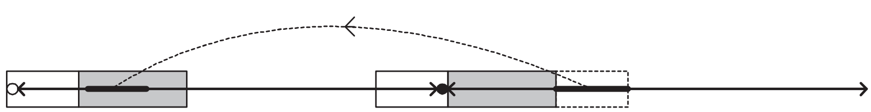
is not possible as it implies that the n-th letter of $\boldsymbol{p}_1$ is a, while the n-th letter of $\boldsymbol{p}_2$ is b.

5g 

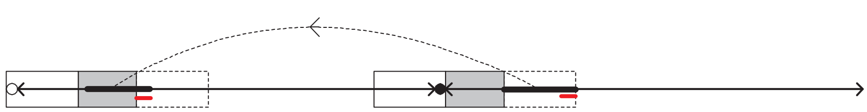
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5h 

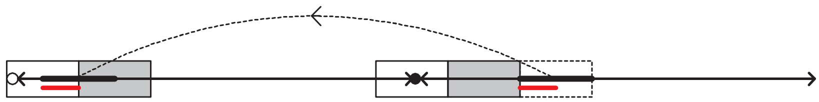
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5i 

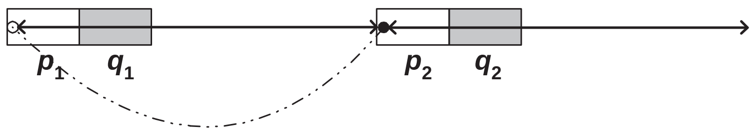
is not possible as it contradicts Lemma 2.

5j 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

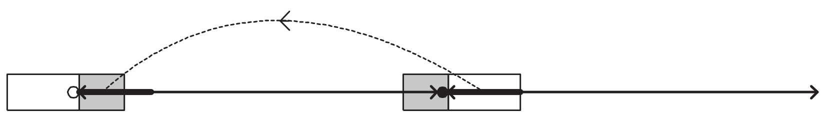
5k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

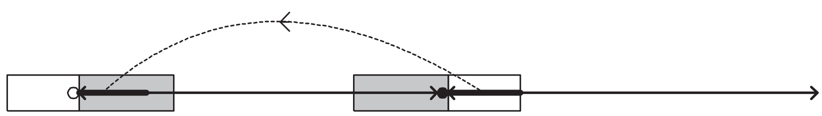
5l 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is a and the first letter of $\boldsymbol{p}_2$ is b.

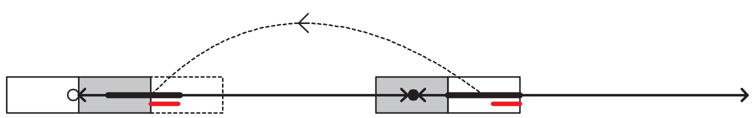
Case (6) – $\boldsymbol{pq}$-$\boldsymbol{qp}$

6a 

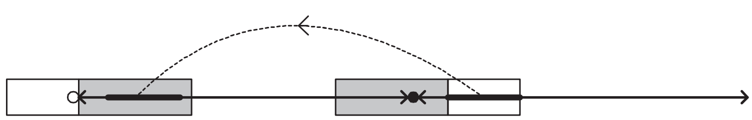
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6b 

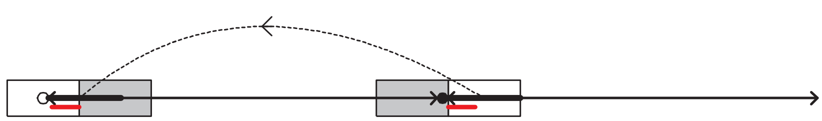
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6c 

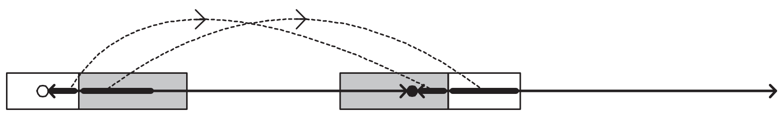
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6d 

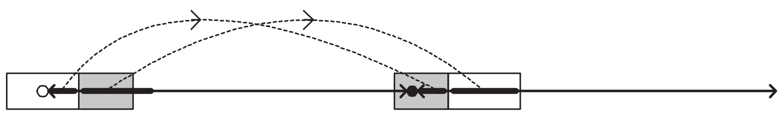
is not possible as it contradicts Lemma 2.

6e 

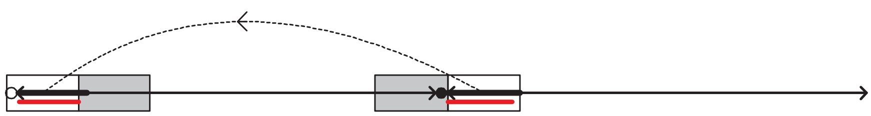
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6f 

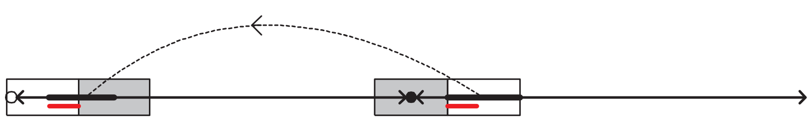
is not possible as it indicates that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6g 

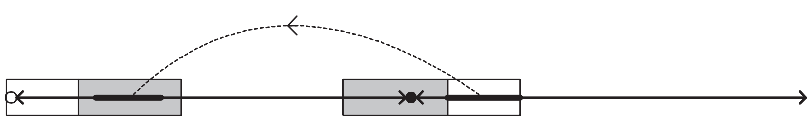
is not possible as it indicates that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6h 

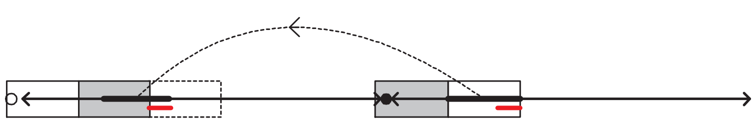
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6i 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

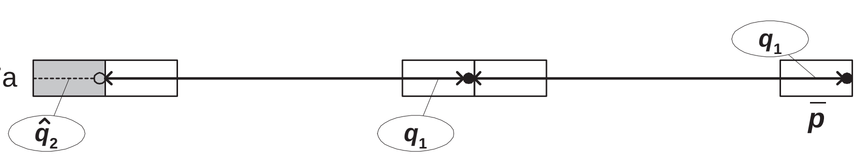
6j 

is not possible as it contradicts Lemma 2.

6k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

Case (7) – $\boldsymbol{qp}$-$\boldsymbol{pp}$

7a 

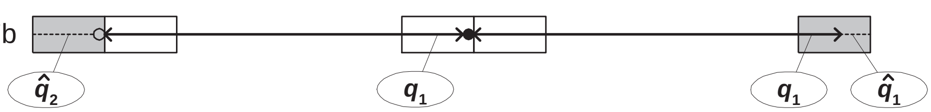
is possible if b is a suffix of $\boldsymbol{p}$ and a a suffix of $\boldsymbol{q}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^{2r+2}$ for some $r \geq 2$, and hence it is derived from a near repetition aa, ab, ba, or bb in $\boldsymbol{x}$ (and so **case 5 of the theorem holds for** $\boldsymbol{q}_2 = \varepsilon$); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $i < \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i sq\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ (and so **case 6 of theorem holds for** $\boldsymbol{q}_2 = \varepsilon$); or

3. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{p}$ and $i \geq \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 7 of the theorem holds for** $\boldsymbol{q}_2 = \varepsilon$); or

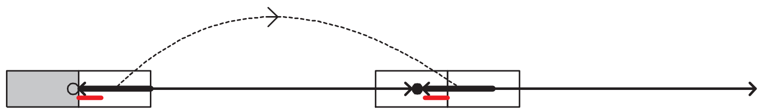
4. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $j = 2i$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 8 of the theorem holds for** $\boldsymbol{q}_2 = \varepsilon$).

7b 

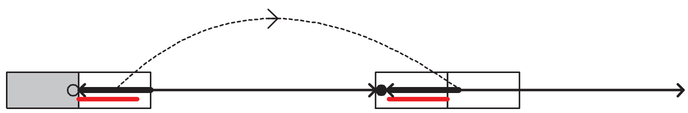
is possible if b is a suffix of $\boldsymbol{p}$ and a a suffix of $\boldsymbol{q}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ either it is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ and $i = 2r{+}1$, and thus derived from aa or ba near repetitions in $\boldsymbol{x}$, or $\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}$ and $j = 2r{+}1$, and thus derived from ab or bb near repetitions in $\boldsymbol{x}$ (and so **case 9 of the theorem holds for** $\boldsymbol{q}_2 = \varepsilon$); or

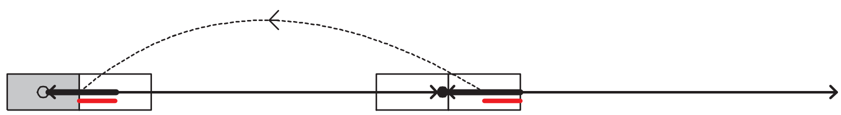
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, $j = 2i{+}1$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and so **case 10 of the theorem holds for** $\boldsymbol{q}_2 = \varepsilon$).

7c 

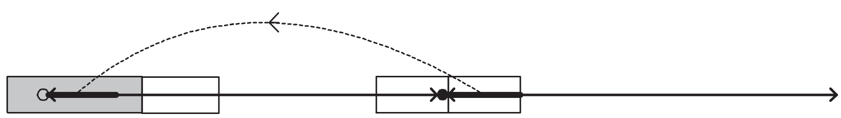
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7d 

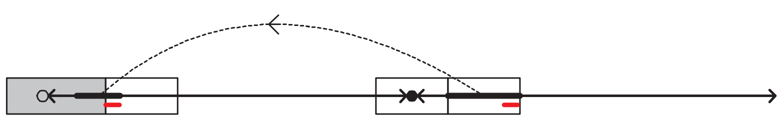
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7e 

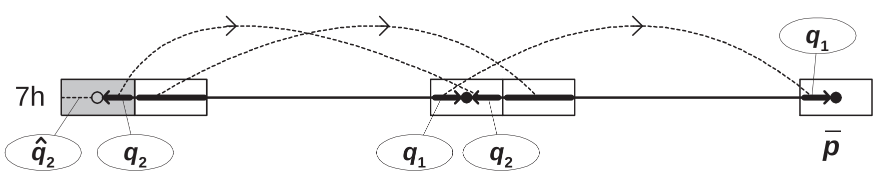
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7f 

is not possible as it contradicts Lemma 2.

7g 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7h 

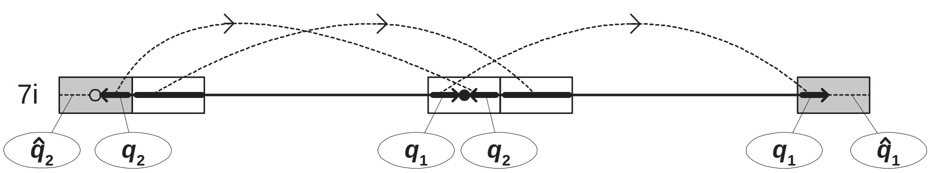
is possible if for some $\boldsymbol{q}_2$, $a\boldsymbol{q}_2$ is a suffix of $\boldsymbol{q}$ and $b\boldsymbol{q}_2$ is a suffix of $\boldsymbol{p}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^{2r+2}$ for some $r \geq 2$, and hence it is derived from a near repetition aa, ab, ba, or bb in $\boldsymbol{x}$ (and so **case 5 of the theorem holds**); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $i < \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i s\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ (and so **case 6 of theorem holds**); or

3. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{p}$ and $i \geq \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 7 of the theorem holds**); or

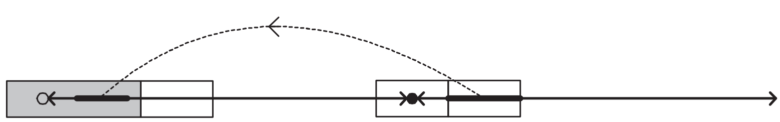
4. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $j = 2i$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}$ that is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 8 of the theorem holds**).

7i


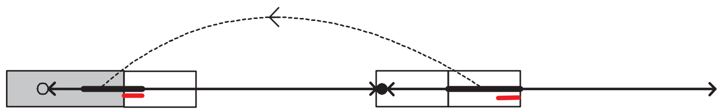
is possible if $\boldsymbol{p} = \boldsymbol{q}_1 b\boldsymbol{q}_2$ for some $\boldsymbol{q}_1$ a prefix of $\boldsymbol{q}$ and some $\boldsymbol{q}_2$ so that $a\boldsymbol{q}_2$ is a suffix of $\boldsymbol{q}$. Either

1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ either it is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ and $i = 2r{+}1$, and thus derived from aa or ba near repetitions in $\boldsymbol{x}$, or $\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}$ and $j = 2r{+}1$, and thus derived from ab or bb near repetitions in $\boldsymbol{x}$ (and so **case 9 of the theorem holds**); or

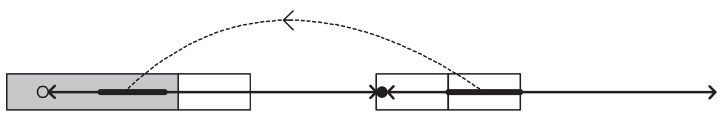
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, $j = 2i{+}1$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and so **case 10 of the theorem holds**).

7j


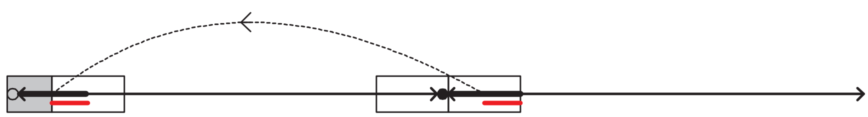
is not possible as it contradicts Lemma 2.

7k 

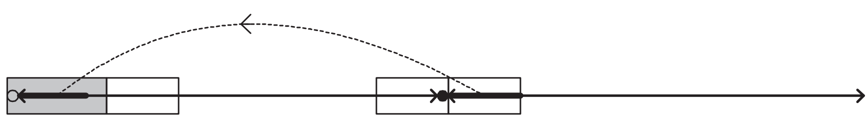
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7l 

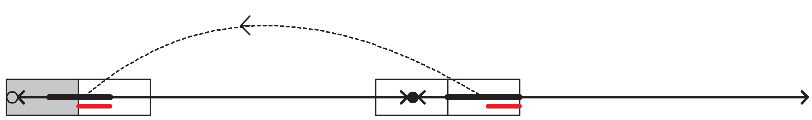
is not possible as it contradicts Lemma 2.

7m 

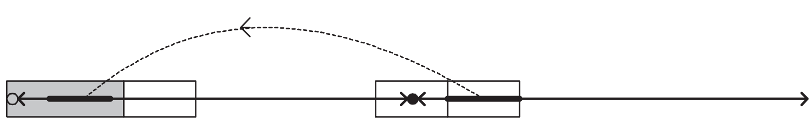
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7n 

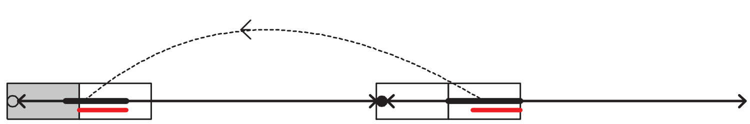
is not possible as it contradicts Lemma 2.

7o 

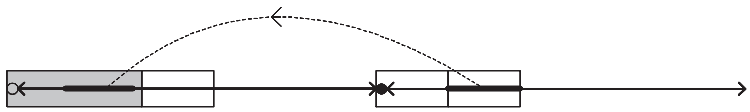
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7p 

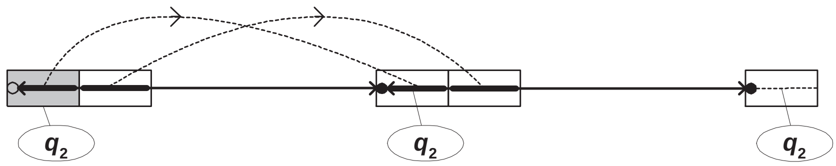
is not possible as it contradicts Lemma 2.

7q 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7r 

is not possible as it contradicts Lemma 2.

7s 

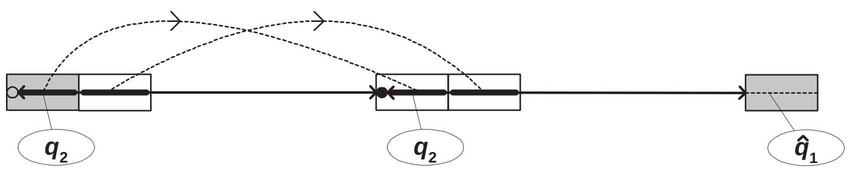
is possible if $\boldsymbol{q} = a\boldsymbol{q}_2$ and $\boldsymbol{p} = b\boldsymbol{q}_2$ for some $\boldsymbol{q}_2$. Either

1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{qp}^{2r+2}$ for some $r \geq 2$, and hence it is derived from a near repetition aa, ab, ba, or bb in $\boldsymbol{x}$ (and so **case 5 of the theorem holds for $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$**); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $i < \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{pp}^i sq\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}b$ near repetition in $\boldsymbol{x}$ (and so **case 6 of theorem holds for $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$**); or

3. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{p}$ and $i \geq \frac{j+1}{2}$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{pp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i}$ that is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 7 of the theorem holds for $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$**); or

4. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and $\overline{\boldsymbol{p}}$ is followed by a restrained copy of $\boldsymbol{q}$ and $j = 2i$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{pp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}$ that is a substring of $\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and hence **case 8 of the theorem holds for $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$**).

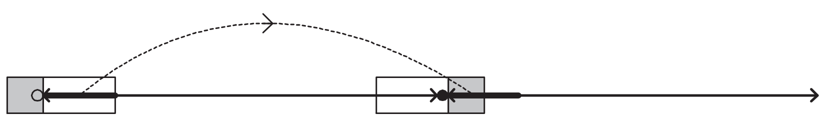
7t 

is possible if $\boldsymbol{q} = a\boldsymbol{q}_2$ and $\boldsymbol{p} = b\boldsymbol{q}_2$ for some $\boldsymbol{q}_2$. Either

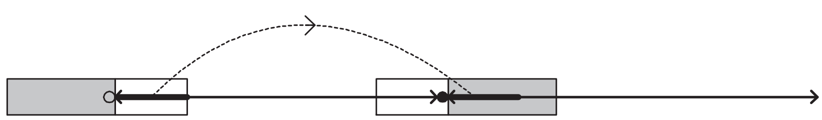
1. there is no restrained copy of $\boldsymbol{q}$ as a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, and then $\boldsymbol{y}[s..s{+}2k{+}1]$ either it is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ and $i = 2r{+}1$, and thus derived from aa or ba near repetitions in $\boldsymbol{x}$, or $\boldsymbol{q}\boldsymbol{p}^j\boldsymbol{q}$ and $j = 2r{+}1$, and thus derived from ab or bb near repetitions in $\boldsymbol{x}$ (and so **case 9 of the theorem holds for** $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$); or

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$, $j = 2i{+}1$. Then $\boldsymbol{y}[s..s{+}2k{+}1]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ that is the expansion of $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ (and so **case 10 of the theorem holds for** $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1 = \varepsilon$).

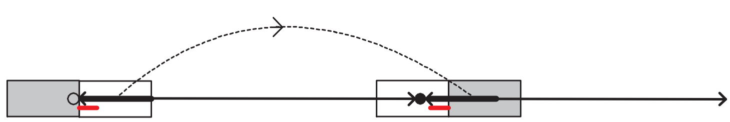
Case (8) – $\boldsymbol{qp}$-$\boldsymbol{pq}$

8a 

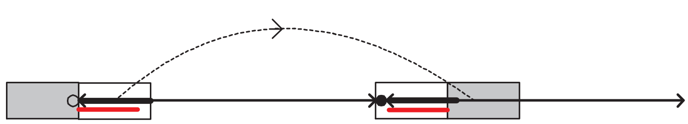
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

8b 

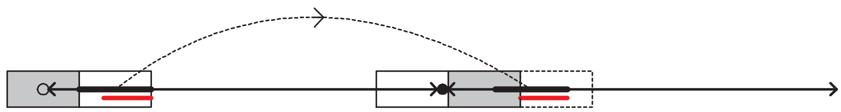
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

8c 

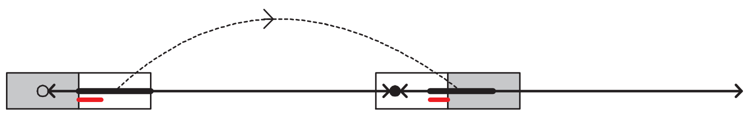
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8d 

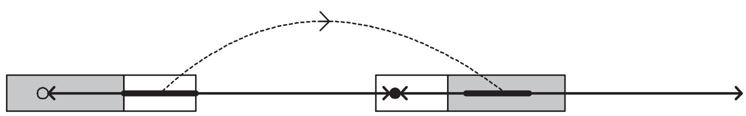
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8e 

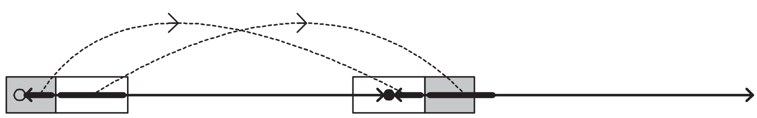
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8f 

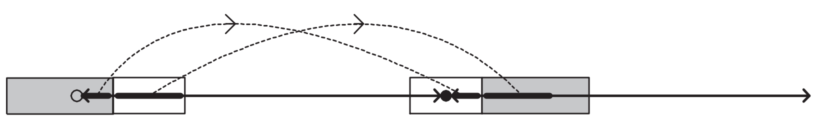
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8g 

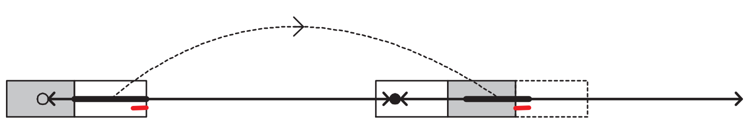
is not possible as it contradicts Lemma 2.

8h 

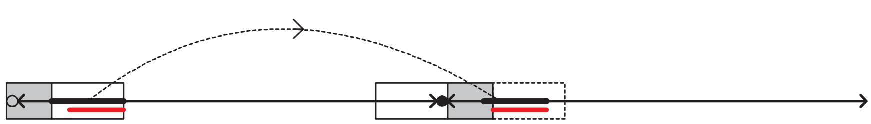
is not possible as it indicates that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

8i 

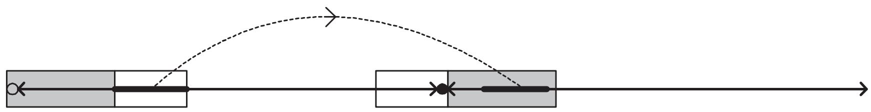
is not possible as it indicates that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

8j 

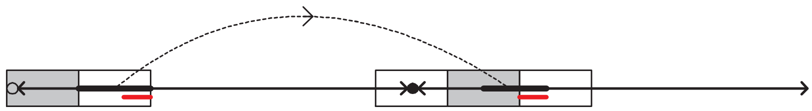
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8k 

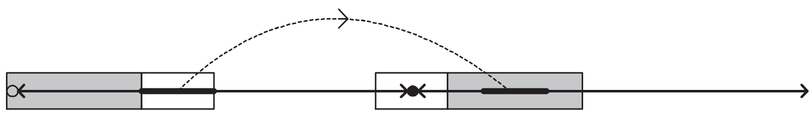
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8l 

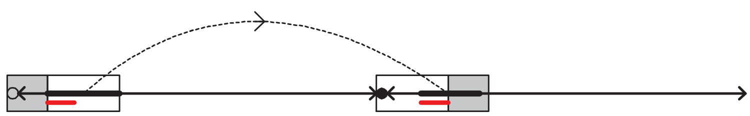
is not possible as it contradicts Lemma 1.

8m 

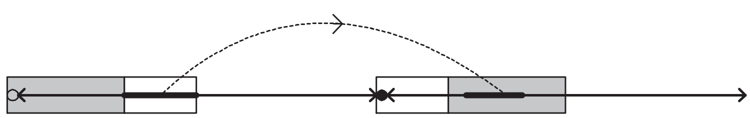
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8n 

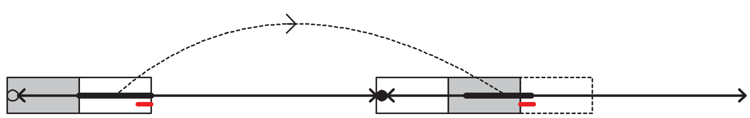
is not possible as it contradicts Lemma 1.

8o 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

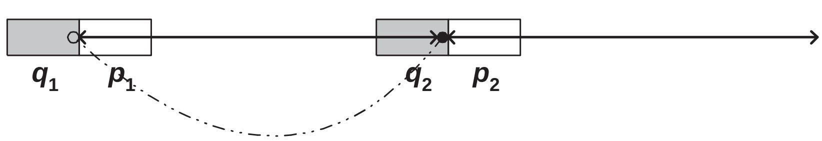
8p 

is not possible as it contradicts Lemma 1.

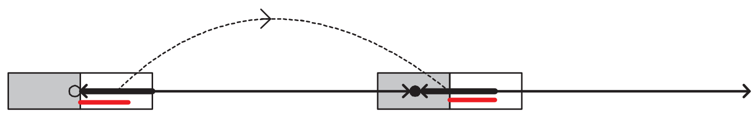
8q 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

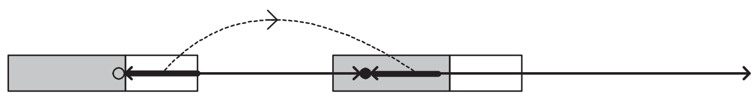
Case (9) – $\boldsymbol{qp}$-$\boldsymbol{qp}$

9a 

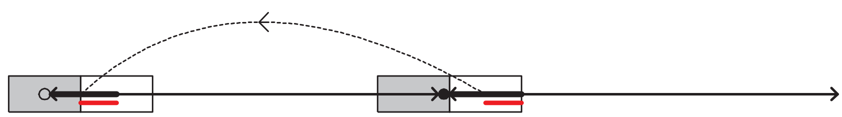
is not possible as it implies that the last letter of $\boldsymbol{q}_1$ is a and the last letter of $\boldsymbol{q}_2$ is b.

9b 

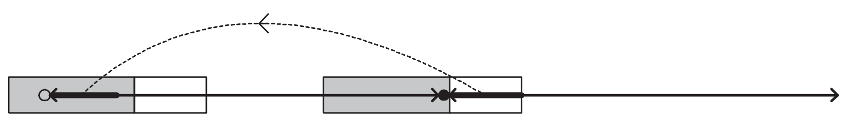
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9c 

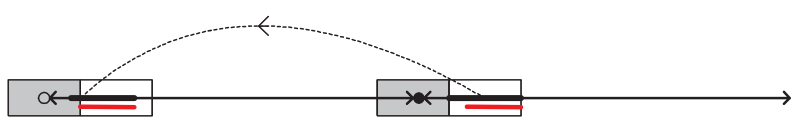
is not possible as it contradicts Lemma 1.

9d 

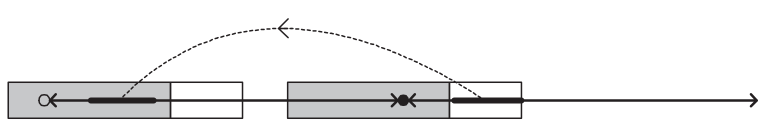
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9e 

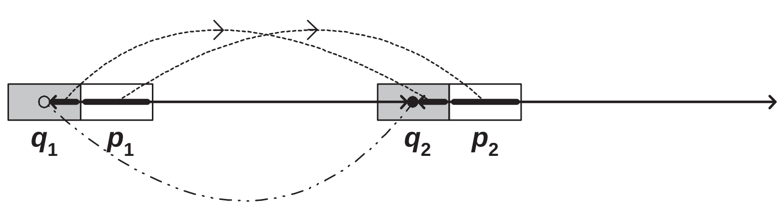
is not possible as it contradicts Lemma 2.

9f 

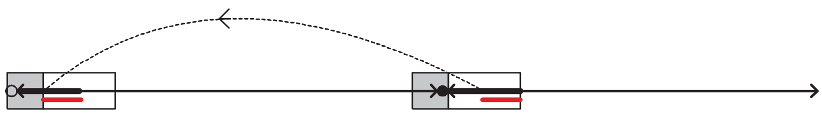
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9g 

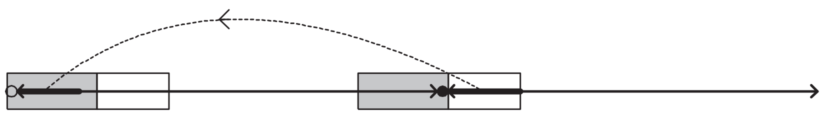
is not possible as it contradicts Lemma 2.

9h 

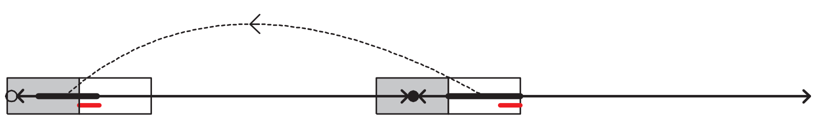
is not possible as it implies that the n-th letter of $\boldsymbol{q}_1$ is a while the n-th letter of $\boldsymbol{q}_2$ is b, where n is the position of $\boldsymbol{y}[s]$ in $\boldsymbol{q}_1$ and the position of $\boldsymbol{y}[s{+}k{+}1]$ is $\boldsymbol{q}_2$.

9i 

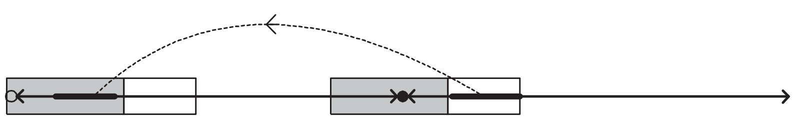
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9j 

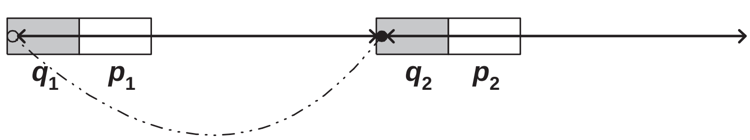
is not possible as it contradicts Lemma 2.

9k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9l 

is not possible as it contradicts Lemma 2.

9m 

is not possible as it implies that the first letter of $\boldsymbol{q}_1$ is a and the first letter of $\boldsymbol{q}_2$ is b.

□

In the following we are going to discuss all possible ways near repetitions of type $\boldsymbol{b}\boldsymbol{u}\boldsymbol{a}\boldsymbol{u}$ can arise. We say that $\boldsymbol{y}[s..s{+}2k{+}1]$ is an $\boldsymbol{b}\boldsymbol{u}\boldsymbol{a}\boldsymbol{u}$ near repetition if $\boldsymbol{y}[s] = b$, $\boldsymbol{y}[s{+}k{+}1] = a$ and $\boldsymbol{y}[s{+}m] = \boldsymbol{y}[s{+}k{+}1{+}m]$ for any $1 \le i \le k$.

Theorem 4 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ an extension, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $\mathcal{S} = \boldsymbol{y}[s..s{+}2k{+}1]$ be a big $b\boldsymbol{u}a\boldsymbol{u}$ near repetition in $\boldsymbol{y}$. Then either*

1. $\boldsymbol{q} = \boldsymbol{p}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 \hat{\boldsymbol{p}}_1$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from aa, ab, ba or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, *and* $\sigma(bb)$ *all contain* $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1}$ *as a substring* ($0 \leq r < i$ *for aa, ab, ba, and* $0 \leq r < j$ *for bb*). $\boldsymbol{p}^{r+1}\boldsymbol{q}\boldsymbol{p}^{r+1} = \boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p} = \hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^r\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1$.

2. $\boldsymbol{q} = \boldsymbol{p}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 \hat{\boldsymbol{p}}_1$, *for some* $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a* $b\boldsymbol{u}aa\boldsymbol{u}b$ *near repetition in* $\boldsymbol{x}$ *the following way:* $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) =$
$\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
$\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.

3. $\boldsymbol{q} = \boldsymbol{q}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a ba near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(ba) = \boldsymbol{p}^j\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ *contains* $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}$ *as a substring.* $\boldsymbol{p}^{i+1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} = \boldsymbol{p}^i\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q} =$
$\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}_1\big]\big]a\boldsymbol{p}_2$.

4. $\boldsymbol{q} = \boldsymbol{q}_1 a \boldsymbol{p}_2$, $\boldsymbol{p} = \hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, $\hat{\boldsymbol{p}}_2$, *and* $\mathcal{S}$ *is derived from a* $b\boldsymbol{u}a\boldsymbol{u}$ *near repetition* ($\boldsymbol{u} \neq \varepsilon$) *in* $\boldsymbol{x}$ *in the following way:* $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{q}$, $\sigma(b\boldsymbol{u}a\boldsymbol{u}) = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}} = \boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\boldsymbol{q} =$
$\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}_1}\boldsymbol{q}_1\big]a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}_1\big]\big]a\boldsymbol{p}_2$.

5. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i \geq 6$ (*for aa, ba*) *or* $j \geq 6$ (*for ab, bb*) *and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ *all contain* $\boldsymbol{q}\boldsymbol{p}^{2r+2}$ *as a substring* ($2 \leq r < \frac{i}{2}{-}1$ *for aa, ba,* $2 \leq r < \frac{j}{2}{-}1$ *for ba, bb*).
$\boldsymbol{q}\boldsymbol{p}^{2r+2} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{p} = \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2$.

6. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i < \frac{j+1}{2}$ *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}b$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}b) =$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}$.

7. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $i \geq \frac{j+1}{2}$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) =$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^{2i-j-1}\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{p}^{2i-j-1}\boldsymbol{q}$.

8. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_2$, $j = 2i$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) =$ $\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{q} =$ $\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]a\boldsymbol{q}_2\boldsymbol{q}$.

9. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $i = 2r{+}1$ *(for aa, ba) or* $j = 2r{+}1$ *(for ab, bb) for some* $1 \le r$, *and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *they all include* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$ *as a substring.* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.

10. $\boldsymbol{p} = \boldsymbol{q}_1 a \boldsymbol{q}_2$, $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1\hat{\boldsymbol{q}}_1$, *for some* $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\hat{\boldsymbol{q}}_1$, $\hat{\boldsymbol{q}}_2$, $j = 2i{+}1$, *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$ $\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]\big]\hat{\boldsymbol{q}}_1$.

Proof Virtually identical to the proof of the previous theorem. $\square$

In the following we are going to discuss all possible ways near repetitions of type $b\boldsymbol{u}aa\boldsymbol{u}b$ can arise. We say that $\boldsymbol{y}[s..s{+}2k{+}3]$ is an $b\boldsymbol{u}aa\boldsymbol{u}b$ near repetition if $\boldsymbol{y}[s] = \boldsymbol{y}[s{+}2k{+}3] = b$, $\boldsymbol{y}[s{+}k{+}1] = \boldsymbol{y}[s{+}k{+}2] = a$ and $\boldsymbol{y}[s{+}m] = \boldsymbol{y}[s{+}k{+}2{+}m]$ for any $1 \le i \le k$.

Theorem 5 *Let* $\boldsymbol{x}$, $\boldsymbol{y}$ *be two-pattern strings of scope* λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ *an extension, and* $\boldsymbol{y} = \sigma(\boldsymbol{x})$. *Let* $\mathcal{S} = \boldsymbol{y}[s..s{+}2k{+}3]$ *be a big* $b\boldsymbol{u}aa\boldsymbol{u}b$ *near repetition in* $\boldsymbol{y}$. *Then either*

1. $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, *and* $\hat{\boldsymbol{p}}_2$. $i \ge 3$ *(for aa, ba, and ab) or* $j \ge 3$ *(for bb).* $\mathcal{S}$ *is derived from one of the near repetitions aa, ab, ba, and bb in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ *all contain* $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$ *as a substring* $(r \ge 3)$. $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r = \boldsymbol{p}\boldsymbol{p}^{r-1}\boldsymbol{q}\boldsymbol{p}^{r-1}\boldsymbol{p} = \hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^{r-1}\boldsymbol{p}_1\big]aa\big[\boldsymbol{p}_2\boldsymbol{p}^{r-1}\boldsymbol{p}_1\big]b\big]\hat{\boldsymbol{p}}_1$.

2. $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, *and* $\hat{\boldsymbol{p}}_2$. $\mathcal{S}$ *is derived from near repetition* $b\boldsymbol{u}aa\boldsymbol{u}b$ *in* $\boldsymbol{x}$ *in the following way:* $\sigma(b\boldsymbol{u}aa\boldsymbol{u}b) =$ $\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$ $\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[b\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]aa\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]b\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.

3. $\boldsymbol{p} = a$, $\boldsymbol{q} = b\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $j = 2r$ *(for ab, bb) or* $i = 2r$ *(for ba, aa),* $r \ge 3$, *and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb configuration in* $\boldsymbol{x}$ *in the*

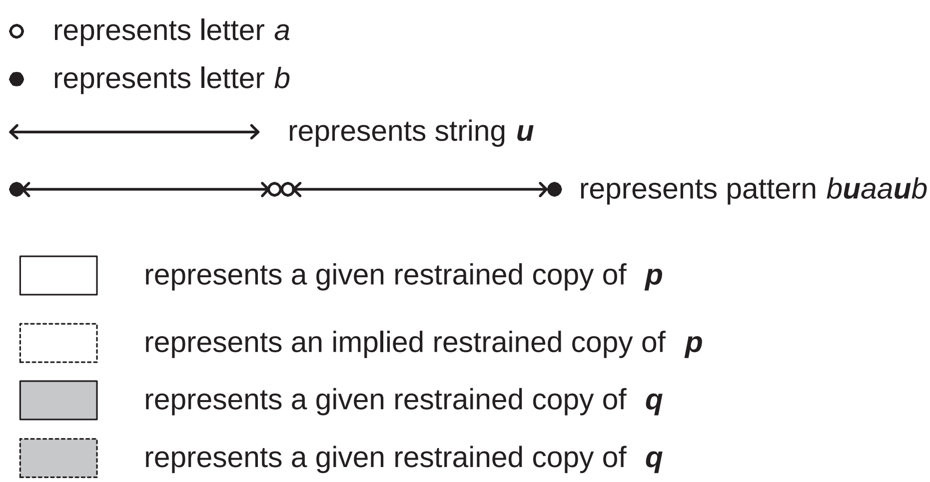
following way: $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *the all contain* $\boldsymbol{qp}^{2r}\boldsymbol{q}$ *as a substring.* $\boldsymbol{qp}^{2r}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^{r-1}\big]aa\big[\boldsymbol{p}^{r-1}\big]b\big]\hat{\boldsymbol{q}}_1$.

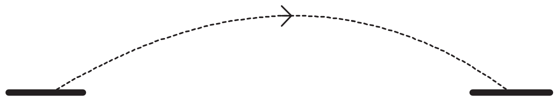
4. $\boldsymbol{p} = a$, $\boldsymbol{q} = b\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $j = 2i{+}2$ *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]\boldsymbol{pp}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]b\big]\hat{\boldsymbol{q}}_1 =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]aa\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]b\big]\hat{\boldsymbol{q}}_1$.

5. $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b\boldsymbol{q}_2 = \boldsymbol{q}_1 b\hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa\boldsymbol{q}_2$, $i = 2r{+}1$ *(for aa, ba) or* $j = 2r{+}1$ *(for ab, bb),* $r \geq 3$, *and* $\mathcal{S}$ *is derived from one of near repetitions aa, ab, ba, or bb in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *they all contain* $\boldsymbol{qp}^{2r+1}\boldsymbol{q}$ *as a substring.* $\boldsymbol{qp}^{2r+1}\boldsymbol{q} = \boldsymbol{qp}^r\boldsymbol{pp}^r\boldsymbol{q} =$
$\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]aa\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]b\big]\hat{\boldsymbol{q}}_1$.

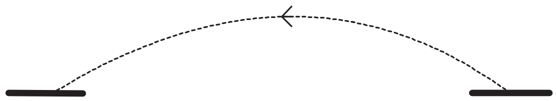
6. $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b\boldsymbol{q}_2 = \boldsymbol{q}_1 b\boldsymbol{q}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa\boldsymbol{q}_2$, $j = 2i{+}1$ *and* $\mathcal{S}$ *is derived from a near repetition* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) =$
$\cdot\cdot\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{qp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{pp}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[b\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]aa\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]b\big]\hat{\boldsymbol{q}}_1$.

Proof We shall conduct the proof in the same spirit as the proof of Theorem 2 by a "brute force" discussion of all possible ways the near repetition $a\boldsymbol{u}b\boldsymbol{u}$ can be placed in $\boldsymbol{y}$ with respect to restrained copies of $\boldsymbol{p}$ and $\boldsymbol{q}$. We shall employ a graphical language similar to the one used in the proof of Theorem 2. Also similarly we define the reflection $\mathcal{R}_{\mathcal{S}} : \boldsymbol{y}[s{+}1..s{+}k]^* \to \boldsymbol{y}[s{+}k{+}3..s{+}2k{+}2]^*$ and its inverse, the antireflection.





represents reflection of a segment to a segment

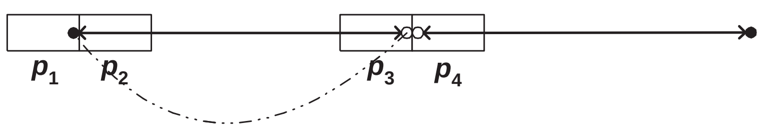


represents antireflection of a segment to a segment

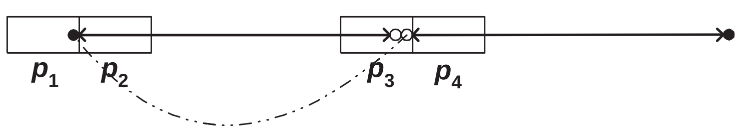
represents two letters at matching positions

Recall that for the near repetition $\mathcal{S}$ to be big, $k > 3\lambda$.

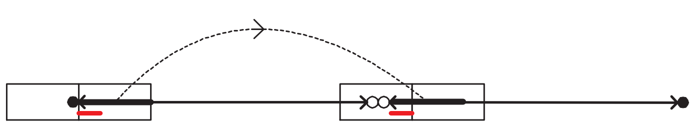
Case (1) – $\boldsymbol{pp}$-$\boldsymbol{pp}$ (i.e. the point $\boldsymbol{y}[s] = b$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$, and the point $\boldsymbol{y}[s{+}k{+}1] = a$ is located in a restrained copy of $\boldsymbol{p}$ followed by another restrained copy of $\boldsymbol{p}$):

1a 

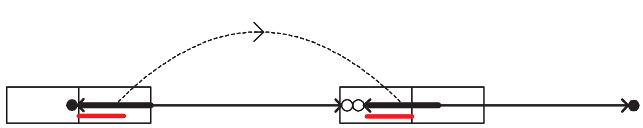
is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is b and the last letter of $\boldsymbol{p}_3$ is a.

1b 

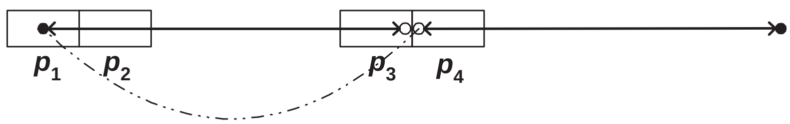
is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is b and the last letter of $\boldsymbol{p}_3$ is a.

1c 

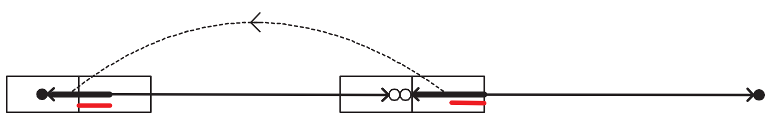
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1d 

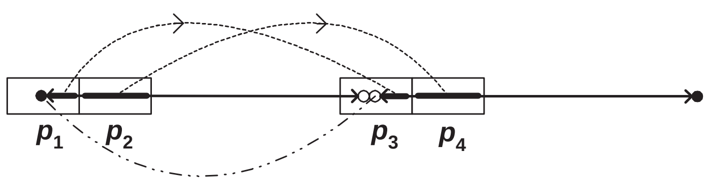
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1e


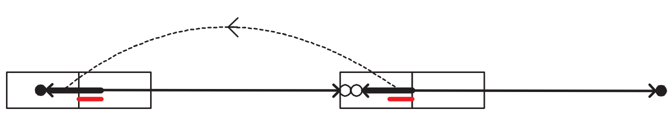
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$ implying a border a for $\boldsymbol{p}$ (as the last letter of $\boldsymbol{p}_3$ is a and the first letter of $\boldsymbol{p}_4$ is a).

1f


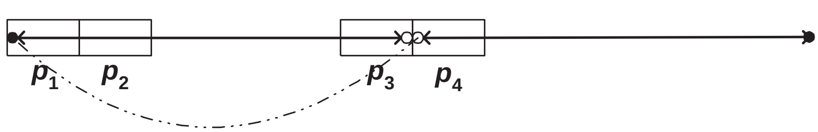
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1g


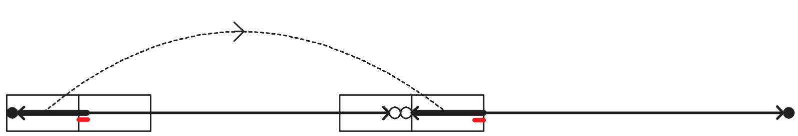
is not possible as it implies that the n-the letter of $\boldsymbol{p}_1$ is b while the n-the letter of $\boldsymbol{p}_3$ is a, where n is the position of $\boldsymbol{y}[s]$ is $\boldsymbol{p}_1$ and the position of $\boldsymbol{y}[s+k+2]$ in $\boldsymbol{p}_3$.

1h


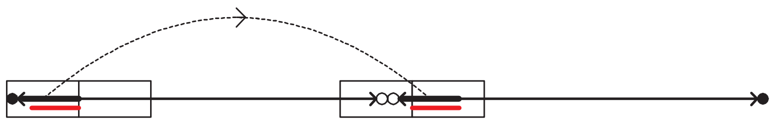
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1i


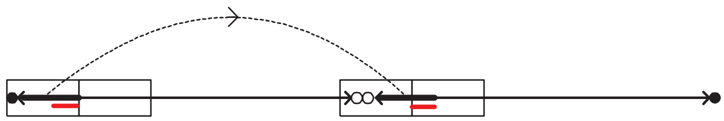
is not possible as it implies that the first letter of $\boldsymbol{p}_3$ is b and the first letter of $\boldsymbol{p}_4$ is a.

1j


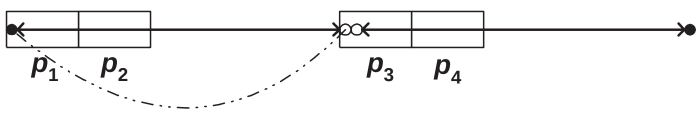
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

1k 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

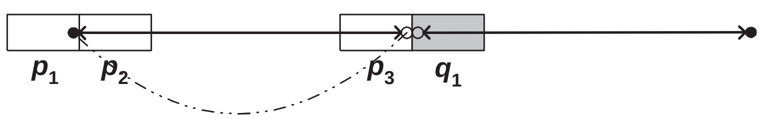
1l 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

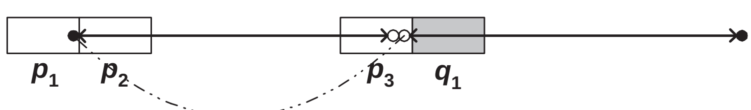
1m 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_3$ is a.

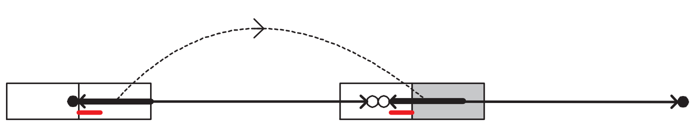
Case (2) – $\boldsymbol{pp}$-$\boldsymbol{pq}$

2a 

is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is b and the last letter of $\boldsymbol{p}_3$ is a.

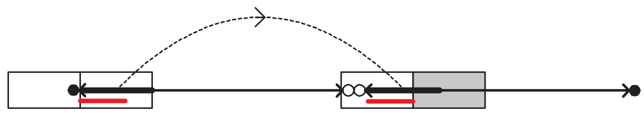
2b 

is not possible as it implies that the last letter of $\boldsymbol{p}_1$ is b and the last letter of $\boldsymbol{p}_3$ is a.

2c 

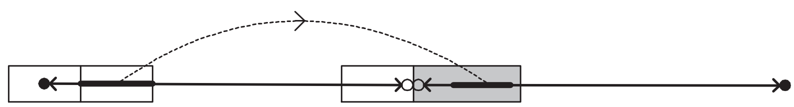
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2d



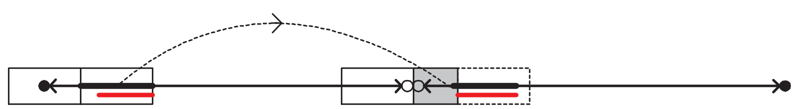
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2e



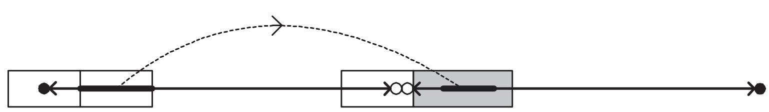
is not possible as it contradicts Lemma 1.

2f



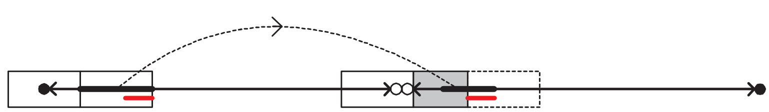
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2g



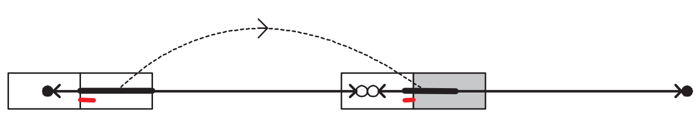
is not possible as it contradicts Lemma 1.

2h



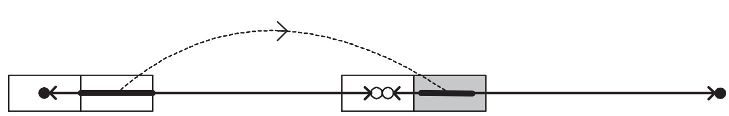
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2i



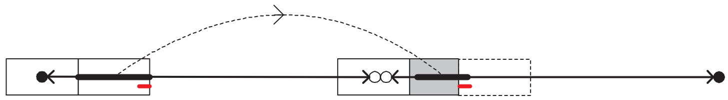
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2j



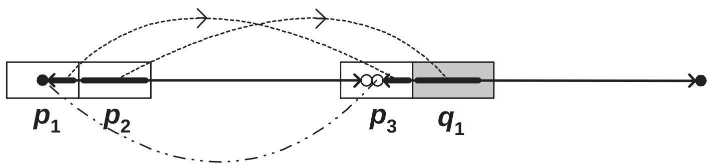
is not possible as it contradicts Lemma 1.

2k



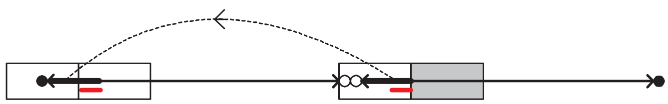
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2l



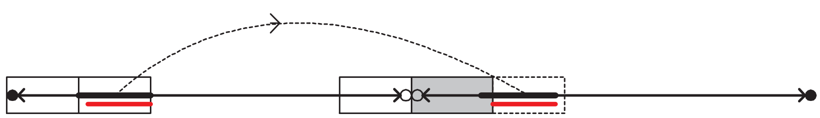
is not possible as it the n-the letter of $\boldsymbol{p}_1$ is b and th n-th letter of $\boldsymbol{p}_3$ is a, where n is the position of $\boldsymbol{y}[s]$ in $\boldsymbol{p}_1$ and the position of $\boldsymbol{y}[s{+}k{+}2]$ in $\boldsymbol{p}_3$.

2m



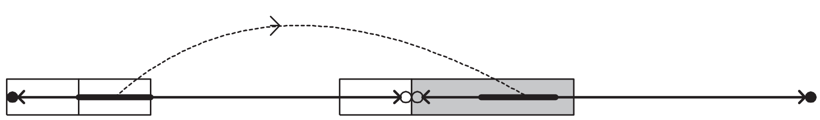
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2n



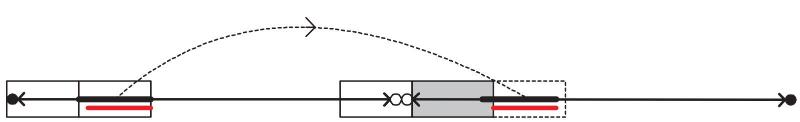
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

2o



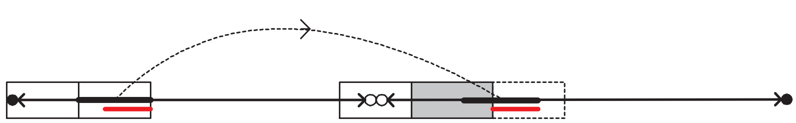
is not possible as it contradicts Lemma 1.

2p

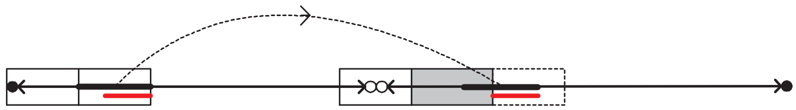


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

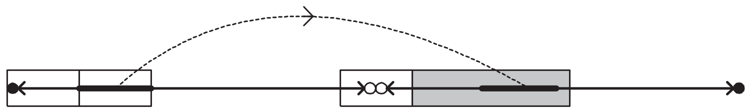
2q



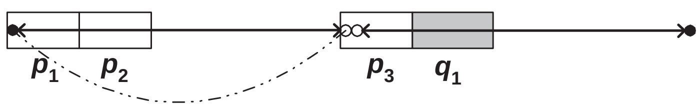
is not possible as it contradicts Lemma 1.

2r


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

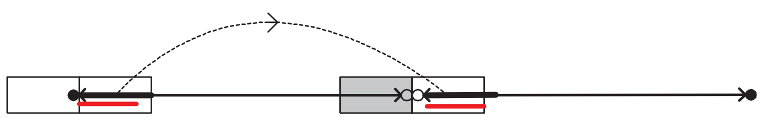
2s


is not possible as it contradicts Lemma 1.

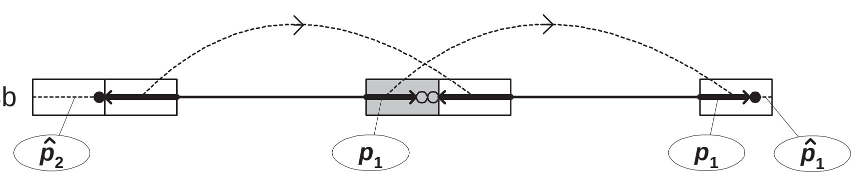
2t


is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_3$ is a.

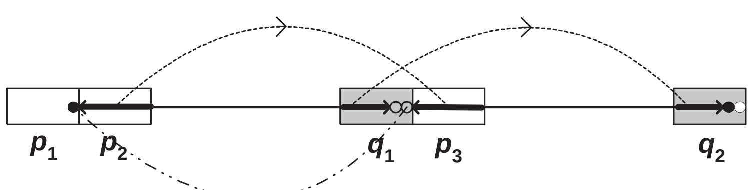
Case (3) – $\boldsymbol{pp}$-$\boldsymbol{qp}$

3a


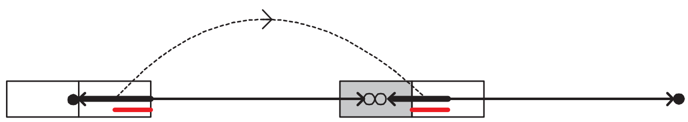
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3b


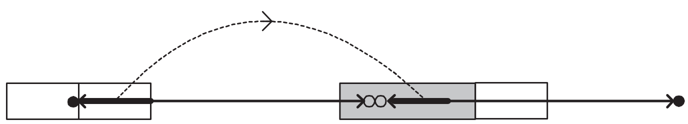
is possible. Then $\hat{\boldsymbol{p}}_2 b = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, and $\hat{\boldsymbol{p}}_2$. This reduces it to the case 3j below for $\boldsymbol{p}_2 = \varepsilon$.

3c


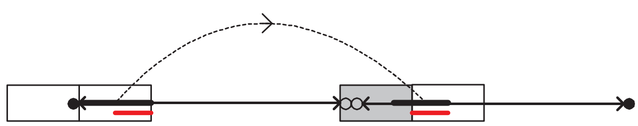
is not possible as it implies that the one-before-last letter of $\boldsymbol{q}_1$ is a and the one-before-last letter of $\boldsymbol{q}_2$ is b.

3d 

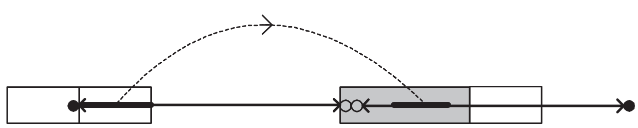
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3e 

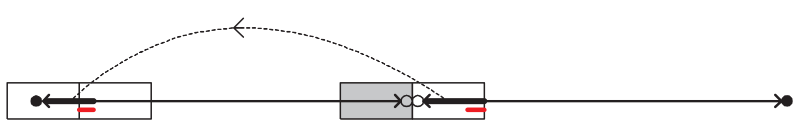
is not possible as it contradicts Lemma 1.

3f 

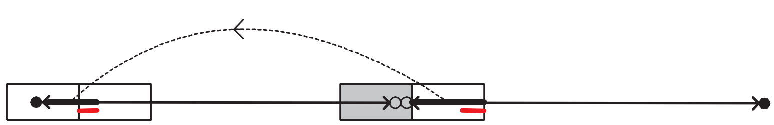
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3g 

is not possible as it contradicts Lemma 1.

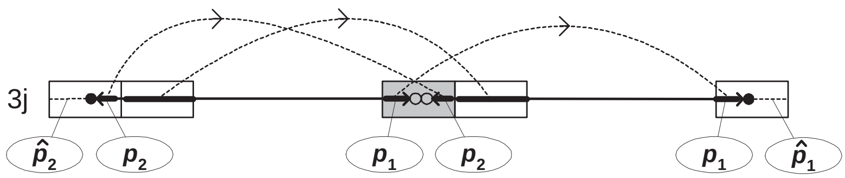
3h 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3i 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3j

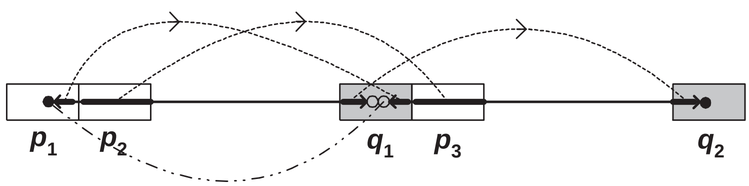


is possible. Then $\hat{\boldsymbol{p}}_2 b \boldsymbol{p}_2 = \boldsymbol{p}_1 b \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 aa \boldsymbol{p}_2$, for some $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, and $\hat{\boldsymbol{p}}_2$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$. Then $\boldsymbol{y}[s..s{+}2k{+}3]$ is a substring of $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$, $r \geq 3$, and thus $\mathcal{S}$ is derived from aa, ab, ba, and bb near repetitions in $\boldsymbol{x}$ and **case 1 of the theorem holds**; or

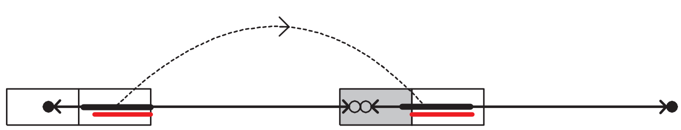
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$. Then $\boldsymbol{y}[s..s{+}2k{+}3]$ is a substring of $\boldsymbol{p}^{i+1}\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{i+1}$ and thus $\mathcal{S}$ is derived from a $b\boldsymbol{u}aa\boldsymbol{u}b$ configuration in $\boldsymbol{x}$ and **case 2 of the theorem holds**.

3k



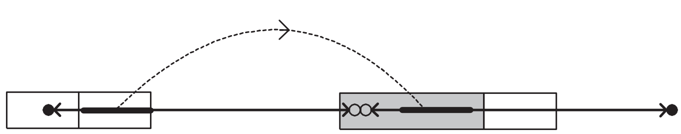
is not possible as it implies that n-the letter of $\boldsymbol{q}_1$ is a and n-th letter of $\boldsymbol{q}_2$ is b, where n is the position of $\boldsymbol{y}[s{+}k{+}2]$ in $\boldsymbol{q}_1$ and the position of $\boldsymbol{y}[s{+}2k{+}3]$ in $\boldsymbol{q}_2$.

3l



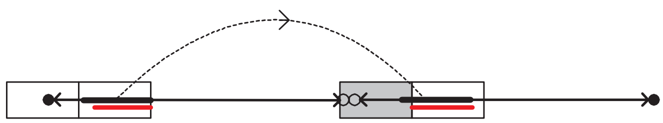
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3m



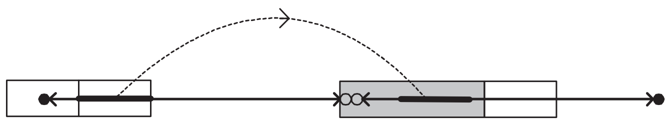
is not possible as it contradicts Lemma 1.

3n



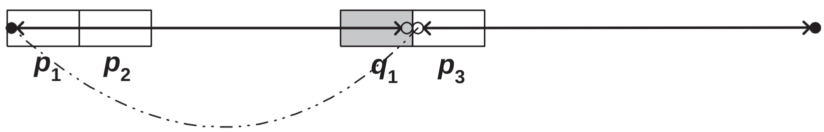
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3o



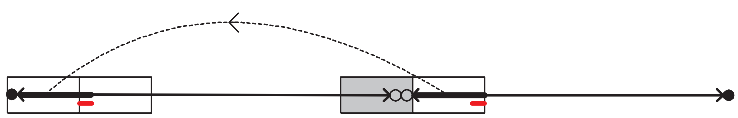
is not possible as it contradicts Lemma 1.

3p



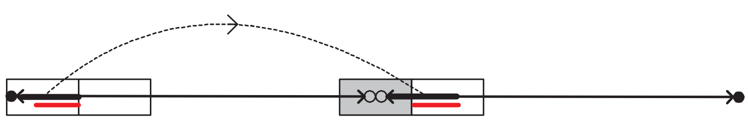
is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and first letter of $\boldsymbol{p}_3$ is a.

3q



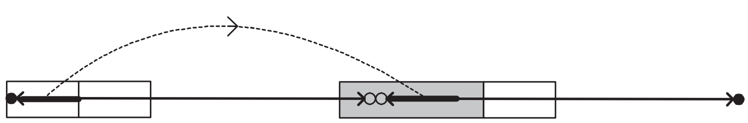
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3r



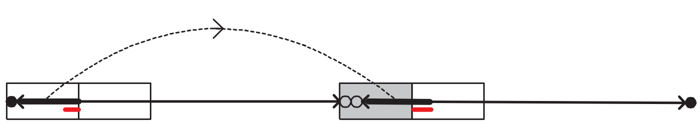
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

3s

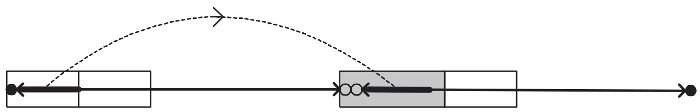


is not possible as it contradicts Lemma 1.

3t

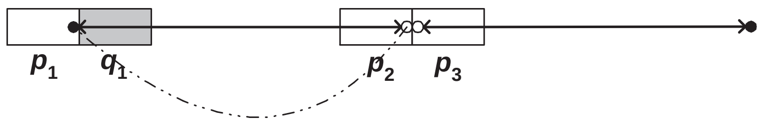


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

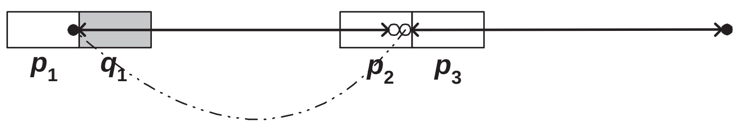
3u 

is not possible as it contradicts Lemma 1.

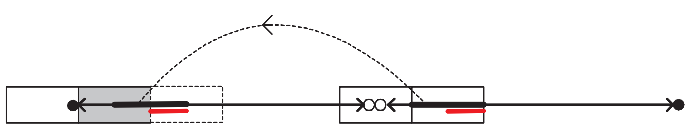
Case (4) – ***pq*-*pp***

4a 

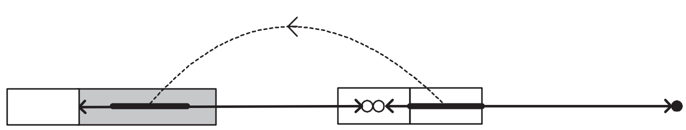
is not possible as it indicates that the last letter of $\boldsymbol{p}_1$ is b while the last letter of $\boldsymbol{p}_2$ is a.

4b 

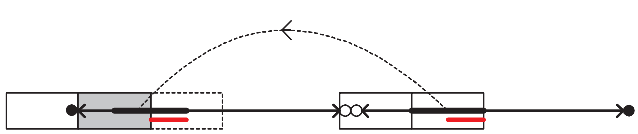
is not possible as it indicates that the last letter of $\boldsymbol{p}_1$ is b while the last letter of $\boldsymbol{p}_2$ is a.

4c 

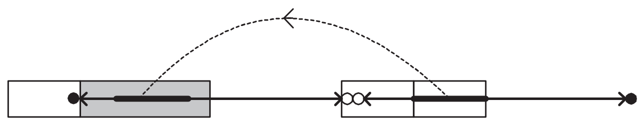
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4d 

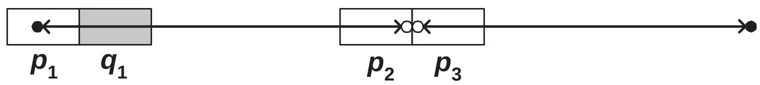
is not possible as it contradicts Lemma 2.

4e 

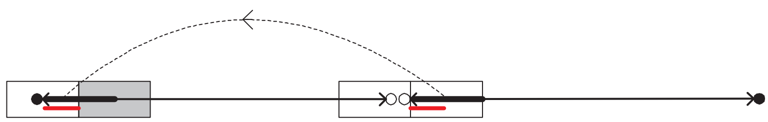
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4f 

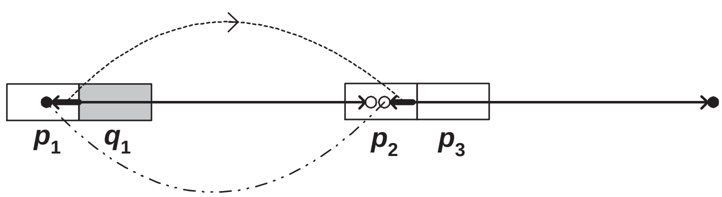
is not possible as it contradicts Lemma 2.

4g 

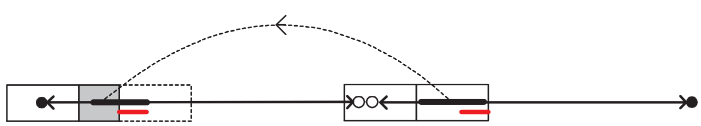
is not possible since the last letter of $\boldsymbol{p}_2$ is a and the first letter of $\boldsymbol{p}_3$ is a, and so $\boldsymbol{p} = a$, which contradicts the fact b occurs in $\boldsymbol{p}_1$.

4h 

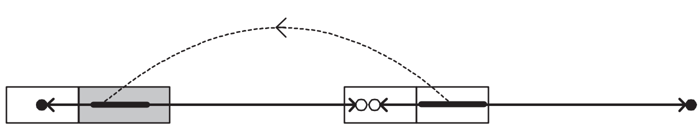
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4i 

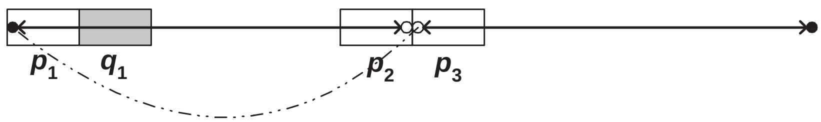
is not possible as it implies that n-the letter of $\boldsymbol{p}_1$ is b and n-th letter of $\boldsymbol{p}_2$ is a, where n is the position of $\boldsymbol{y}[s]$ in $\boldsymbol{p}_1$ and the position of $\boldsymbol{y}[s+k+2]$ in $\boldsymbol{p}_2$.

4j 

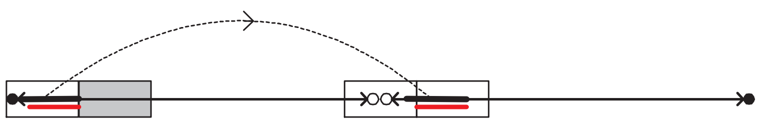
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

4k 

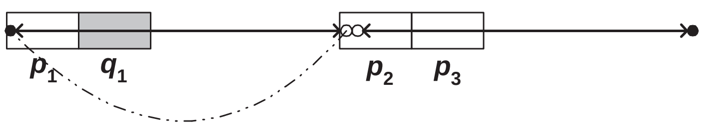
is not possible as it contradicts Lemma 2.

4l 

is not possible as the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_3$ is a.

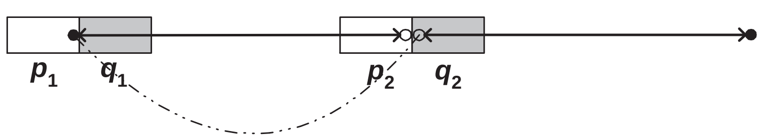
4m 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

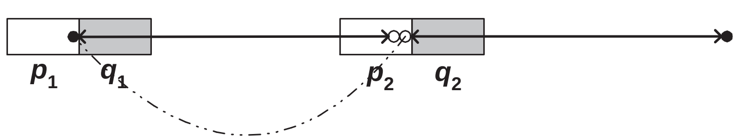
4n 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_2$ is a.

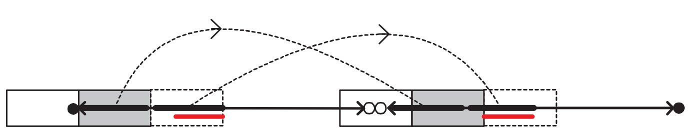
Case (5) – $\boldsymbol{pq}$-$\boldsymbol{pq}$

5a 

is not possible as it indicates that the last letter of $\boldsymbol{p}_1$ is b while the last letter of $\boldsymbol{p}_2$ is a.

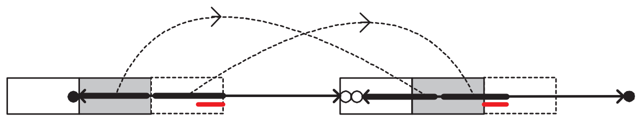
5b 

is not possible as it indicates that the last letter of $\boldsymbol{p}_1$ is b while the last letter of $\boldsymbol{p}_2$ is a.

5c 

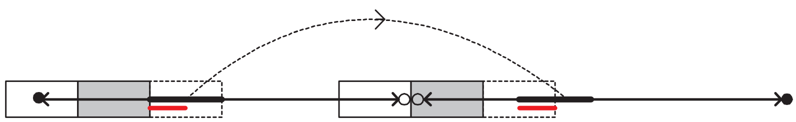
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5d



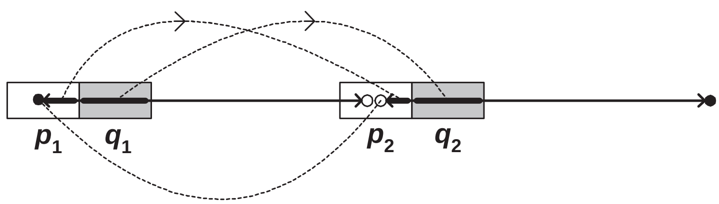
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5e



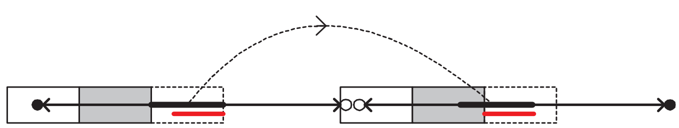
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5f



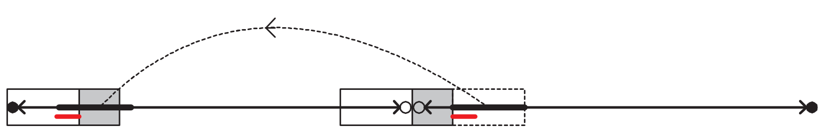
is not possible as it implies that n-the letter of $\boldsymbol{p}_1$ is b and n-th letter of $\boldsymbol{p}_2$ is a, where n is the position of $\boldsymbol{y}[s]$ in $\boldsymbol{p}_1$ and the position of $\boldsymbol{y}[s+k+2]$ in $\boldsymbol{p}_2$.

5g



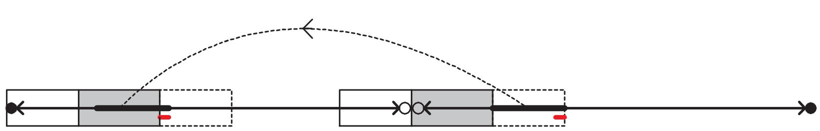
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5h

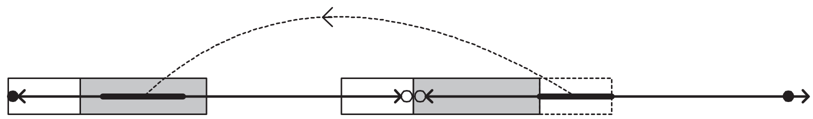


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

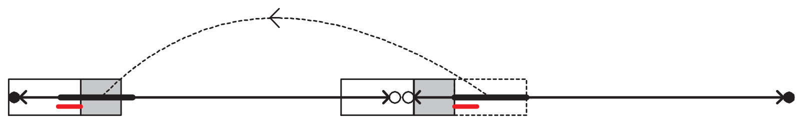
5i



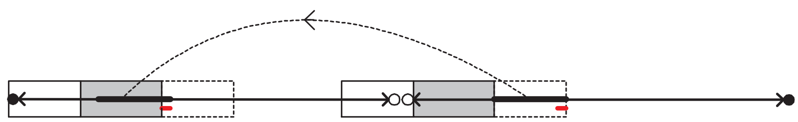
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5j


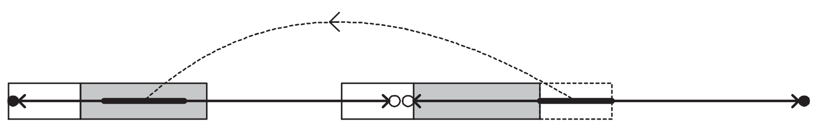
is not possible as it contradicts Lemma 2.

5k


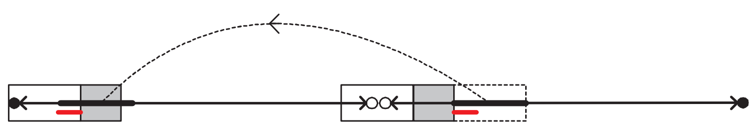
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5l


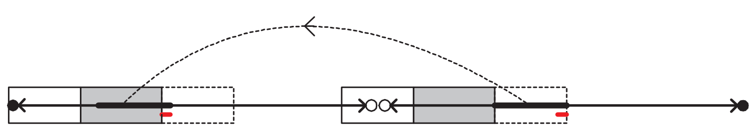
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5m


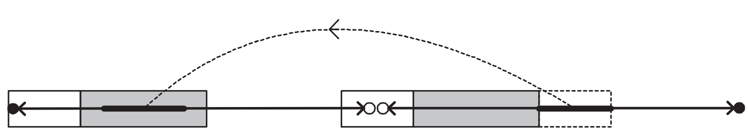
is not possible as it contradicts Lemma 2.

5n


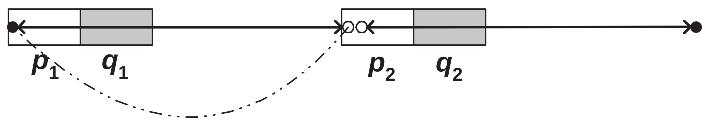
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

5o


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

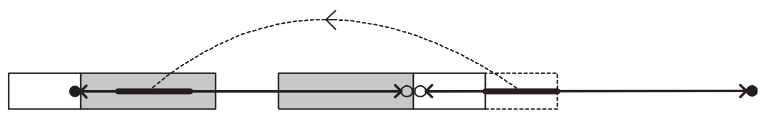
5p


is not possible as it contradicts Lemma 2.

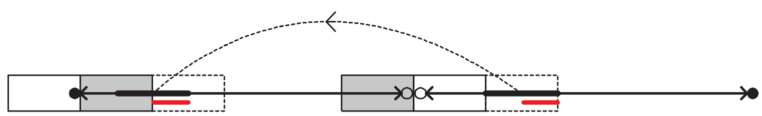
5q 

is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_2$ is a.

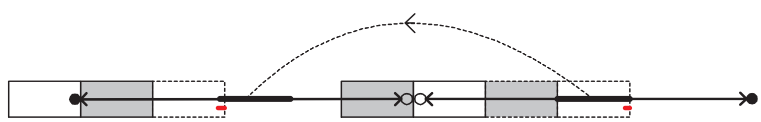
Case (6) – $\boldsymbol{pq}$-$\boldsymbol{qp}$

6a 

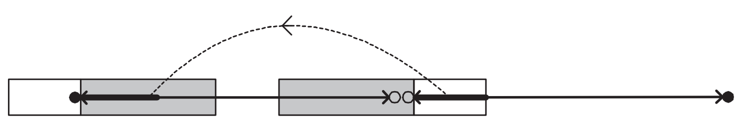
is not possible as it contradicts Lemma 2.

6b 

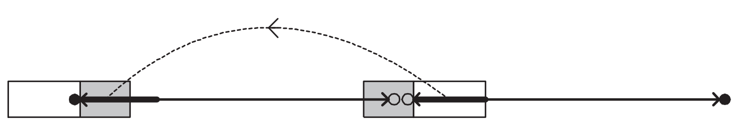
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6c 

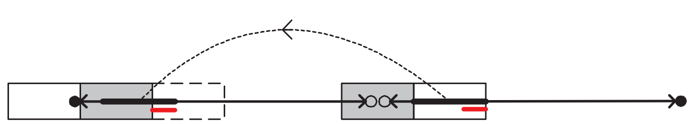
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6d 

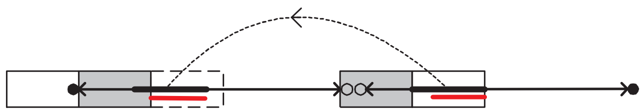
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6e 

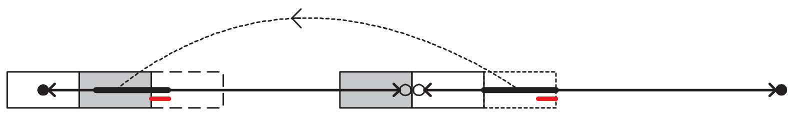
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6f 

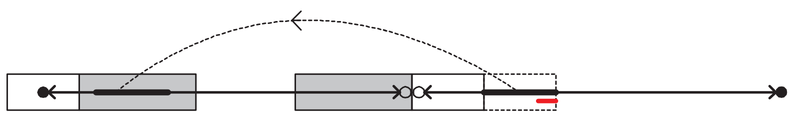
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6g 

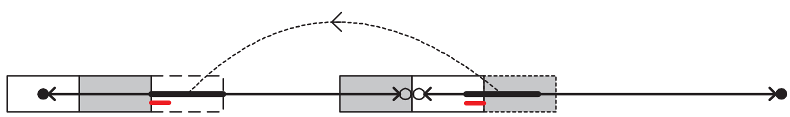
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6h 

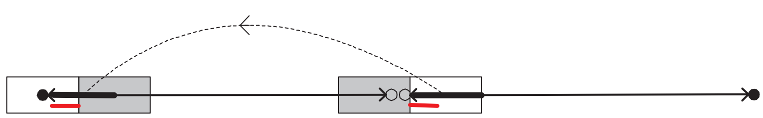
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6i 

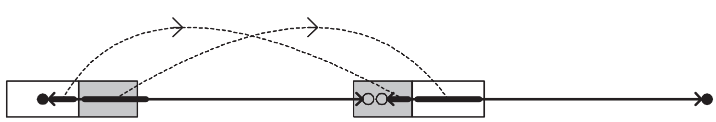
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6j 

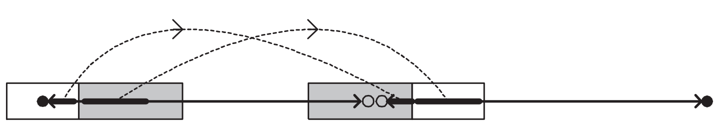
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6k 

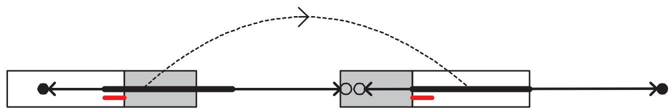
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6l 

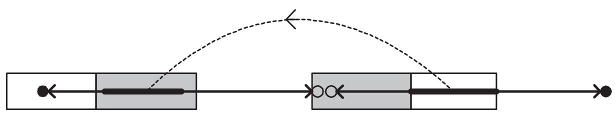
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

6m 

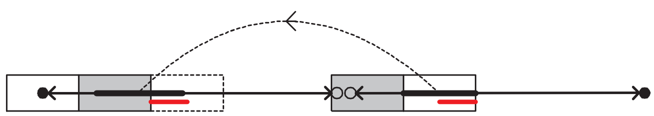
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

6n 

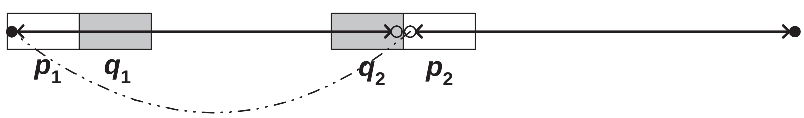
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6o 

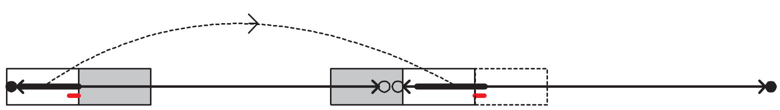
is not possible as it contradicts Lemma 2.

6p 

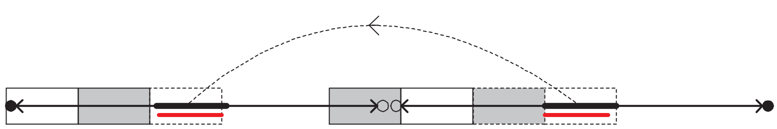
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6q 

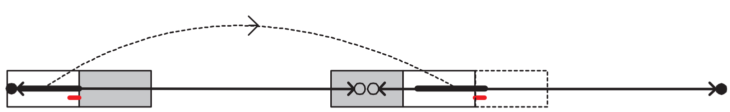
is not possible as it implies that the first letter of $\boldsymbol{p}_1$ is b and the first letter of $\boldsymbol{p}_2$ is a.

6r 

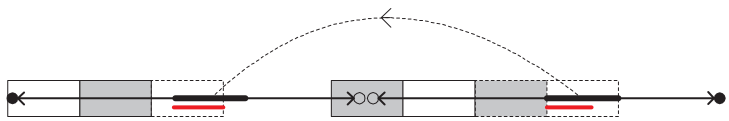
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

6s 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

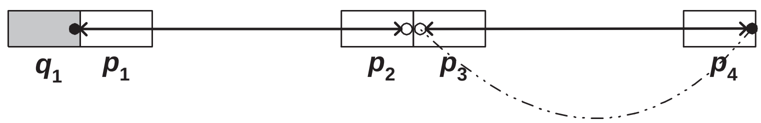
6t 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

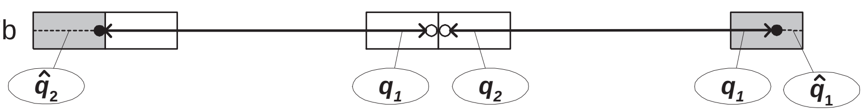
6u 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

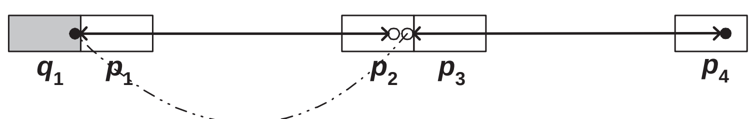
Case (7) – $\boldsymbol{qp}$-$\boldsymbol{pp}$

7a 

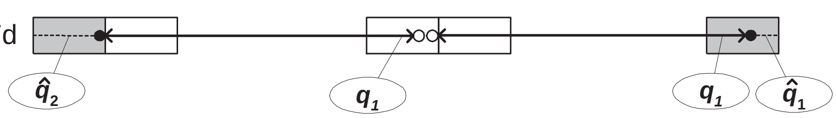
is not possible as it implies that the last letter of $\boldsymbol{p}_2$ is a and the last letter of $\boldsymbol{p}_4$ is b.

7b 

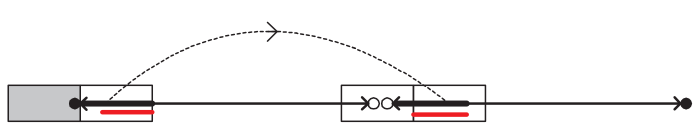
is possible only if $\boldsymbol{p} = a$. $\boldsymbol{q} = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $\boldsymbol{q}_1 = \boldsymbol{q}_2 = \varepsilon$. This reduces it to case 7h below for $\boldsymbol{q}_2 = \boldsymbol{q}_1 = \varepsilon$.

7c 

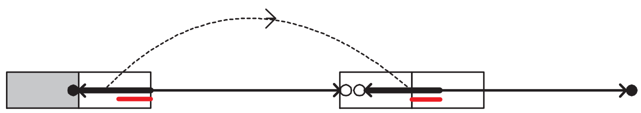
is not possible as it implies that the n-th letter of $\boldsymbol{p}_2$ is a and the n-th letter of $\boldsymbol{p}_4$ is b, where n is the position of $\boldsymbol{y}[s{+}k{+}1]$ is $\boldsymbol{p}_2$ and the position of $\boldsymbol{y}[s{+}2k{+}3]$ in $\boldsymbol{p}_4$.

7d 

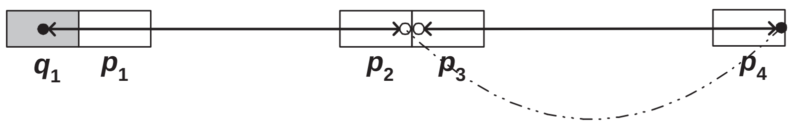
is possible if $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa$. This reduces it to case 7j below for $\boldsymbol{q}_2 = \varepsilon$.

7e 

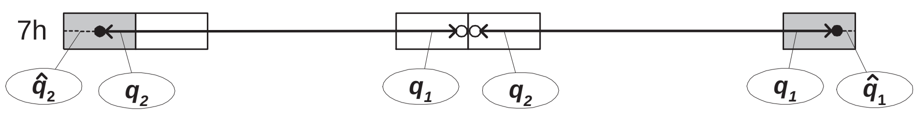
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7f


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7g


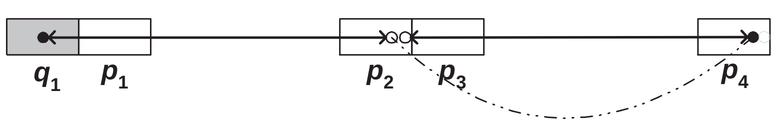
is not possible as it implies that the last letter of $\boldsymbol{p}_2$ is a and the last letter of $\boldsymbol{p}_4$ is b.

7h


is possible only if $\boldsymbol{p} = a$. $\boldsymbol{q} = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 b$, $\boldsymbol{q}_1 = \boldsymbol{q}_2 = \varepsilon$. Either

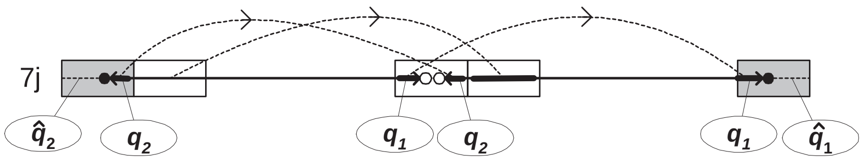
1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$. Then $\boldsymbol{y}[s..s{+}2k{+}3]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q}$, $r \geq 2$. Thus $\mathcal{S}$ is derived from aa, ab, ba, and bb near repetitions and so **case 3 of the theorem holds**.

2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$. Then $j = 2i{+}2$ and $\boldsymbol{y}[s..s{+}2k{+}3]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\boldsymbol{p}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}$ and thus $\mathcal{S}$ is derived from a $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ and so **case 4 of the theorem holds**.

7i


is not possible as it implies that the n-th letter of $\boldsymbol{p}_2$ is a and the n-th letter of $\boldsymbol{p}_4$ is b, where n is the position of $\boldsymbol{y}[s{+}k{+}1]$ is $\boldsymbol{p}_2$ and the position of $\boldsymbol{y}[s{+}2k{+}3]$ in $\boldsymbol{p}_4$.

7j

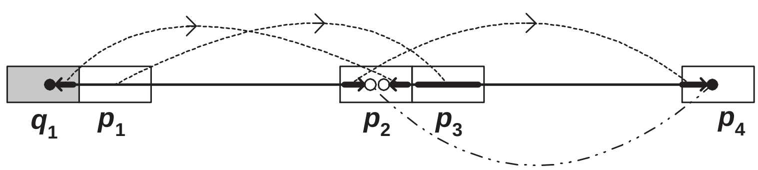


is possible if $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 b \boldsymbol{q}_2 = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 aa \boldsymbol{q}_2$. Either

1. there is no restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}1..s{+}2k{+}3]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$, $r \geq 2$, and so $\mathcal{S}$ is derived from one of near repetitions aa, ab, ba, or bb and so **case 5 of the theorem holds**; or

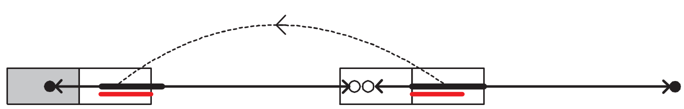
2. there is a restrained copy of $\boldsymbol{q}$ that is a segment of $\boldsymbol{y}[s{+}1..s{+}k]$ and so $\boldsymbol{y}[s..s{+}1..s{+}2k{+}3]$ is a substring of $\boldsymbol{q}\boldsymbol{p}^{i} q \bar{\boldsymbol{u}}\boldsymbol{p}^{j} \boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}$. Thus $j = 2i{+}1$ and so $\boldsymbol{q}\boldsymbol{p}^{i} q \bar{\boldsymbol{u}}\boldsymbol{p}^{j} \boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^{i} q \bar{\boldsymbol{u}}\boldsymbol{p}^{j} \boldsymbol{q}\bar{\boldsymbol{u}}\boldsymbol{p}^{i}\boldsymbol{q}$. So $\mathcal{S}$ is derived from a $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ near repetition in $\boldsymbol{x}$ and **case 6 of the theorem holds**.

7k



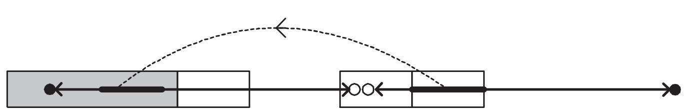
is not possible as it implies that the n-th letter of $\boldsymbol{p}_2$ is a and the n-th letter of $\boldsymbol{p}_4$ is b, where n is the position of $\boldsymbol{y}[s{+}k{+}1]$ is $\boldsymbol{p}_2$ and the position of $\boldsymbol{y}[s{+}2k{+}3]$ in $\boldsymbol{p}_4$.

7l



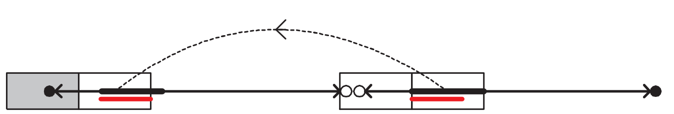
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7m

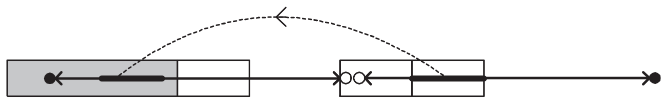


is not possible as it contradicts Lemma 1.

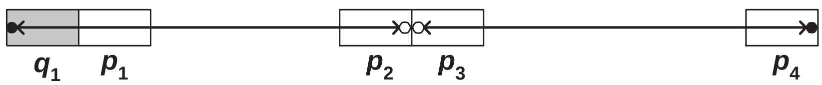
7n



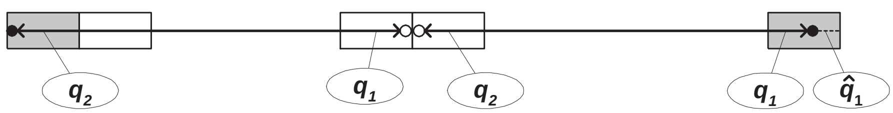
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7o 

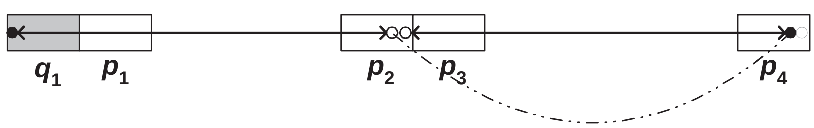
is not possible as it contradicts Lemma 1.

7p 

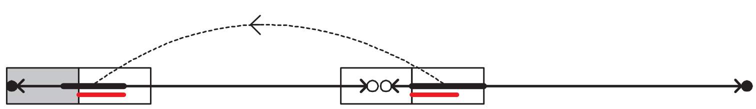
is not possible as it implies that the last letter of $\boldsymbol{p}_2$ is a and the last letter of $\boldsymbol{p}_4$ is b.

7q 

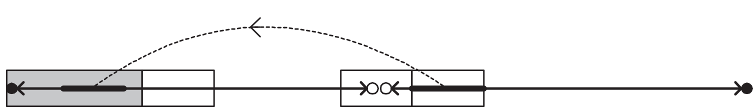
is possible only if $\boldsymbol{p} = a$. $\boldsymbol{q} = \boldsymbol{q}_1 b \hat{\boldsymbol{q}}_1 = b\boldsymbol{q}_2$, $\boldsymbol{q}_1 = \boldsymbol{q}_2 = \varepsilon$. This reduces it to case 7h above.

7r 

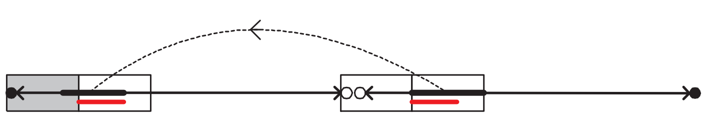
is not possible as it implies that the one-before-last letter of $\boldsymbol{p}_2$ is a and the one-before-last letter of $\boldsymbol{p}_4$ is b.

7s 

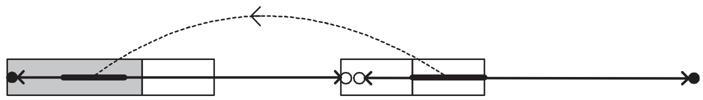
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

7t 

is not possible as it contradicts Lemma 2.

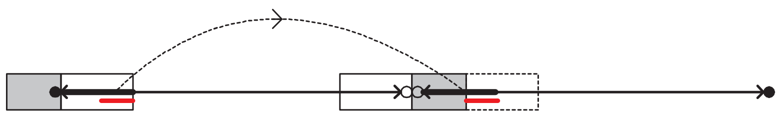
7u 

is not possible as it contradicts the primitiveness of ***p***.

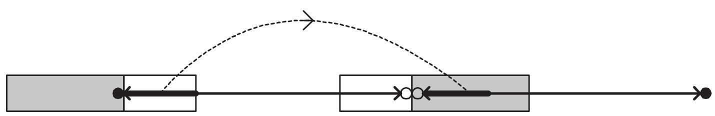
7v 

is not possible as it contradicts Lemma 2.

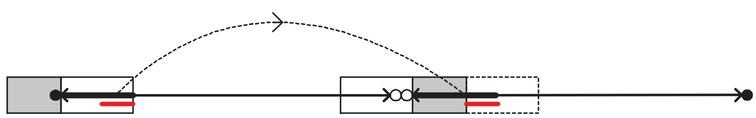
Case (8) – ***qp-pq***

8a 

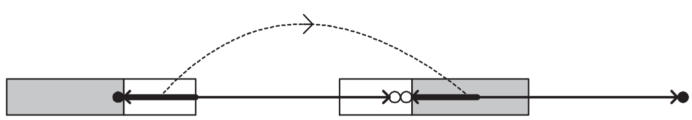
is not possible as it contradicts the primitiveness of ***p***.

8b 

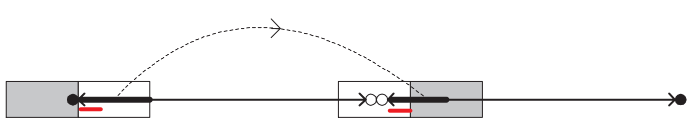
is not possible as it contradicts Lemma 1.

8c 

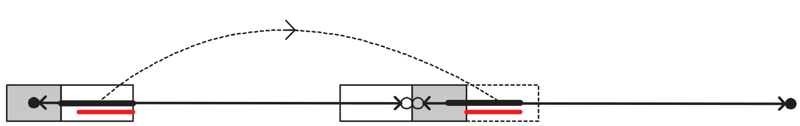
is not possible as it contradicts the primitiveness of ***p***.

8d 

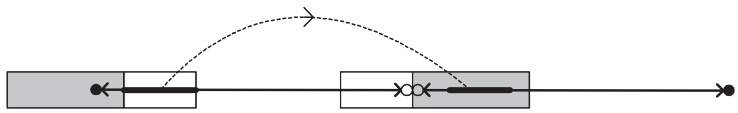
is not possible as it contradicts Lemma 1.

8e 

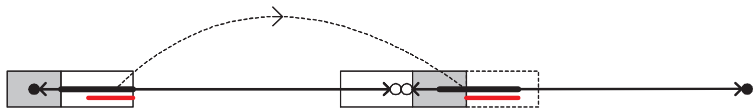
is not possible as it contradicts the primitiveness of ***p***.

8f 

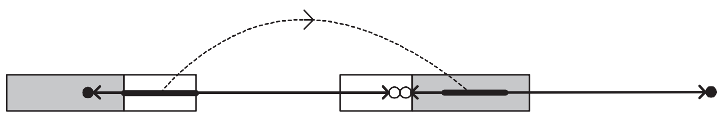
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8g 

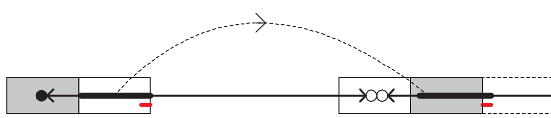
is not possible as it contradicts Lemma 1.

8h 

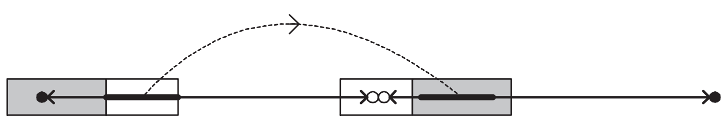
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8i 

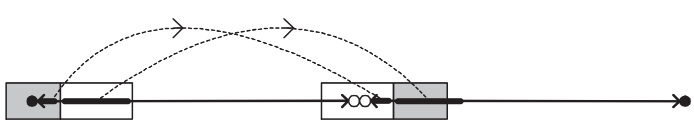
is not possible as it contradicts Lemma 1.

8j 

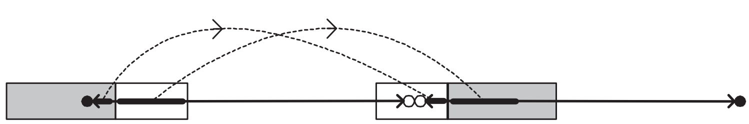
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8k 

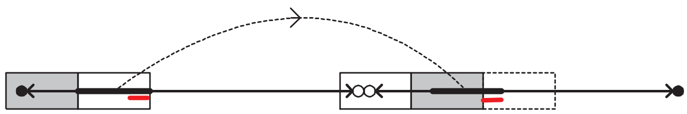
is not possible as it contradicts Lemma 1.

8l 

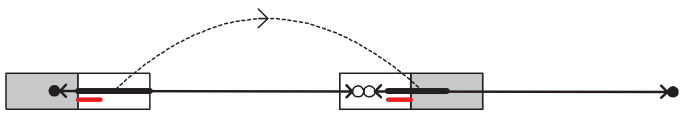
is not possible as it implies that $\boldsymbol{q}$ is a prefix of $\boldsymbol{p}$.

8m 

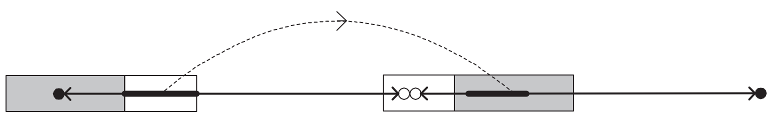
is not possible as it implies that $\boldsymbol{p}$ is a prefix of $\boldsymbol{q}$.

8n


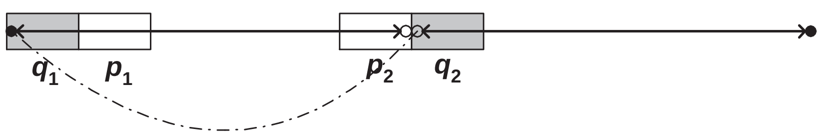
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8o


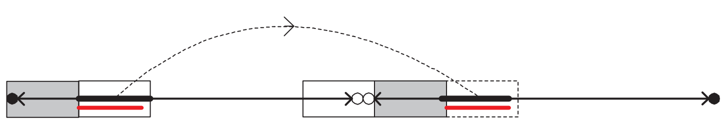
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

8p


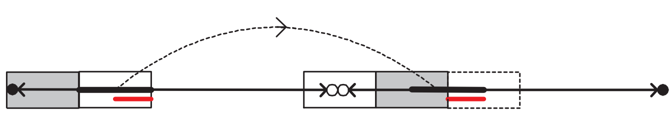
is not possible as it contradicts Lemma 1.

8q


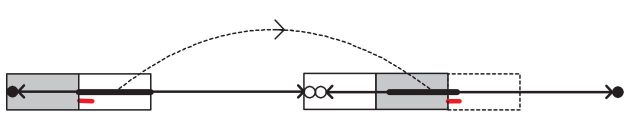
is not possible as it implies that the first letter of $\boldsymbol{q}_1$ is b and the first letter of $\boldsymbol{q}_2$ is a.

8r


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

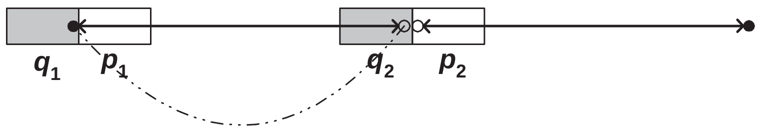
8s


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

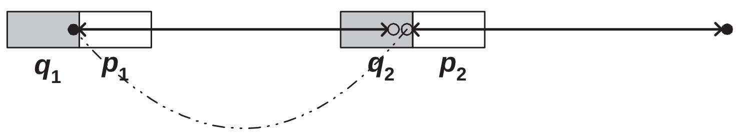
8t


is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

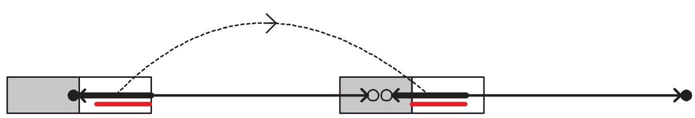
Case (9) – $\boldsymbol{qp}$-$\boldsymbol{qp}$

9a 

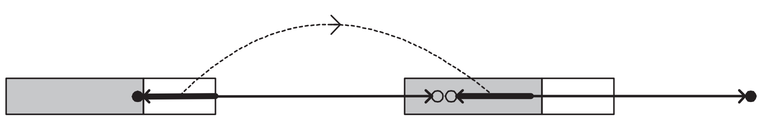
is not possible as it implies that the last letter of $\boldsymbol{q}_1$ is b and the last letter of $\boldsymbol{q}_2$ is a.

9b 

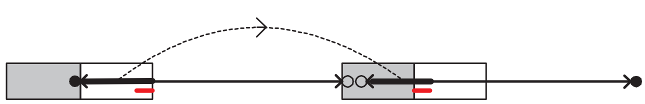
is not possible as it implies that the last letter of $\boldsymbol{q}_1$ is b and the last letter of $\boldsymbol{q}_2$ is a.

9c 

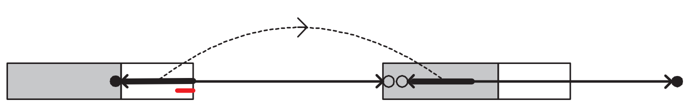
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9d 

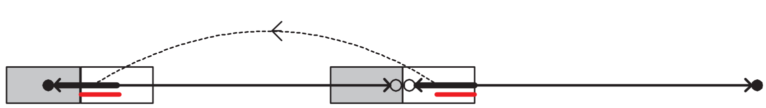
is not possible as it contradicts Lemma 1.

9e 

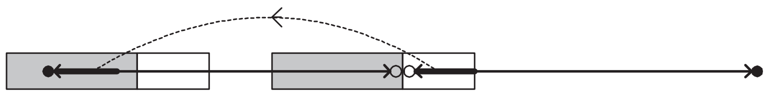
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9f 

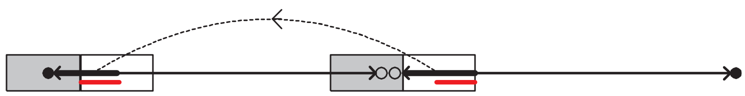
is not possible as it contradicts Lemma 1.

9g 

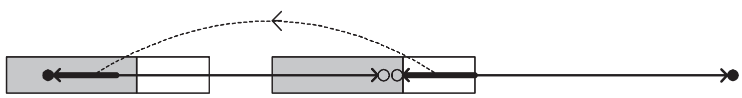
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9h 

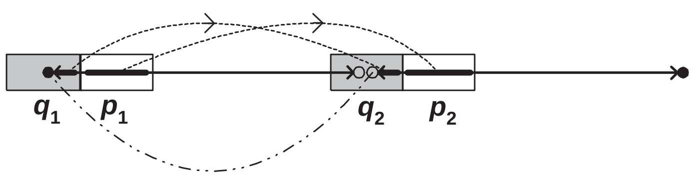
is not possible as it contradicts Lemma 2.

9i 

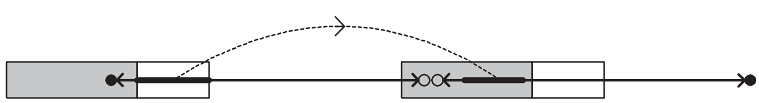
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9j 

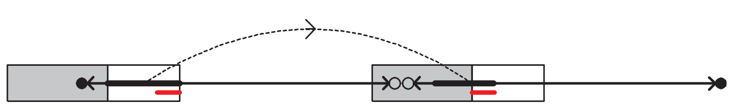
is not possible as it contradicts Lemma 2.

9k 

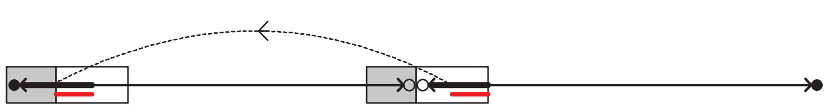
is not possible as it implies that the n-th letter of $\boldsymbol{q}_1$ is b and the n-th letter of $\boldsymbol{q}_2$ is a, where n is the position of $\boldsymbol{y}[s]$ is $\boldsymbol{q}_1$ and the position of $\boldsymbol{y}[s{+}k{+}2]$ in $\boldsymbol{q}_2$.

9l 

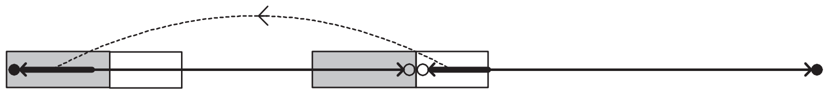
is not possible as it contradicts Lemma 1.

9m 

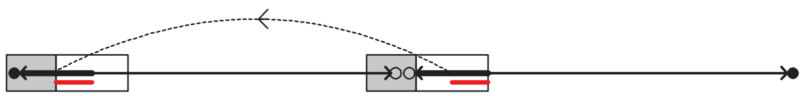
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9n 

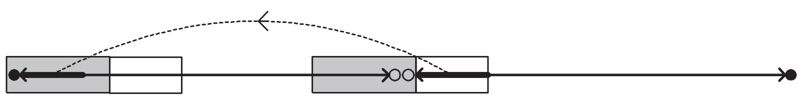
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9o 

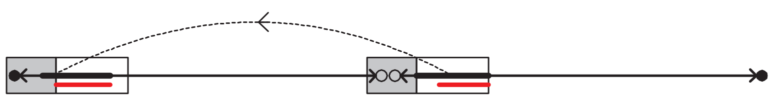
is not possible as it contradicts Lemma 2.

9p 

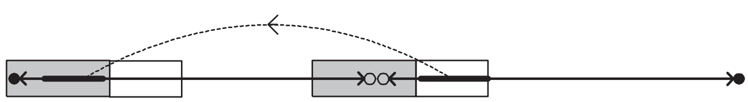
is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9q 

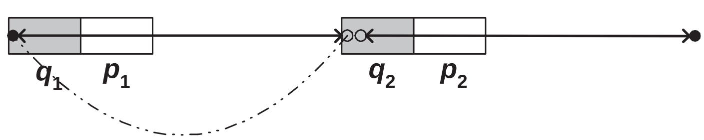
is not possible as it contradicts Lemma 2.

9r 

is not possible as it contradicts the primitiveness of $\boldsymbol{p}$.

9s 

is not possible as it contradicts Lemma 2.

9t 

is not possible as it implies that the first letter of $\boldsymbol{q}_1$ is b and the first letter of $\boldsymbol{q}_2$ is a.

□

In the following we are going to discuss all possible ways near repetitions of type $a\boldsymbol{u}bb\boldsymbol{u}a$ can arise. We say that $\boldsymbol{y}[s..s+2k+3]$ is an $a\boldsymbol{u}bb\boldsymbol{u}a$ near repetition if $\boldsymbol{y}[s] = \boldsymbol{y}[s+2k+3] = a$, $\boldsymbol{y}[s+k+1] = \boldsymbol{y}[s+k+2] = b$ and $\boldsymbol{y}[s+m] = \boldsymbol{y}[s+k+2+m]$ for any $1 \leq i \leq k$.

Theorem 6 *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two-pattern strings of scope λ, $\sigma = [\boldsymbol{p}, \boldsymbol{q}, i, j]_\lambda$ an extension, and $\boldsymbol{y} = \sigma(\boldsymbol{x})$. Let $\mathcal{S} = \boldsymbol{y}[s..s{+}2k{+}3]$ be a big $a\boldsymbol{u}bb\boldsymbol{u}a$ near repetition in $\boldsymbol{y}$. Then either*

1. $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, *and* $\hat{\boldsymbol{p}}_2$. $i \geq 3$ *(for aa, ba, and ab) or* $j \geq 3$ *(for bb).* $\mathcal{S}$ *is derived from one of the near repetitions aa, ab, ba, and bb in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$ *all contain* $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r$ *as a substring* ($r \geq 3$). $\boldsymbol{p}^r\boldsymbol{q}\boldsymbol{p}^r = \boldsymbol{p}\boldsymbol{p}^{r-1}\boldsymbol{q}\boldsymbol{p}^{r-1}\boldsymbol{p} = \hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^{r-1}\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^{r-1}\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1$.

2. $\hat{\boldsymbol{p}}_2 a \boldsymbol{p}_2 = \boldsymbol{p}_1 a \hat{\boldsymbol{p}}_1$, $\boldsymbol{q} = \boldsymbol{p}_1 bb \boldsymbol{p}_2$, *for some* $\boldsymbol{p}_1$, $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_2$, *and* $\hat{\boldsymbol{p}}_2$. $\mathcal{S}$ *is derived from near repetition* $a\boldsymbol{u}bb\boldsymbol{u}a$ *in* $\boldsymbol{x}$ *in the following way:* $\sigma(a\boldsymbol{u}bb\boldsymbol{u}a) =$
$\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q} = \boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
$\boldsymbol{p}^{j-i-1}\hat{\boldsymbol{p}}_2\big[a\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]bb\big[\boldsymbol{p}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{p}_1\big]a\big]\hat{\boldsymbol{p}}_1\boldsymbol{p}^{j-i-1}\boldsymbol{q}$.

3. $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $j = 2r$ *(for ab, bb) or* $i = 2r$ *(for ba, aa),* $r \geq 3$, *and* $\mathcal{S}$ *is derived from aa, ab, ba, or bb configuration in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *the all contain* $\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q}$ *as a substring.* $\boldsymbol{q}\boldsymbol{p}^{2r}\boldsymbol{q} = \hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^{r-1}\big]bb\big[\boldsymbol{p}^{r-1}\big]a\big]\hat{\boldsymbol{q}}_1$.

4. $\boldsymbol{p} = b$, $\boldsymbol{q} = a\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{q}}_2 a$, $j = 2i{+}2$ *and* $\mathcal{S}$ *is derived from a* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *near repetition in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]\boldsymbol{p}\boldsymbol{p}\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]a\big]\hat{\boldsymbol{q}}_1 =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]bb\big[\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-2}\big]a\big]\hat{\boldsymbol{q}}_1$.

5. $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \hat{\boldsymbol{q}}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $i = 2r{+}1$ *(for aa, ba) or* $j = 2r{+}1$ *(for ab, bb),* $r \geq 3$, *and* $\mathcal{S}$ *is derived from one of near repetitions aa, ab, ba, or bb in* $\boldsymbol{x}$ *in the following way:* $\sigma(aa)$, $\sigma(ab)$, $\sigma(ba)$, $\sigma(bb)$, *they all contain* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q}$ *as a substring.* $\boldsymbol{q}\boldsymbol{p}^{2r+1}\boldsymbol{q} = \boldsymbol{q}\boldsymbol{p}^r\boldsymbol{p}\boldsymbol{p}^r\boldsymbol{q} =$
$\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^r\big]\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^r\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.

6. $\boldsymbol{q} = \hat{\boldsymbol{q}}_2 a \boldsymbol{q}_2 = \boldsymbol{q}_1 a \boldsymbol{q}_1$, $\boldsymbol{p} = \boldsymbol{q}_1 bb \boldsymbol{q}_2$, $j = 2i{+}1$ *and* $\mathcal{S}$ *is derived from a near repetition* $\cdot a\boldsymbol{u}b\boldsymbol{u}a$ *in* $\boldsymbol{x}$ *in the following way:* $\sigma(\cdot a\boldsymbol{u}b\boldsymbol{u}a) =$
$\cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^j\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^i\boldsymbol{q} = \cdot\cdot\boldsymbol{q}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{p}\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q} =$
$\cdot\cdot\hat{\boldsymbol{q}}_2\big[a\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]bb\big[\boldsymbol{q}_2\boldsymbol{p}^i\boldsymbol{q}\overline{\boldsymbol{u}}\boldsymbol{p}^{j-i-1}\boldsymbol{q}_1\big]a\big]\hat{\boldsymbol{q}}_1$.

Proof Virtually identical to the proof of the previous theorem. $\square$