Saturated ideals obtained via restricted iterated collapse of huge cardinals

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December 1987

Technical Report 87-09

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Abstract.

A uniform method to define a (restricted iterated) forcing notion to collapse a huge cardinal to a small one to obtain models with various types of highly saturated ideals over small cardinals is presented. The method is discussed in great technical details in the first chapter, while in the second chapter the application of the method is shown on three different models: Model I with an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_1$ that satisfies Chang's conjecture, Model II with an $\aleph_1$-complete $\aleph_3$-saturated ideal over $\aleph_3$, and Model III with an $\aleph_1$-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal over $\aleph_1$.

Introduction

"There is no κ^+-complete κ^+-saturated ideal over κ (κ an uncountable cardinal)" is a straightforward generalization of the classical result of Ulam (see [U] or [J]) "there is no non-trivial σ-additive measure on $\aleph_1$". Solovay (see [S]) proved that if "there exists a κ-complete κ-saturated ideal over κ", then κ is a large cardinal (Mahlo). So if one wants to generalize the notion of saturated ideals to an ideal over a smaller cardinal κ, either completeness of such ideal must be less than κ, or saturatedness must be at least κ^+. From Solovay's work (see [S]) follows that the consistency strength of the existence of an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_1$ is at least the existence of a measurable cardinal. Later improved by Mitchell (see [Mi]), the consistency strength is in fact at least the existence of a certain sequence of measurable cardinals. Until Kunen's paper [K$_2$], there was no model known with an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_1$. Kunen used a collapse of huge cardinal to obtain such a model. Variations of his method were used by Magidor (see [M]), Laver (see [L]), and Forman-Laver (see [FL]) to obtain various saturated ideals over $\aleph_1$ or $\aleph_3$.

In this paper the author tried to unify all these variations. In Chapter 1 an exposition of the method (a restricted iterated forcing) with most of details worked out is presented. Only the general knowledge of forcing and iterated forcing is assumed. The aspects of restricted forcing (keeping forcing terms "small" so the resulting posets are not getting too "big"), extension and covering properties of elementary embeddings (i.e. when an elementary embedding $j{:}V{\to}M$ can be extended to one from a generic extension of V to a generic extension of M, and when a subset of a generic extension of M is a set from the generic extension of M) are the main thrust of the first chapter. Also the circumstances which give rise to a particular ideal (which can be made saturated depending on the forcing used) are discussed as well.

Starting with an elementary embedding $j{:}V{\to}M$ with the critical point κ, restricted iterated forcing is used to obtain a poset $B = P * Q$ so that B is a regular suborder of $j(P)$, and in any generic extension of V via B there is an ideal $\mathcal{I}$ over κ so that $\wp(\kappa)/\mathcal{I}$ can be embedded into Boolean completion of $j(P)/B$, and hence it inherits the saturatedness of $j(P)/B$. The extension of the elementary embedding j to one from $V[G]$ to $M[H]$ can satisfy (if the circumstances are right) the "transfer" property, i.e. for every object X from $V[G]$ of certain size $j''X{\in}M[H]$, and of course $j(X){\in}M[H]$. In many situations $j''X$ becomes a "subobject" of $j(X)$, lending itself to prove properties like Chang's conjecture, or the graph one proven by Forman-Laver.

In Chapter 2 three different models for various saturated ideals are produced via restricted iterated collapse of a huge cardinal. In the first chapter the author tried to set up the machinery of restricted iterated forcing so that only certain properties of the forcing to be iterated must be checked for all pieces to fit together to get the posets P, Q, and B so that B can be regularly embedded into $j(P)$. Some additional properties of the model V^B then follow from properties of P and/or $j(P)/B$.

Lately the field has been quite active by efforts of Forman, Magidor, and Shelah (see [FMS]) who obtained a model where MM (Martin's Maximum Axiom) holds by collapsing "just" a supercompact cardinal to $\aleph_1$. Some of the consequences of MM are that $2^{\aleph}_0 = \aleph_2$ and the non-stationary ideal over $\aleph_1$ is $\aleph_2$-saturated. Forman, Magidor, and Shelah (private communication) using a forcing similar to the one used to produce a model where MM holds, obtained a model where GCH holds and the non-stationary ideal over $\aleph_1$ is "somewhere" $\aleph_2$-saturated (i.e. the restriction of the ideal to a stationary subset of $\aleph_1$ is $\aleph_2$-saturated). It seems at the moment that huge cardinals can give rise to some "fancy" saturated ideals, while supercompact cardinals can give similar results as far as $\aleph_2$-saturatedness is concerned.

Let us mention an interesting open problem. Although a supercompact cardinal is enough to get an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_1$, and a huge cardinal cardinal is enough to get an $\aleph_1$-complete $\aleph_3$-saturated ideal over $\aleph_3$, a model with an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_2$ is not known. Note that if $\aleph_\omega < \kappa < \aleph_{\omega_1}$, then there is no an $\aleph_1$-complete $\aleph_2$-saturated ideal over κ (see [F$_2$]). For completeness, Woodin (see [W]) obtained a model of ZFC with an $\aleph_1$-complete ideal over $\aleph_1$ which has a dense set of size $\aleph_1$ via the axiom of determinacy. He can now obtain (private communication) enough of determinacy by collapsing a huge cardinal to $\aleph_1$ to use similar construction to get a model of ZFC with an $\aleph_1$-complete $\aleph_1$-dense ideal over $\aleph_1$, but it is a completely different approach from the one presented in this paper.

The motivation of the author to undertake writing of this paper was twofold: first is the scarcity of published literature in this area (no wonder due to the enormous technicality of the subject), and second the non-existence of expository literature in the topic allowing non-experts to understand and use the methods. The author sincerely hopes that this paper will succeed in at least partially filling both gaps.

Notation and basic definitions.

For all basic notations about sets see [J] and [K$_1$], about forcing and iterated forcing see [J],[K$_1$], and [B]. For definitions and properties of large cardinals see [MK] and [SRK].

We are using as standard set-theoretical notation as possible. To distinguish formulas from text, they are enclosed between ‶ 〃, e.g. ‶x∈X〃. WLOG abbreviates "without loss of generality", $\emptyset$ denotes the empty set. Lower case Greek letters are reserved for ordinal numbers. *Ord* denotes the class of all ordinal numbers. If X is a set, then $|X|$ denotes its size (cardinality). If f is a function, $dom(f)$ denotes its domain, while $rng(f)$ denotes its range. If X is a set, then $f``X$ denotes the range of the function $f|X$ (f restricted to X). For a set X, $\wp(X)$ denotes the power set of X, while $[X]^{<\gamma}$ denotes the set of all subsets of X of size $< \gamma$, and $[X]^{\leq\gamma}$ denotes the set of all subsets of X of size $\leq \gamma$, and $[X]^{\gamma}$ denotes the set of all subsets of X of size γ. If X and Y are sets, then ${}^XY$ denotes the set of all functions from X into Y, and for an ordinal γ, ${}^{<\gamma}X = \bigcup\{{}^{\alpha}X : \alpha < \gamma\}$, while ${}^{\leq\gamma}X = \bigcup\{{}^{\alpha}X : \alpha \leq \gamma\}$. If V is the set universe, the cumulative hierarchy of sets $\langle V_\alpha : \alpha \in Ord\rangle$ is defined by $V_0 = \emptyset$, $V_{\alpha+1} = V_\alpha \cup \wp(V_\alpha)$, and $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$ for α limit. Then $V = \bigcup\{V_\alpha : \alpha \in Ord\}$. A set X is said to have rank α, if $X \in V_{\alpha+1} - V_\alpha$.

Let P be a poset (i.e. a set partially ordered by $\leq$). Let $p \in P$, and let $D \subset P$. Then $p << D$ iff $p \leq d$ for every $d \in D$. D is dense in P iff for any $p \in P$ there is $d \in D$ so that $d \leq p$, while D is said to be dense below p iff for any $p' \leq p$ there is $d \in D$ so that $d \leq p'$. $p, p' \in P$ are compatible, we shall denote it by $p\supset\subset q$, if there is $q\in P$ so that $q\leq p,p'$. $p\supset\subset q$ will denote that p and q are incompatible. D is an antichain, iff D consists of pairwise incompatible elements. P satisfies the $\underline{\kappa\text{-c.c.}}$ iff for every $X\in[P]^\kappa$ there are $p,q\in X$ so that they are compatible in P. P satisfies the $\underline{(\kappa,\kappa,\mu)\text{-c.c.}}$ iff for any $X\in[P]^\kappa$ there is $Y\in[X]^\kappa$ so that for any $Z\in[Y]^\mu$ there is $z\in P$ so that $z<<Z$. P satisfies the $\underline{(\kappa,\kappa,<\mu)\text{-c.c.}}$ iff P satisfies the (κ,κ,γ)-c.c. for every $\gamma<\mu$. D is directed iff every two elements of D are compatible. D is centered iff for any $D_0\in[D]^{<\omega}$ there is $p\in P$ so that $p<<D_0$. P is $\underline{\kappa\text{-centered}}$ iff P is a union of κ centered posets. P is said to be $\underline{\lambda\text{-closed}}$ ($\underline{<\lambda\text{-closed}}$) if for every $\xi\leq\lambda$ ($\xi<\lambda$) and every descending sequence $\langle p_\alpha:\alpha\leq\xi\rangle$ of elements of P there is $p\in P$ so that $p\leq p_\alpha$ for every $\alpha\leq\xi$.

$\mathcal{I}$ is a $\underline{\kappa\text{-complete ideal over }\lambda}$ iff (i) $\mathcal{I}\subset\wp(\lambda)$; and (ii) if $X\subset Y\subset\lambda$, and $Y\in\mathcal{I}$, then $X\in\mathcal{I}$; and (iii) if $\{X_\alpha:\alpha<\xi\}$, and $\xi<\kappa$, then $\bigcup\{X_\alpha:\alpha<\xi\}\in\mathcal{I}$; (iv) $\emptyset\in\mathcal{I}$; and (v) $\lambda\notin\mathcal{I}$; and (vi) for every $\alpha\in\lambda$, $\{\alpha\}\in\mathcal{I}$. (Note: usually a κ-complete ideal is defined as one satisfying (i)-(iv), a proper ideal is one satisfying (v), non-principal ideal as one satisfying (vi). Since we shall deal only with proper, non-principal ideals, we included these properties right in the definition.) $\mathcal{I}^+=\{X\subset\kappa:X\notin\mathcal{I}\}$. $\mathcal{I}$ is $\underline{\kappa\text{-saturated}}$ iff $\{X_\alpha:\alpha<\kappa\}\subset\mathcal{I}^+$, then $X_\alpha\cap X_\beta\notin\mathcal{I}$ for some $\alpha\neq\beta<\kappa$. $\mathcal{I}$ is $\underline{(\kappa,\kappa,\mu)\text{-saturated}}$ iff for every $X\in[\mathcal{I}^+]^\kappa$ there is $Y\in[X]^\kappa$ so that for every $Z\in[Y]^\mu$, $\bigcap Z\notin\mathcal{I}$. $\mathcal{I}$ is $\underline{\kappa\text{-centered}}$ iff $\mathcal{I}=\bigcup\{\mathcal{I}^+_\alpha:\alpha<\kappa\}$, and each $\mathcal{I}^+_\alpha$ is centered, i.e. whenever $X_0,\ldots,X_n\in\mathcal{I}^+_\alpha$, then $\bigcap\{X_i:i\leq n\}\notin\mathcal{I}$.

If $M\subset V$ is a transitive model of ZFC, then $\mathcal{U}$ is a $\underline{\text{non-principal } M\text{-}\kappa\text{-complete } M\text{-ultrafilter over } \lambda}$ iff (i) $\mathcal{U}\subset\wp(\lambda)\cap\mathcal{M}$; and (ii) if $X\subset Y\subset\lambda$, and $X\in\mathcal{Y}$, and $Y\in M$, then $Y\in\mathcal{U}$; and (iii) if $\{X_\alpha:\alpha<\xi\}\in M$, and $\xi<\kappa$, then $\bigcap\{X_\alpha:\alpha<\xi\}\in\mathcal{U}$; and (iv) if $X\subset\lambda$, $X\in M$ and $X\notin\mathcal{U}$, then $\lambda-X\in\mathcal{U}$; and (v) for every $\alpha\in\lambda$, $\{\alpha\}\notin\mathcal{U}$.

If P is a forcing notion (i.e. a poset), $p\in P$, then $p\Vdash^V_P$ "ϕ" (read p forces over V that ϕ) means that for any G P-generic over V, so that $p\in G$, $V[G]\models$ "ϕ". Symbol 1_P denotes the greatest element of P (if it exists). $\Vdash^V_P$ "ϕ" means that $p\Vdash^V_P$ "ϕ" for any $p\in P$, which is equivalent to $1_P\Vdash^V_P$ "ϕ" if P has the greatest element, and also it is equivalent to $V[G]\models$ "ϕ" for any G P-generic over V. If P and Q are posets, $P\simeq Q$ denotes that they are isomorphic. $P\subset\subset Q$ denotes that P is a complete suborder (see Def. 13). $P\times Q$ is a poset of ordered pairs $\langle p,q\rangle$, $p\in P$, $q\in Q$ ordered by $\langle p,q\rangle\leq\langle p',q'\rangle$ iff $p\leq p'$ and $q\leq q'$. Then $P\subset\subset P\times Q$, as well as $Q\subset\subset P\times Q$.

As much as possible we shall adhere to denoting forcing terms (i.e. names, see Def. 5) with $\circ$ accent. E.g. $\mathring{X}\in V^P$ denotes a V^P-term. When forcing with P, every object X from the ground model V has a canonical name by which V is embedded into $V[G]$. For simplicity we shall use the same notation for the canonical name for X as for X itself. The notion of iterated forcing $P*\mathring{Q}$ represents a poset of pairs $\langle p,\mathring{q}\rangle$ so that $p\in P$, and $\Vdash^V_P$ "$\mathring{q}\in\mathring{Q}$ & $\mathring{Q}$ is a poset", with the order defined by $\langle p,\mathring{q}\rangle\leq\langle p',\mathring{q}'\rangle$ iff $p\leq p'$ and $p'\Vdash^V_P$ "$\mathring{q}\leq\mathring{q}'$". It is standard (see e.g. [J], [K$_1$]) that if $P\subset\subset Q$, then there is $(Q/P)\in V^P$ so that $P*(Q/P)\simeq Q$. A sequence $\langle P_\alpha:\alpha\leq\kappa\rangle$ is a forcing iteration iff for every $\alpha<\kappa$, there is $\mathring{R}_\alpha\in V^{P_\alpha}$ so that $P_{\alpha+1}=P_\alpha*\mathring{R}_\alpha$. For α limit $p\in P_\alpha$ iff p is an α-sequence so that $p|\beta\in P_\beta$ and $p|\beta\Vdash^V_{P_\beta}$ "$p(\beta)\in\mathring{R}_\beta$" and $supp(p)=\{\beta\in\alpha: p|\beta\Vdash^V_{P_\beta}$ "$p(\beta)\neq 1_{\mathring{R}_\beta}$"$\}$ satisfies some specified properties. For example, if at every limit stage we require that only the sequences with finite support are taken, it is called finite support iteration. Different ideals for support may be used (see [K$_1$]).

Let B be a Boolean algebra, and let ϕ be a formula (using V^B-terms). The symbol $\|\phi\|_B$ denotes the Boolean value of ϕ (see [J]). Symbols O_B, 1_B denote the least and greatest elements of B. $Comp(P)$ for a poset P denotes its Boolean completion. Let $p\in P$. Then $p\Vdash^V_P$ "ϕ" iff $p\leq\|\phi\|_{Comp(P)}$.

Let formula ϕ define a set. Let P be a poset. Let M be a model of ZFC. Then ϕ^P denotes the V^P-term for the set ϕ defines in a generic extension of V via P, and ϕ^M denotes the set ϕ defines in M.

An uncountable regular cardinal α is inaccessible (we shall abbreviate it by inacc.) iff $2^\lambda<\alpha$ for every $\lambda<\alpha$.

If M is a model of ZFC, $j{:}V\to M$ is an elementary embedding iff for any formula $\phi(X_0,\ldots,X_n)$ with $n{+}1$ free variables and no constants, and any $A_0,\ldots,A_n\in V$, $V\models$ "$\phi(A_0,\ldots,A_n)$" iff

$M\models$ "$\phi(j(A_0),\ ...\ ,j(A_n))$". An ordinal κ is the critical point of j iff $j(\alpha)=\alpha$ for all $\alpha<\kappa$, and $j(\kappa)>\kappa$ (such κ must be at least a measurable cardinal - see [J]). If $V_\alpha = M_\alpha$ for all $\alpha\le\kappa$, then $j(x)=x$ for every $x\in V_\kappa$.

Chapter 1.

Def. 1: Let $j{:}V\to M$ be an elementary embedding with critical point κ. Let $M\subset V$ and let j be definable in V. j is **huge** if ${}^{j(\kappa)}M\subset M$ (where ${}^{j(\kappa)}M$ is defined in V).

Def. 2: Let ρ be an ordinal. $X\subset Ord$ is **ρ-Easton** if $|X\cap\gamma|<\gamma$ for all regular $\gamma>\rho$.

Lemma 3: Let λ be Mahlo, $\rho\ge\omega$, and let $X\subset\lambda$ be ρ-Easton. Then X is bounded below λ, i.e. $|X|<\lambda$.

Proof: $A=\{\gamma:\rho\le\gamma<\lambda\ \&\ \lambda \text{ regular}\}$ is stationary in λ since λ is Mahlo. For every $\gamma\in A$ define $f(\gamma)$ as the least ν so that $X\cap\gamma\subset\nu$. Since X is ρ-Easton, f is regressive and so by Fodor's theorem there are a stationary $B\subset A$ and $\sigma<\lambda$ so that $X\cap\gamma\subset f(\gamma)=\sigma$ for all $\gamma\in B$. Hence $X\subset\sigma$. $\square$

Lemma 4: Let $j{:}V\to M$ be huge with critical point κ. For all $\alpha\le j(\kappa)$, all $\rho\ge\omega$, and all $X\subset j(\kappa)$

(4.1) "α is a cardinal" iff $M\models$ "α is a cardinal";
(4.2) "α is regular" iff $M\models$ "α is regular";
(4.3) "α is weakly inaccessible" iff $M\models$ "α is weakly inaccessible";
(4.4) "α is inaccessible" iff $M\models$ "α is inaccessible";
(4.5) "α is Mahlo" iff $M\models$ "α is Mahlo";
(4.6) "X is ρ-Easton and $X\in M$" iff $M\models$ "X is ρ-Easton".
(4.6) "X is a ρ-Easton subset of $j(\kappa)$" iff $M\models$ "X is a ρ-Easton subset of $j(\kappa)$".

Proof: Left to the reader. $\square$

Def. 5: Let P be a poset. Then $V_0^P=\emptyset$, $V_{\alpha+1}^P=V_\alpha^P\cup\wp(V_\alpha^P\times P)$, $V_\alpha^P=\bigcup\{V_\beta^P : \beta<\alpha\}$ for α limit, and $V^P=\bigcup\{V_\alpha^P : \alpha\in Ord\}$.

Lemma 6: Let P be a poset so that $P\in V_\lambda$, λ a regular cardinal. Then

(6.1) $V_\alpha^P\in V_\lambda$ for every $\alpha<\lambda$;
(6.2) $V_{\alpha+n}^P\subset V_{\alpha+3n}$ for every $\alpha\ge\lambda$, α limit, and every $n\in\omega$.

Proof: Left to the reader. $\square$

Lemma 7: Let P be a poset. If $X\in V^P$ and $X\in V_\alpha$, then $X\in V_\alpha^P$.

Proof: Left to the reader. $\square$

Lemma 8: Let P be a poset. If $X\in V^P$, then $X\cap V_\alpha\in V_{\alpha+1}^P$.

Proof: Left to the reader. $\square$

Lemma 9: The maximum principle.
Let P be a poset, A an antichain in P (i.e. a set of mutually incompatible elements of P). For each $a\in A$, let $\mathring{X}_a\in V_{\alpha_a}^P$, α_a ordinals. Then there is a $\mathring{X}\in V_{\alpha+1}^P$, $\alpha=\bigcup\{\alpha_a : a\in A\}$, so that $a\Vdash_P$ "$\mathring{X}=\mathring{X}_a$" for every $a\in A$.

Proof: Define $\mathring{X}$ by $\langle\mathring{Y},p\rangle\in\mathring{X}$ iff for some $a\in A$, $\mathring{Y}\in dom(\mathring{X}_a)$, $p\le a$ and $p\Vdash_P$ "$\mathring{Y}\in\mathring{X}_a$". Then $dom(\mathring{X})=\bigcup\{dom(\mathring{X}_a) : a\in A\}\subset\bigcup\{V_{\alpha_a}^P : a\in A\}=V_\alpha^P$. Now, to prove that $a\Vdash_P$ "$\mathring{X}=\mathring{X}_a$" for each $a\in A$ is fairly standard (see e.g. [K$_1$]) and hence left to the reader. $\square$

Lemma 10: Let P be a poset, $p \in P$, $\overset{\circ}{X} \in V^P$ and $\alpha + n \geq 1$, where $\alpha = 0$, or α is a limit ordinal, and $n \in \omega$. Let $p \Vdash_P$ "$\overset{\circ}{X}$ has rank at most $\alpha + n$". Then there are $q \leq p$ and $\overset{\circ}{Y} \in V^P_{\alpha+2n}$ so that $q \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{Y}$".

Proof: By contradiction. Let α, n be the least such that the negation holds.

(1) If $n = 0$ (so α is limit), then $p \Vdash_P$ "$(\exists \beta < \alpha)(\overset{\circ}{X}$ has rank at most $\beta)$". There are $q \leq p$ and $\beta < \alpha$ so that $q \Vdash_P$ "$\overset{\circ}{X}$ has rank at most β". By the minimality of α, n there are $\bar{q} \leq q$ and $\overset{\circ}{Y} \in V^P_\beta$ such that $\bar{q} \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{Y}$", a contradiction.

(2) So $n \geq 1$.
Let $t = \langle \overset{\circ}{x}, q \rangle \in \overset{\circ}{X}$. Define $D_t = \{r \leq q : (r$ incompatible with $p)$ or $(r \leq p$ and for some $\overset{\circ}{z} \in V^P_{\alpha+2n-2}$, $r \Vdash_P$ "$\overset{\circ}{z} = \overset{\circ}{x}$"$)\}$. We claim that D_t is dense below q.
Let $q' \leq q$. We are to show that there is $q'' \leq q'$ so that $q'' \in D_t$. If q' is incompatible with p, then q' is in D_t and we are done. If on the other hand q' is compatible with p, then there is $\bar{q} \leq p, q'$. Then $\bar{q} \Vdash_P$ "$\overset{\circ}{x} \in \overset{\circ}{X}$ and has rank at most $\alpha + n - 1$". By the minimality of α, n there are $q'' \leq \bar{q}$ and $\overset{\circ}{z} \in V^P_{\alpha+2n-2}$ so that $q'' \Vdash_P$ "$\overset{\circ}{x} = \overset{\circ}{z}$". Therefore $q'' \in D_t$ and we are done. The claim is proven.
Let A_t be a maximal antichain in D_t. For every $a \in A_t$ there is $\overset{\circ}{z}_a \in V^P_{\alpha+2n-2}$ so that $a \Vdash_P$ "$\overset{\circ}{x} = \overset{\circ}{z}_a$" whenever a is compatible with p. By Lemma 9 there is $\overset{\circ}{R_t} \in V^P_{\alpha+2n-1}$ so that $a \Vdash_P$ "$\overset{\circ}{R_t} = \overset{\circ}{z}_a$" for every $a \in A_t$, hence $a \Vdash_P$ "$\overset{\circ}{R_t} = \overset{\circ}{x}$" whenever $a \in A_t$ si compatible with p. Define $\overset{\circ}{Y} = \{\langle \overset{\circ}{R_t}, q \rangle : \langle \overset{\circ}{x}, q \rangle \in \overset{\circ}{X}\}$. Then $\overset{\circ}{Y} \in V^P_{\alpha+2n}$.
Let G be P-generic over V so that $p \in G$. If $t = \langle \overset{\circ}{x}, q \rangle \in \overset{\circ}{X}$ and $q \in G$, then some $a \in A_t$ is in G (and hence compatible with p) and since $a \Vdash_P$ "$\overset{\circ}{R_t} = \overset{\circ}{x}$", $(\overset{\circ}{x})^G = (\overset{\circ}{R_t})^G$. Thus $(\overset{\circ}{X})^G = \{(\overset{\circ}{x})^G : t = \langle \overset{\circ}{x}, q \rangle \in \overset{\circ}{X}$ & $q \in G\} = \{(\overset{\circ}{x})^G : \langle \overset{\circ}{R_t}, q \rangle \in \overset{\circ}{Y}$ & $q \in G\} = \{(\overset{\circ}{R_t})^G : \langle \overset{\circ}{R_t}, q \rangle \in \overset{\circ}{Y}$ & $q \in G\} = (\overset{\circ}{Y})^G$. Therefore there is some $q \leq p$ so that $q \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{Y}$" and $\overset{\circ}{Y} \in V^P_{\alpha+2}$, a contradiction. $\square$

Lemma 11: Let P be a λ-c.c. poset, λ a regular cardinal, $p \in P$ and $\overset{\circ}{X} \in V^P$. Let $p \Vdash_P$ "$\overset{\circ}{X}$ has rank at most λ". Then there is $\overset{\circ}{Y} \in V^P_\lambda$ so that $p \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{Y}$".

Proof: By Lemma 10, $D = \{q \leq p : (\exists \overset{\circ}{Y} \in V^P_\lambda)(q \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{Y}$"$)\}$ is dense below p. Let A be a maximal antichain in D. For every $a \in A$ there is $\overset{\circ}{X}_a \in V^P_\lambda$ be so that $a \Vdash_P$ "$\overset{\circ}{X} = \overset{\circ}{X}_a$". Since λ is regular and $|A| < \lambda$ (as P satisfies the λ-c.c.), there is $\beta < \lambda$ so that $\overset{\circ}{X}_a \in V^P_\beta$ for each $a \in A$. By Lemma 9, there is $\overset{\circ}{Y} \in V^P_{\beta+1} \subset V^P_\lambda$ so that $a \Vdash_P$ "$\overset{\circ}{Y} = \overset{\circ}{X}_a = \overset{\circ}{X}$" for every $a \in A$. Hence $p \Vdash_P$ "$\overset{\circ}{Y} = \overset{\circ}{X}$". $\square$

Lemma 12: Let P be a λ-c.c. poset, λ a regular cardinal. Let $p \in P$, $X \in V$, $\overset{\circ}{Y} \in V^P$, and $p \Vdash_P$ "$\overset{\circ}{Y} \subset X$ & $|\overset{\circ}{Y}| \leq \lambda$". Then there are $q \leq p$ and $\hat{Y} \in V^P$ so that $|\hat{Y}| \leq \lambda$ and $q \Vdash_P$ "$\overset{\circ}{Y} = \hat{Y}$".

Proof: Let $f : \rho \to X$ be a bijection. There are $\overset{\circ}{g} \in V^P$, $\xi \leq \lambda$ and $q \leq p$ so that $q \Vdash_P$ "$\overset{\circ}{g} : \xi \to \overset{\circ}{Y}$". Hence $q \Vdash_P$ "$(\exists \beta \in \rho)(\overset{\circ}{g}(\alpha) = f(\beta))$", for any $\alpha < \xi$. Let $D_\alpha = \{r \leq q : (\exists \beta \in \rho)(r \Vdash_P$ "$\overset{\circ}{g}(\alpha) = f(\beta)$"$)\}$. Then D_α is dense below q. Let A_α be a maximal antichain in D_α. Define $\overset{\circ}{h} \in V^P$ by $\langle \langle \alpha, f(\beta) \rangle, r \rangle \in \overset{\circ}{h}$ iff $r \in A_\alpha$ and $r \Vdash_P$ "$\overset{\circ}{g}(\alpha) = f(\beta)$". Since $|A_\alpha| \leq \lambda$ for every $\alpha < \xi$, and λ is a regular cardinal in V, $|\overset{\circ}{h}| \leq \lambda$. It is left to the reader to check that (1) $\Vdash_P$ "$\overset{\circ}{h} \subset \xi \times X$", (2) $q \Vdash_P$ "$\overset{\circ}{h}$ is a function", (3) $q \Vdash_P$ "$dom(\overset{\circ}{h}) = \xi$", (4) $q \Vdash_P$ "$\overset{\circ}{h} \subset \overset{\circ}{g}$", and so $q \Vdash_P$ "$\overset{\circ}{h} = \overset{\circ}{g}$".
Now define $\hat{Y} \in V^P$ by $\hat{Y} = \{\langle f(\beta), r \rangle : (\exists \alpha < \xi)(\langle \langle \alpha, f(\beta) \rangle, r \rangle \in \overset{\circ}{h})\}$. Thus $|\hat{Y}| \leq \lambda$ and it is left to the reader to check that (5) $q \Vdash_P$ "$\overset{\circ}{h}$"$\xi \subset \hat{Y}$", (9) $q \Vdash_P$ "$\hat{Y} \subset \overset{\circ}{h}$"$\xi$". Thus $q \Vdash_P$ "$\hat{Y} = \overset{\circ}{h}$"$\xi = \overset{\circ}{g}$"$\xi = \overset{\circ}{Y}$". $\square$

Def. 13: P, Q be posets. A mapping $i : P \to Q$ is a <u>complete (regular) embedding</u> of P into Q, iff

(13.1) for every $p, q{\in}P$, if $p \leq q$ in P, then $i(p) \leq i(q)$ in Q;

(13.2) for every $p, q{\in}P$, if $p \mathrel{\supset\!\!\!\subset} q$ in P, then $i(p) \mathrel{\supset\!\!\!\subset} i(q)$ in Q;

(13.3) for every $q{\in}Q$ there is $p{\in}P$ so that whenever $p'{\in}P$ and $p' \mathrel{\supset\!\!\!\subset} p$ in P, then $i(p') \mathrel{\supset\!\!\!\subset} q$ in Q (we shall denote this relationship between p and q by $p{\prec_P}q$ in Q).

$P{\subset\!\!\subset}Q$ denotes that P is a <u>complete suborder</u> of Q, i.e. the identity is a complete embedding of P into Q.

Note: (13.3) can be replaced by: for every A, a maximal antichain (a set of mutually incompatible elements) in P, $i''A$ is a maximal antichain in Q.

Def. 14: Let P be a poset. We shall say that P is <u>separative</u> if for every $p, q{\in}P$, whenever $p \not\leq q$, then there is $p'{\in}P$ so that $p' \leq p$ and $p' \mathrel{\supset\subset} q$.

Lemma 15: Let P, Q be posets such that $P{\subset\!\!\subset}Q$ and P is separative. Let $p, p_1, p_2{\in}P$, $q{\in}Q$. Let $p{\prec_P}q$ and $q \leq p_1, p_2$ in Q. Then $p \leq p_1, p_2$ in P.

Proof: Assume that $p \not\leq p_1$ in P. Then by separativeness of P, there is $P_3{\in}P$ such that $p_3 \leq p$ and $p_3 \mathrel{\supset\subset} p_1$ in P. Thus $p_3 \mathrel{\supset\!\!\!\subset} p$ in P. Since $p{\prec_P}q$, $p_3 \mathrel{\supset\!\!\!\subset} q$ in Q. Therefore $p_3 \mathrel{\supset\!\!\!\subset} p_1$ in Q, and since $P{\subset\!\!\subset}Q$, $p_3 \mathrel{\supset\!\!\!\subset} p_1$ in P, a contradiction. Thus $p \leq p_1$, and by the same argument $p \leq p_2$. $\square$

Lemma 16: Let $j{:}V{\to}M$ be an elementary embedding. Let P be a poset. Then $j|P{:}P{\to}j(P)$ satisfies (13.1) and (13.2).

Proof: Left to the reader. $\square$

Lemma 17: Let P, Q be posets such that $P{\subset\!\!\subset}Q$. Let $\phi(x_1, ..., x_n)$ be an upward absolute formula. Let $\mathring{X}_1,...,\mathring{X}_n \in V^P$. Let $p{\in}P$, $q{\in}Q$ and $q \leq p$ in Q. Let $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $q \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$".

Proof: Let $q_1 \leq q$ in Q. Let G be Q-generic over V so that $q_1{\in}G$. Since $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$", $V[G{\cap}P]{\models}$ "$\phi(\mathring{X}_1^{G\cap P},...,\mathring{X}_n^{G\cap P})$". Since each $\mathring{X}_i{\in}V^P$, $\mathring{X}_i^{G\cap P} = \mathring{X}_i^G$, and by upward absolutness of ϕ, $V[G]{\models}$ "$\phi(\mathring{X}_1^G,...,\mathring{X}_n^G)$". Thus for some $q_2{\in}G$ $q_2 \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $q_2 \mathrel{\supset\!\!\!\subset} q_1$ in Q, and so there is $q_3{\in}Q$, $q_3 \leq q_2, q_1$ and $q_3 \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Hence $q \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". $\square$

Lemma 18: Let P, Q be posets such that $P{\subset\!\!\subset}Q$. Let $\phi(x_1, ..., x_n)$ be a downward absolute formula. Let $\mathring{X}_1,...,\mathring{X}_n \in V^P$. Let $p{\in}P$ and let $p \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$".

Proof: Let $p_1 \leq p$ in P. Then $p_1 \leq p$ in Q. Let G be Q-generic over V so that $p_1{\in}G$. Since $p_1 \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$", $V[G]{\models}$ "$\phi(\mathring{X}_1^G,...,\mathring{X}_n^G)$". Since each $\mathring{X}_i{\in}V^P$, $\mathring{X}_i^{G\cap P} = \mathring{X}_i^G$, and by downward absolutness of ϕ, $V[G{\cap}P]{\models}$ "$\phi(\mathring{X}_1^{G\cap P},...,\mathring{X}_n^{G\cap P})$". Thus for some $p_2{\in}G{\cap}P$ $p_2 \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $p_2 \mathrel{\supset\!\!\!\subset} p_1$ in P, and so there is $p_3{\in}P$, $p_3 \leq p_2, p_1$ and $p_3 \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Hence $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". $\square$

Lemma 19: Let P, Q be posets such that $P{\subset\!\!\subset}Q$. Let $\phi(x_1, ..., x_n)$ be a downward absolute formula. Let $\mathring{X}_1,...,\mathring{X}_n \in V^P$. Let $p{\in}P$, $q{\in}Q$, $p{\prec_P}q$, and let $q \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$".

Proof: Let $p_1 \leq p$ in P. Then $p_1 \mathrel{\supset\!\!\!\subset} p$ in P, and so $p_1 \mathrel{\supset\!\!\!\subset} q$ in Q. Let $q_1{\in}Q$ be such that $q_1 \leq p_1, q$ in Q. Let G be Q-generic over V so that $q_1{\in}G$. Since $q_1 \Vdash_Q$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$", $V[G]{\models}$ "$\phi(\mathring{X}_1^G,...,\mathring{X}_n^G)$". Since each $\mathring{X}_i{\in}V^P$, $\mathring{X}_i^{G\cap P} = \mathring{X}_i^G$, and by downward absolutness of ϕ, $V[G{\cap}P]{\models}$ "$\phi(\mathring{X}_1^{G\cap P},...,\mathring{X}_n^{G\cap P})$". Thus for some $p_2{\in}G{\cap}P$ $p_2 \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Then $p_2 \mathrel{\supset\!\!\!\subset} p_1$ in P, and so there is $p_3{\in}P$, $p_3 \leq p_2, p_1$ and $p_3 \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". Hence $p \Vdash_P$ "$\phi(\mathring{X}_1,...,\mathring{X}_n)$". $\square$

Def. 20: Let P,Q be posets so that $P\subset\subset Q$. Let $\mathring{R}\in V^P$ so that $\Vdash_P$ "$\mathring{R}$ *is a poset*". Define $Q\otimes_P\mathring{R} = \{\langle p,\mathring{q}\rangle : q\in Q,\ \mathring{r}\in V^P,\ q \Vdash_Q$ "$\mathring{r}\in\mathring{R}$"$\}$, $\langle q_1,\mathring{r}_1\rangle \le \langle q_2,\mathring{r}_2\rangle$ iff $q_1\le q_2$ in Q, and $q_1 \Vdash_Q$ "$\mathring{r}_1\le\mathring{r}_2$ *in* $\mathring{R}$".

Lemma 21: Let P,Q be posets such that $P\subset\subset Q$. Let $\mathring{R}\in V^P$ so that $\Vdash_P$ "$\mathring{R}$ *is a poset*". Then $Q\otimes_P\mathring{R}$ is dense in $Q*\mathring{R}$.

Proof: Obviously $Q\otimes_P\mathring{R}$ is a suborder of $Q*\mathring{R}$. Let $\langle q,\mathring{r}\rangle\in Q*\mathring{R}$. Then $q\in Q$, $\mathring{r}\in V^Q$ and $q\Vdash_Q$ "$\mathring{r}\in\mathring{R}$". Let G be Q-generic over V so that $q\in G$. Since $\mathring{R}\in V^P$ and $G\cap P$ is P-generic over V, $\mathring{R}^G=\mathring{R}^{G\cap P}\in V[G\cap P]$. Thus $\mathring{r}^G\in V[G\cap P]$, and so there is $\mathring{r}_1\in V^P$ such that $\mathring{r}_1^{G\cap P}=\mathring{r}^G$. Thus for some $q_1\in G$, $q_1\Vdash_Q$ "$\mathring{r}_1=\mathring{r}$". Then $q_1\supset\!\subset q$ in Q, and so there is $q_3\in Q$ such that $q_2\le q_1,q$ in Q and $q_2\Vdash_Q$ "$\mathring{r}_1=\mathring{r}$". Then $\langle q_2,\mathring{r}_1\rangle\in Q\otimes_P\mathring{R}$ and $\langle q_2,\mathring{r}_1\rangle\le\langle q,\mathring{r}\rangle$ in $Q*\mathring{R}$. $\square$

Lemma 22: Let P,Q,R be posets so that $P\subset\subset Q\subset\subset R$. Let $\mathring{S}\in V^P$ and let $\Vdash_P$ "$\mathring{S}$ *is s poset*". Let Q be separative. Then $Q\otimes_P\mathring{S}\subset\subset R\otimes_P\mathring{S}$.

Proof: Obviously $Q\otimes_P\mathring{S}\subset R\otimes_P\mathring{S}$.
First we verify that (13.1) holds. Let $\langle q_1,\mathring{s}_1\rangle\le\langle q_2,\mathring{s}_2\rangle$ in $Q\otimes_P\mathring{S}$. Then $q_1\le q_2$ in Q and $q_1\Vdash_Q$ "$\mathring{s}_1\le\mathring{s}_2$ *in* $\mathring{S}$". By Lemma 17 $q_1\Vdash_R$ "$\mathring{s}_1\le\mathring{s}_2$ *in* $\mathring{S}$". As $q_1\le q_2$ in R, $\langle q_1,\mathring{s}_1\rangle\le\langle q_1,\mathring{s}_1\rangle$ in $R\otimes_P\mathring{S}$.
Next we verify that (13.2) holds. Let $\langle q_1,\mathring{s}_1\rangle\supset\!\subset\langle q_2,\mathring{s}_2\rangle$ in $R\otimes_P\mathring{S}$. There is $\langle r,\mathring{s}\rangle\in R\otimes_P\mathring{S}$ so that $\langle r,\mathring{s}\rangle\le\langle q_1,\mathring{s}_1\rangle,\ \langle q_2,\mathring{s}_2\rangle$ in $R\otimes_P\mathring{S}$. Thus $r\le q_1,q_2$ in R and $r\Vdash_R$ "$\mathring{s}\le\mathring{s}_1,\ \mathring{s}_2$ *in* $\mathring{S}$". Since $Q\subset\subset R$, there is $q\in Q$ so that $q\prec_Q r$. By Lemma 5 $q\le q_1,q_2$ in Q. Since "$s\le s_1$ *in* S" is an absolute formula, by Lemma 19 $q\Vdash_Q$ "$\mathring{s}\le\mathring{s}_1,\ \mathring{s}_2$ *in* $\mathring{S}$". By Lemma 17 $q\Vdash_R$ "$\mathring{s}\le\mathring{s}_1,\ \mathring{s}_2$ *in* $\mathring{S}$". So $\langle q,\mathring{s}\rangle\in Q\otimes_P\mathring{S}$, and thus $\langle q,\mathring{s}\rangle\le\langle q_1,\mathring{s}_1\rangle,\ \langle q_2,\mathring{s}_2\rangle$ in $Q\otimes_P\mathring{S}$.
Verify that (13.3) holds. Let $\langle r,\mathring{s}\rangle\in R\otimes_P\mathring{S}$. Then $\mathring{s}\in V^P$, $r\in R$ and $r\Vdash_R$ "$\mathring{s}\in\mathring{S}$". Since $Q\subset\subset R$, there is $q\in Q$ such that $q\prec_Q r$. Since "$x\in X$" is absolute, by Lemma 19 $q\Vdash_Q$ "$\mathring{s}\in\mathring{S}$", and by Lemma 17 $q\Vdash_R$ "$\mathring{s}\in\mathring{S}$". So $\langle q,\mathring{s}\rangle\in Q\otimes_P\mathring{S}$. We shall show that $\langle q,\mathring{s}\rangle\prec\langle r,\mathring{s}\rangle$:
let $\langle q_1,\mathring{s}_1\rangle\in Q\otimes_P\mathring{S}$ so that $\langle q_1,\mathring{s}_1\rangle\supset\!\subset\langle q,\mathring{s}\rangle$ in $Q\otimes_P\mathring{S}$. Thus there is $\langle q_2,\mathring{s}_2\rangle\in Q\otimes_P\mathring{S}$ such that $\langle q_2,\mathring{s}_2\rangle\le\langle q_1,\mathring{s}_1\rangle,\ \langle q,\mathring{s}\rangle$ in $Q\otimes_P\mathring{S}$. Hence $q_2\le q_1,q$ in Q, and $q\Vdash_Q$ "$\mathring{s}_2\le\mathring{s}_1,\ \mathring{s}$ *in* $\mathring{S}$". Since $q_2\supset\!\subset q$ in Q, $q_2\supset\!\subset r$ in R. Let $r_1\in R$ so that $r_1\le r,q_2$ in R. Since "$s_2\le s_1, s$ *in* S" is absolute, by Lemma 17 $r_1\Vdash_R$ "$\mathring{s}_2\le\mathring{s}_1,\ \mathring{s}$ *in* $\mathring{S}$". Then $\langle r_1,\mathring{s}_2\rangle\in R\otimes_P\mathring{S}$ and $\langle r_1,\mathring{s}_2\rangle\le\langle r,\mathring{s}\rangle,\ \langle q_1,\mathring{s}_1\rangle$ in $R\otimes_P\mathring{S}$. Hence $\langle q_1,\mathring{s}_1\rangle\supset\!\subset\langle r,\mathring{s}\rangle$ in $R\otimes_P\mathring{S}$. $\square$

Lemma 23: Let P,Q be posets so that $P\subset\subset Q$. Let $\mathring{S},\ \mathring{R}\in V^P$ so that $\Vdash_P$ "$\mathring{R}\subset\subset\mathring{S}$". Then $Q\otimes_P\mathring{R}\subset\subset Q\otimes_P\mathring{S}$.

Proof: Obviously $Q\otimes_P\mathring{R}$ is a suborder of $Q\otimes_P\mathring{S}$.
Let's verify (13.2). Let $\langle q_1,\mathring{r}_1\rangle,\ \langle q_2,\mathring{r}_2\rangle\in Q\otimes_P\mathring{S}$ be so that $\langle q_1,\mathring{r}_1\rangle\supset\!\subset\langle q_2,\mathring{r}_2\rangle$ in $Q\otimes_P\mathring{S}$. There is $\langle q_3,\mathring{r}_3\rangle\in Q\otimes_P\mathring{S}$ so that $\langle q_3,\mathring{r}_3\rangle\le\langle q_1,\mathring{r}_1\rangle,\ \langle q_2,\mathring{r}_2\rangle$ in $Q\otimes_P\mathring{S}$. Hence $q_3\le q_1,q_2$ in Q and $q_3\Vdash_Q$ "$\mathring{s}_3\le\mathring{r}_1,\mathring{r}_2$ *in* $\mathring{S}$". So $q_3\Vdash_Q$ "$\mathring{r}_1\supset\!\subset\mathring{r}_2$ *in* $\mathring{S}$". And thus $q_3\Vdash_Q$ "$\mathring{r}_1\supset\!\subset\mathring{r}_2$ *in* $\mathring{R}$". It follows that $q_3\Vdash_Q$ "$(\exists\,\mathring{r}_3)(\mathring{r}_3\le\mathring{r}_1,\mathring{r}_2$ *in* $\mathring{R})$". There is $\mathring{r}_3\in V^Q$ so that $q_3\Vdash_Q$ "$\mathring{r}_3\le\mathring{r}_1,\mathring{r}_2$ *in* $\mathring{R}$". Then $\langle q_3,\mathring{r}_3\rangle\in Q*\mathring{R}$. By Lemma 21 there is $\langle q_4,\mathring{r}_4\rangle\in Q\otimes_P\mathring{R}$ so that $\langle q_4,\mathring{r}_4\rangle\le\langle q_3,\mathring{r}_3\rangle$ in $Q*\mathring{R}$. $q_4\Vdash_Q$ "$\mathring{r}_4\le\mathring{r}_1,\mathring{r}_2$ *in* $\mathring{R}$". Therefore $\langle q_4,\mathring{r}_4\rangle\le\langle q_1,\mathring{r}_1\rangle,\ \langle q_2,\mathring{r}_2\rangle$ in $Q\otimes_P\mathring{R}$, and so $\langle q_1,\mathring{r}_1\rangle\supset\!\subset\langle q_2,\mathring{r}_2\rangle$ in $Q\otimes_P\mathring{R}$.
Let's verify (13.3). Let $\langle q,\mathring{s}\rangle\in Q\otimes_P\mathring{S}$. Then $q\Vdash_Q$ "$\mathring{s}\in\mathring{S}$". So $q\Vdash_Q$ "$(\exists\mathring{r}\in\mathring{R})(\mathring{r}\prec_{\mathring{R}}\mathring{s})$", since

$q \Vdash_Q$ "$\mathring{R} \subset\subset \mathring{S}$". Thus there is $\mathring{r} \in V^Q$ such that $q \Vdash_Q$ "$\mathring{r} \prec_{\mathring{R}} \mathring{s}$", and so $\langle q,\mathring{r}\rangle \in Q*\mathring{R}$. By Lemma 21 there is $\langle q_1,\mathring{r}_1\rangle \in Q\otimes_P\mathring{R}$ so that $\langle q_1,\mathring{r}_1\rangle \le \langle q,\mathring{r}\rangle$ in $Q*\mathring{R}$. We shall show that $\langle q_1,\mathring{r}_1\rangle$ is the element in $Q\otimes_P\mathring{R}$ we are looking for: let $\langle q_2,\mathring{r}_2\rangle \in Q\otimes_P\mathring{R}$ so that $\langle q_2,\mathring{r}_2\rangle \mathrel{\sqsupset\!\!\sqsubset} \langle q_1,\mathring{r}_1\rangle$ in $Q\otimes_P\mathring{R}$. Then there is $\langle q_3,\mathring{r}_3\rangle \in Q\otimes_P\mathring{R}$ so that $\langle q_3,\mathring{r}_3\rangle \le \langle q_2,\mathring{r}_2\rangle, \langle q_1,\mathring{r}_1\rangle$ in $Q\otimes_P\mathring{R}$. Then $q_3 \le q_1, q_2$ in Q and $q_3 \Vdash_Q$ "$\mathring{r}_3 \le \mathring{r}_1,\mathring{r}_2$ *in* $\mathring{R}$". Also $q_3 \Vdash_Q$ "$\mathring{r}_1 \le \mathring{r} \prec_{\mathring{R}} \mathring{s}$". Thus $q_3 \Vdash_Q$ "$\mathring{r}_3 \mathrel{\sqsupset\!\!\sqsubset} \mathring{s}$ *in* $\mathring{S}$", and so $q_3 \Vdash_Q$ "$(\exists\, \mathring{s}_1)(\mathring{s}_1 \le \mathring{r}_3,\mathring{s}$ *in* $\mathring{S})$". Thus there is $\mathring{s}_1 \in V^Q$ so that $q_3 \Vdash_Q$ "$\mathring{s}_1 \le \mathring{r}_3,\mathring{s}$ *in* $\mathring{S}$". It follows that $\langle q_3,\mathring{s}_1\rangle \in Q*\mathring{S}$. By Lemma 21 there is $\langle q_4,\mathring{s}_2\rangle \in Q\otimes_P\mathring{S}$ so that $\langle q_4,\mathring{s}_2\rangle \le \langle q_3,\mathring{s}_1\rangle$. Then $q_4 \Vdash_Q$ "$\mathring{s}_2 \le \mathring{r}_3,\mathring{s}$ *in* $\mathring{S}$". Thus $\langle q_4,\mathring{s}_2\rangle \le \langle q_2,\mathring{r}_2\rangle, \langle q,\mathring{s}\rangle$ in $Q\otimes_P\mathring{S}$. Hence It follows that $\langle q_2,\mathring{r}_2\rangle \mathrel{\sqsupset\!\!\sqsubset} \langle q,\mathring{s}\rangle$ in $Q\otimes_P\mathring{S}$. $\square$

Lemma 24: Let P, Q be posets. Let $\mathring{R}\in V^P$ and let $\Vdash_P$ "$\mathring{R}$*is a separative poset*". Let Q be separative. Then $Q\otimes_P\mathring{R}$ is separative.

Proof: We shall prove that $Q*\mathring{R}$ is separative; since $Q\otimes_P\mathring{R}$ is dense in $Q*\mathring{R}$ it follows that $Q\otimes_P\mathring{R}$ must also be separative, too.
Let $\langle q_1,\mathring{r}_1\rangle \not\le \langle q_2,\mathring{r}_2\rangle$ in $Q*\mathring{R}$. There are two possible cases:
(i) $q_1 \not\le q_2$ in Q. Then by separativness of Q, there is $q_3\in Q$ such that $q_3 \le q_1$ and $q_3 \supset\!\subset q_2$ in Q. Then $\langle q_3,\mathring{r}_1\rangle \in Q*\mathring{R}$, $\langle q_3,\mathring{r}_1\rangle \le \langle q_1,\mathring{r}_1\rangle$, and $\langle q_3,\mathring{r}_1\rangle \supset\!\subset \langle q_2,\mathring{r}_2\rangle$ in $Q*\mathring{R}$.
(ii) $q_1 \le q_2$ in Q. Then $q_1 \not\Vdash_Q$ "$\mathring{r}_1 \le \mathring{r}_2$ *in* $\mathring{R}$". So there is $q_3\in Q$, $q_3 \le q_1$ in Q, so that $q_3 \Vdash_Q$ "$\mathring{r}_1 \not\le \mathring{r}_2$ *in* $\mathring{R}$", and so $q_3 \Vdash_Q$ "$(\exists\, \mathring{r}_3)(\mathring{r}_3 \le \mathring{r}_1$ *and* $\mathring{r}_3 \supset\!\subset \mathring{r}_2$ *in* $\mathring{R})$". So for some $\mathring{r}_3 \in V^Q$, $q_3 \Vdash_Q$ "$\mathring{r}_3 \le \mathring{r}_1$ *and* $\mathring{r}_3 \supset\!\subset \mathring{r}_2$ *in* $\mathring{R}$". Thus, $\langle q_3,\mathring{r}_3\rangle \in Q*\mathring{R}$. Hence $\langle q_3,\mathring{r}_3\rangle \le \langle q_1,\mathring{r}_1\rangle$, and $\langle q_3,\mathring{r}_3\rangle \supset\!\subset \langle q_2,\mathring{r}_2\rangle$ in $Q*\mathring{R}$. Hence $Q*\mathring{R}$ is separative. $\square$

Lemma 25: Let $M\subset V$ be a transitive model of ZFC.

(25.1) Let $P\in M$ be a poset and ϕ a restricted formula with n free variable and no constants. Let $p\in P$ and let $\mathring{X}_1, \dots, \mathring{X}_n\in M^P$. Then $p \Vdash_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$" iff $p \Vdash^M_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$".

(25.2) If $P\in M_\lambda$ and ${}^{<\lambda}M_\lambda\subset M$ and P satisfies the λ-c.c. in V, then "G is P-generic over V" iff "G is P-generic over M".

Proof:

(25.1) Let $p \Vdash^M_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$" and assume that $p \not\Vdash_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$". Then for some $q \le p$, $q \Vdash_P$ "$\neg\phi(\mathring{X}_1, \dots, \mathring{X}_n)$". Choose G, a P-generic filter over V (and so over M as well) so that $q\in G$. Then $V[G]\models$ "$\neg\phi(X_1, ..., X_n)$" and so (as ϕ is restricted and $X_1, ..., X_n\in M[G]$) $M[G]\models$ "$\neg\phi(X_1, ..., X_n)$", where $X_i = (\mathring{X}_i)^G$ for $i = 1, ..., n$. Since $p\in G$, $M[G]\models$ "$\phi(X_1, ..., X_n)$", a contradiction. Hence $p \Vdash_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$".
The opposite direction: let $p \Vdash_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$". Assume that $p \not\Vdash^M_P$ "$\phi(\mathring{X}_1, \dots, \mathring{X}_n)$". Then for some $q \le p$, $q \Vdash^M_P$ "$\neg\phi(\mathring{X}_1, \dots, \mathring{X}_n)$". By the same argument as above, $q \Vdash_P$ "$\neg\phi(\mathring{X}_1, \dots, \mathring{X}_n)$", a contradiction as $q \le p$.

(25.2) One direction is easy and so is left to the reader. For the other direction assume that G is P-generic over M. Let D be dense in P. Let A be a maximal antichain in D. Since $|A| < \lambda$, $A\in M$ and so $G\cap A \ne \emptyset$. Hence $G\cap D \ne \emptyset$. $\square$

Lemma 26: Let $j{:}V\to M$ be an elementary embedding, $M\subset V$, and let j be definable in V. Let $P\in V$ be a poset. Let G be P-generic over V and let H be $j(P)$-generic over M so that if $p\in G$, then $j(p)\in H$. Then

(26.1) there is an elementary embedding $\hat{\jmath}{:}V[G]\to M[H]$ definable in $V[H]$ and extending j;

(26.2) if ${}^\lambda M\subset M$ and $j(P)$ satisfies the λ-c.c. in V, λ regular, then if $X\in V$, $Y\in V[H]$, $Y\subset X$, $|Y| < \lambda$, and $Y\subset M[H]$, then $Y\in M[H]$;

(26.3) if $M_\alpha = V_\alpha$ for all $\alpha \le \lambda$, λ a limit ordinal, and if $P \in V_\lambda$, then $V[H]_\alpha = M[H]_\alpha$ for all $\alpha \le \lambda$.

Proof:

(26.1) Define $\hat{j}((\mathring{X})^G) = (j(\mathring{X}))^H$.
$\hat{j}$ is well-defined, because if $(\mathring{X})^G = (\mathring{Y})^G$, then for some $p \in G$, $p \Vdash_{\overline{P}}$ "$\mathring{X} = \mathring{Y}$", and so $j(p) \Vdash_{\overline{j(P)}}^{M}$ "$j(\mathring{X}) = j(\mathring{Y})$" by elementarity of j. Since $j(p) \in H$, $\hat{j}((\mathring{X})^G) = (j(\mathring{X}))^H = (j(\mathring{Y}))^H = \hat{j}((\mathring{Y})^G)$. $\hat{j}$ is elementary, for if $V[G] \models$ "$\phi((\mathring{X}_1)^G, \ldots ,(\mathring{X}_n)^G)$", then $p \Vdash_{\overline{P}}$ "$\phi(\mathring{X}_1, \ldots ,\mathring{X}_n)$" for some $p \in G$, and so $j(p) \Vdash_{\overline{j(P)}}^{M}$ "$\phi(j(\mathring{X}_1), \ldots ,j(\mathring{X}_n))$". Since $j(p) \in H$, $M[H] \models$ "$\phi((j(\mathring{X}_1))^H, \ldots ,(j(\mathring{X}_n))^H)$", so $M[H] \models$ "$\phi(\hat{j}((\mathring{X}_1)^G), \ldots ,\hat{j}((\mathring{X}_n)^G))$".
$\hat{j}$ extends j, for if $X \in V$, then $(X)^G = X$ and so $\hat{j}(X) = j(X)$.

(26.2) Let $\mathring{Y} \in V^{j(P)}$ so that $Y = (\mathring{Y})^H$. By Lemma 12 we can assume WLOG that $|\mathring{Y}| \le \lambda$. Let $\langle \mathring{y}, q \rangle \in \mathring{Y}$. Then $(\mathring{y})^H \in (\mathring{Y})^H = Y \subset M[H]$, so there is $\hat{y} \in M^{j(P)}$ such that $(\hat{y})^H = (\mathring{y})^H$. Define $\hat{Y} = \{\langle \hat{y}, q \rangle : \langle \mathring{y}, q \rangle \in \mathring{Y}\}$. Then $\hat{Y} \subset M^{j(P)} \subset M$ and $|\hat{Y}| = |\mathring{Y}| \le \lambda$, so $\hat{Y} \in {}^\lambda M \subset M$, hence $\hat{Y} \in M$. Since $\hat{Y} \in V^{j(P)}$, $\hat{Y} \in M^{j(P)}$ and thus $(\hat{Y})^H \in M[H]$. $(\hat{Y})^H = \{(\hat{y})^H : \langle \hat{y}, q \rangle \in \hat{Y}\} = \{(\mathring{y})^H : \langle \mathring{y}, q \rangle \in \mathring{Y}\} = (\mathring{Y})^H = Y$. Hence $Y \in M[H]$.

(26.3) will be proven by induction:

(i) $(V[H])_0 = (M[H])_0 = \emptyset$.

(ii) Assume that $(V[H])_\alpha = (M[H])_\alpha$ and $\alpha < \lambda$.
Let $(\mathring{X})^H \in (V[H])_{\alpha+1}$. Then $(\mathring{X})^H \subset (V[H])_\alpha = (M[H])_\alpha$. By Lemma 10, we can assume WLOG that $\mathring{X} \in V^P_{\alpha+2}$, and by Lemma 6, $V^P_{\alpha+2} \subset V_\lambda = M_\lambda$. Hence $(\mathring{X})^H \in M[H]$ and so $(\mathring{X})^H \in (M[H])_{\alpha+1}$. Thus $(V[H])_{\alpha+1} \subset (M[H])_{\alpha+1}$, and so $(V[H])_{\alpha+1} = (M[H])_{\alpha+1}$.

(iii) if $(V[H])_\beta = (M[H])_\beta$ for all $\beta < \alpha \le \lambda$, α limit, then $(V[H])_\alpha = (M[H])_\alpha$. $\square$

Lemma 27: Let $j{:}V \to M$ be huge with critical point κ. Let $P \in V_\kappa$ be a poset. Then

(27.1) $j(p) = p$ for all $p \in P$;

(27.2) $j(P) = P$;

(27.3) G is P-generic over V iff G is P-generic over M;

(27.4) for any G P-generic over V, there is a nearly huge $\hat{j}{:}V[G] \to M[G]$ extending j.

Proof: (27.1) and (27.2) are easy and so they are left to the reader to prove.

(27.3) follows directly from Lemma 25.

(27.4) follows from Lemma 26. $\square$

Properties 28: Let $C(\gamma, \delta)$, $\gamma < \delta$ cardinals, define in V a poset.

(28.1) $C(\gamma, \delta) \subset V_\delta$ for all cardinals $\gamma < \delta$, δ inacc.;

(28.2) $C(\gamma, \tau) = \{s \cap V_\tau : s \in C(\gamma, \delta)\}$, for all cardinals $\gamma < \tau \le \delta$, τ, δ inacc.;

(28.3) for every $s \in C(\gamma, \delta)$, $s \le s \cap V_\tau$ in $C(\gamma, \delta)$, for all cardinals $\gamma < \tau \le \delta$, τ, δ inacc.;

(28.4) for every $s \in C(\gamma, \delta)$, $s \cap V_\tau \prec_{C(\gamma,\tau)} s$ in $C(\gamma, \delta)$, for all cardinals $\gamma < \tau \le \delta$, τ, δ inacc.;

(28.5) $C(\gamma, \tau) \subset\subset C(\gamma, \delta)$, for all cardinals $\gamma < \tau \le \delta$, τ, δ inacc.;

(28.6) $C(\gamma, \delta)$ is separative for all cardinals $\gamma \le \delta$, δ inacc.

Properties 29: Let $I = \langle I_\alpha : \alpha \le \kappa, \alpha \text{ limit } \rangle$ be a sequence such that:

(29.1) I_α is an ideal on α containing all finite subsets of α, for all limit $\alpha \le \kappa$;

(29.2) $I_\alpha \subset I_\beta$ for all limit $\alpha \le \beta \le \kappa$;

(29.3) if α is inaccessible, than $x \in I_\alpha$ implies that $|x| < \alpha$.

Lemma 30: Let $|\{\alpha \in \kappa : \alpha \text{ inaccessible }\}| = \kappa$. Let $I = \langle I_\alpha : \alpha \text{ limit}, \alpha \le \kappa \rangle$ be a sequence of ideals satisfying (29.1) - (29.3). Let C define a poset and satisfy (28.1) - (28.6). Then there exists an (iterated forcing) sequence $\langle P_\alpha : \alpha \le \kappa \rangle$ such that

(30.1) $P_0 = C(\omega_0,\kappa)$;

(30.2) $P_{\alpha+1} = P_\alpha * \{\emptyset\}$ whenever $\alpha<\kappa$ is not inaccessible;

(30.3) $P_{\alpha+1} = P_\alpha \otimes_{P_\alpha\uparrow V_\alpha} C(\alpha,\kappa)^{P_\alpha\uparrow V_\alpha}$ whenever $\alpha<\kappa$ is inaccessible, where $C(\alpha,\kappa)^{P_\alpha\uparrow V_\alpha}$ denotes $C(\alpha,\kappa)$ as defined in the extension by $P_\alpha\uparrow V_\alpha$;
(this is a sound definition since it follows from (30.7) - (30.10) that $P_\alpha\uparrow V_\alpha \subset\subset P_\alpha$, see (3*) below) where for any β inacc. such that $\alpha\le\beta\le\kappa$ we define $P_\alpha\uparrow V_\beta = \{p\uparrow V_\beta : p\in P_\alpha\}$, and $p\uparrow V_\beta$ is defined as follows: $(p\uparrow V_\beta)(0) = p(0)\cap V_\beta$, if $p(\xi)=\emptyset$, then $(p\uparrow V_\beta)(\xi)=\emptyset$, and when $p(\xi)\neq\emptyset$ (hence ξ is inacc.), then $(p\uparrow V_\beta)(\xi)\in V_\beta^{P_\xi\uparrow V_\xi}$ so that $\Vdash_{P_\xi\uparrow V_\xi}$ "$(p\uparrow V_\beta)(\xi) = p(\xi)\cap V_\beta^{P_\xi\uparrow V_\xi}$" (such $P_\xi\uparrow V_\xi$-name in V_β exists by Lemma 11 as ξ is inacc.). The ordering is defined by $p\uparrow V_\beta \le q\uparrow V_\beta$ in $P_\alpha\uparrow V_\beta$ if $p\uparrow V_\beta\le q\uparrow V_\beta$ in P_α (this is a sound definition since by (30.7) $P_\alpha\uparrow V_\beta\subset P_\alpha$)

(30.4) For $\alpha\le\kappa$ limit, P_α consists of all limits of conditions of $\langle P_\beta : \beta<\alpha\rangle$ with support from I_α;

And furthermore the sequence $\langle P_\alpha : \alpha\le\kappa\rangle$ satisfies:

(30.5) P_α is separative for every $\alpha\le\kappa$;

(30.6) $P_\alpha\subset V_\kappa$ for every $\alpha<\kappa$;

for every inacc. $\alpha\le\kappa$, any inacc. β such that $\alpha\le\beta\le\kappa$:

(30.7) $P_\alpha\uparrow V_\beta \subset P_\alpha$;

(30.8) $P_\alpha\uparrow V_\beta$ is separative;

(30.9) $p \supset\subset q$ in $P_\alpha\uparrow V_\beta$ iff $p\supset\subset q$ in P_α for any $p,q\in P_\alpha\uparrow V_\beta$;

(30.10) $p\uparrow V_\beta \prec_{P_\alpha\uparrow V_\beta} p$ in P_α, for any $p\in P_\alpha$;

(30.11) $p\uparrow V_\beta \ge p$ in P_α, for any $p\in P_\alpha$;

(30.12) $\{p|supp(p) : p\in P_\alpha\uparrow V_\alpha\}\subset V_\alpha$.

Proof:

(1*) For any $\alpha\le\kappa$, any $\gamma<\alpha$, and any β such that $\gamma<\beta\le\kappa$, $(p|\gamma)\uparrow V_\beta = (p\uparrow V_\beta)|\gamma$, for any $p\in P_\alpha$.
$((p|\gamma)\uparrow V_\beta)(0) = (p|\gamma)(0)\cap V_\beta = p(0)\cap V_\beta = (p\uparrow V_\beta)(0) = ((p\uparrow V_\beta)|\gamma)(0)$. If $p(\xi)=\emptyset$, then $(p|\gamma)(\xi)=\emptyset$ and so $((p|\gamma)\uparrow V_\beta)(\xi) = \emptyset = (p\uparrow V_\beta)(\xi) = ((p\uparrow V_\beta)|\gamma)(\xi)$. If $p(\xi)\neq\emptyset$, then ξ is inacc. and $\xi<\gamma$. Then $((p|\gamma)\uparrow V_\beta)(\xi)\in V_\beta^{P_\xi\uparrow V_\xi}$ so that $\Vdash_{P_\xi\uparrow V_\xi}$ "$((p|\gamma)\uparrow V_\beta)(\xi) = (p|\gamma)(\xi)\cap V_\beta^{P_\xi\uparrow V_\xi}$", hence $\Vdash_{P_\xi\uparrow V_\xi}$ "$((p|\gamma)\uparrow V_\beta)(\xi) = p(\xi)\cap V_\beta^{P_\xi\uparrow V_\xi}$", thus $((p|\gamma)\uparrow V_\beta)(\xi)$ and $(p\uparrow V_\beta)(\xi)$ are names for the same object.

(2*) For any $\alpha\le\kappa$, any $\alpha<\gamma\le\beta\le\kappa$, γ,β inacc., $p\uparrow V_\gamma = (p\uparrow V_\beta)\uparrow V_\gamma = (p\uparrow V_\gamma)\uparrow V_\beta$, for any $p\in P_\alpha$.
$(p\uparrow V_\gamma)(0) = p(0)\cap V_\gamma$. $((p\uparrow V_\gamma)\uparrow V_\beta)(0) = (p\uparrow V_\gamma)(0)\cap V_\beta = (p(0)\cap V_\gamma)\cap V_\beta = p(0)\cap V_\gamma$.
$((p\uparrow V_\beta)\uparrow V_\gamma)(0) = (p\uparrow V_\beta)(0)\cap V_\gamma = (p(0)\cap V_\beta)\cap V_\gamma = p(0)\cap V_\gamma$. Thus $(p\uparrow V_\gamma)(0) = ((p\uparrow V_\gamma)\uparrow V_\beta)(0) = ((p\uparrow V_\beta)\uparrow V_\gamma)(0)$. If $p(\xi)=\emptyset$, then $(p\uparrow V_\gamma)(\xi)=\emptyset$, $((p\uparrow V_\gamma)\uparrow V_\beta)(\xi)=\emptyset$, and $((p\uparrow V_\beta)\uparrow V_\gamma)(\xi)=\emptyset$. Thus $(p\uparrow V_\gamma)(\xi) = ((p\uparrow V_\gamma)\uparrow V_\beta)(\xi) = ((p\uparrow V_\beta)\uparrow V_\gamma)(\xi)$. If $p(\xi)\neq\emptyset$, then ξ is inacc. and $\xi\le\gamma$, and so $((p\uparrow V_\gamma)(\xi)\in V_\gamma^{P_\xi\uparrow V_\xi}$ so that $\Vdash_{P_\xi\uparrow V_\xi}$ "$(p\uparrow V_\gamma)(\xi) = p(\xi)\cap V_\gamma^{P_\xi\uparrow V_\xi}$", and $((p\uparrow V_\beta)(\xi)\in V_\beta^{P_\xi\uparrow V_\xi}$ so that $\Vdash_{P_\xi\uparrow V_\xi}$ "$(p\uparrow V_\beta)(\xi) = p(\xi)\cap V_\beta^{P_\xi\uparrow V_\xi}$", therefore $((p\uparrow V_\beta)\uparrow V_\gamma)(\xi)\in V_\gamma^{P_\xi\uparrow V_\xi}$ so that $\Vdash_{P_\xi\uparrow V_\xi}$ "$((p\uparrow V_\beta)\uparrow V_\gamma)(\xi) = (p\uparrow V_\beta)(\xi)\cap V_\gamma^{P_\xi\uparrow V_\xi}$", and hence $\Vdash_{P_\xi\uparrow V_\xi}$ "$((p\uparrow V_\beta)\uparrow V_\gamma)(\xi) = (p(\xi)\cap V_\beta^{P_\xi\uparrow V_\xi})\cap V_\gamma^{P_\xi\uparrow V_\xi} = p(\xi)\cap V_\gamma^{P_\xi\uparrow V_\xi}$". So $(p\uparrow V_\gamma)(\xi)$ and $((p\uparrow V_\beta)\uparrow V_\gamma)(\xi)$ are names for the same object. Similarly for $((p\uparrow V_\gamma)\uparrow V_\beta)$.

(3*) For any $\alpha\le\kappa$, any inacc. β such that $\alpha<\beta\le\kappa$, $P_\alpha\uparrow V_\beta\subset\subset P_\alpha$.
To verify (13.1) notice that by (30.7) and by the definiton of the order on $P_\alpha\uparrow V_\beta$, $P_\alpha\uparrow V_\beta$ is a suborder of P_α.
(13.2) is in fact (30.9).
(13.3) follows from (30.10).

(4*) For any $\alpha \le \kappa$, any inacc. β and inacc. γ such that $\alpha < \gamma \le \beta \le \kappa$, $P_\alpha{\upharpoonright}V_\gamma \subset\subset P_\alpha{\upharpoonright}V_\beta$.

Let $p \in P_\alpha{\upharpoonright}V_\gamma$. Then $p = q{\upharpoonright}V_\gamma$ for some $q \in P_\alpha$. By (30.7) $p \in P_\alpha$, so $p{\upharpoonright}V_\beta \in P_\alpha{\upharpoonright}V_\beta$. By (2*) $p{\upharpoonright}V_\beta = p$, so $p \in P_\alpha{\upharpoonright}V_\beta$. Hence $P_\alpha{\upharpoonright}V_\gamma \subset P_\alpha{\upharpoonright}V_\beta$, and by the definition (see 30.3) it is a suborder.

Let $p, q \in P_\alpha{\upharpoonright}V_\beta$ so that $p \supset\!\!\!\subset q$ in $P_\alpha{\upharpoonright}V_\beta$, then $p \supset\!\!\!\subset q$ in P_α by (3*) as $P_\alpha{\upharpoonright}V_\beta \subset\subset P_\alpha$, and so $p \supset\!\!\!\subset q$ in $P_\alpha{\upharpoonright}V_\gamma$ by (3*) as $P_\alpha{\upharpoonright}V_\gamma \subset\subset P_\alpha$.

Let $p \in P_\alpha{\upharpoonright}V_\beta$. Consider $p{\upharpoonright}V_\gamma$. Let $p' \in P_\alpha{\upharpoonright}V_\gamma$. Let $p' \supset\!\!\!\subset p{\upharpoonright}V_\gamma$ in $P_\alpha{\upharpoonright}V_\gamma$. Then $p' \supset\!\!\!\subset p$ in P_α by (30.10), hence $p' \supset\!\!\!\subset p$ in $P_\alpha{\upharpoonright}V_\beta$ as by (3*) $P_\alpha{\upharpoonright}V_\beta \subset\subset P_\alpha$.

(5*) For any $\alpha \le \kappa$, any inacc. β and inacc. γ such that $\alpha < \gamma \le \beta \le \kappa$, if $p, q \in P_\alpha$ so that $p \le q$, then $p{\upharpoonright}V_\beta \le q{\upharpoonright}V_\beta$.

If not, then $p{\upharpoonright}V_\beta \not\le q{\upharpoonright}V_\beta$. By (30.8) there is $p_1 \in P_\alpha{\upharpoonright}V_\beta$ so that $p_1 \le p{\upharpoonright}V_\beta$ and $p_1 \supset\subset q{\upharpoonright}V_\beta$. Since $p_1 \supset\!\!\!\subset p{\upharpoonright}V_\beta$, $p_1 \supset\!\!\!\subset p$ in P_α, by (30.10). Thus $p_1 \supset\!\!\!\subset q$ in P_α. By (30.11) $q \le q{\upharpoonright}V_\beta$ and so $p_1 \supset\!\!\!\subset q{\upharpoonright}V_\beta$, a contradiction.

The proper proof will be conducted by induction over the level of iteration.

Define $P_0 = C(\omega_0, \kappa)$. Then $P_0 \subset V_\kappa$ and is separative by (28.1), (28.6). Hence (30.1),(30.5), and (30.6) are satisfied.

For $\alpha < \kappa$ not inacc. define $P_{\alpha+1} = P_\alpha * \{\emptyset\}$. Since by the induction hypothesis $P_\alpha \subset V_\kappa$ and is separative, so is $P_{\alpha+1}$. Thus (30.2),(30.5), and (30.6) are satisfied.

For $\alpha < \kappa$, α inacc. define $P_{\alpha+1} = P_\alpha \otimes_{P_\alpha{\upharpoonright}V_\alpha} C(\alpha,\kappa)^{P_\alpha{\upharpoonright}V_\alpha}$.

Let's explain this part a bit more. Let $\mathring{C} \in V^{P_\alpha{\upharpoonright}V_\alpha}$ be a name for $C(\alpha,\kappa)$ as defined in $V^{P_\alpha{\upharpoonright}V_\alpha}$. The forcing conditions of $P_{\alpha+1}$ are $\langle p, \mathring{s}\rangle$ such that $p \in P_\alpha$, $\mathring{s} \in V^{P_\alpha{\upharpoonright}V_\alpha}$, and $p \Vdash_{P_\alpha}$ "$\mathring{s} \in \mathring{C}$". By the induction hypothesis (30.10), $p{\upharpoonright}V_\alpha \prec_{P_\alpha{\upharpoonright}V_\alpha} p$, and so by Lemma 19 (as "$x \in X$" is absolute), $p{\upharpoonright}V_\alpha \Vdash_{P_\alpha{\upharpoonright}V_\alpha}$ "$\mathring{s} \in \mathring{C}$". Thus, $p{\upharpoonright}V_\alpha \Vdash_{P_\alpha{\upharpoonright}V_\alpha}$ "$\mathring{s}$ has rank at most κ", and because $|P_\alpha{\upharpoonright}V_\alpha| \le \alpha$ (follows from (30.12) as α is inacc.), and $\alpha < \kappa$, by Lemma 11 there is $\mathring{s}_1 \in V_\kappa^{P_\alpha{\upharpoonright}V_\alpha}$ so that $p{\upharpoonright}V_\alpha \Vdash_{P_\alpha{\upharpoonright}V_\alpha}$ "$\mathring{s}_1 = \mathring{s}$". By Lemma 17 (as "$x = y$" is absolute), $p{\upharpoonright}V_\alpha \Vdash_{P_\alpha}$ "$\mathring{s}_1 = \mathring{s}$". By (30.10) $p \Vdash_{P_\alpha}$ "$\mathring{s}_1 = \mathring{s}$". Hence $\langle p, \mathring{s}_1\rangle \in P_\alpha \otimes_{P_\alpha{\upharpoonright}V_\alpha} C(\alpha,\kappa)^{P_\alpha{\upharpoonright}V_\alpha}$ and $\langle p, \mathring{s}_1\rangle = \langle p, \mathring{s}\rangle$. Thus we can restrict our conditions to those with the second coordinate from $V_\kappa^{P_\alpha{\upharpoonright}V_\alpha}$, and so $P_{\alpha+1} \subset V_\kappa$. By Lemma 24 it also is separative as P_α is by the induction hypothesis, and $C(\alpha,\kappa)^{P_\alpha{\upharpoonright}V_\alpha}$ is separative in the generic extension via $P_\alpha{\upharpoonright}V_\alpha$ by (28.6). Hence (30.3), (30.5), and (30.6) are satisfied.

Let's look at the limit case.

Let $\alpha < \kappa$, α limit. Consider the set A of all limits of $\langle P_\xi : \xi < \alpha\rangle$ (full limits in Kunen's terminology [K$_1$], or inverse limits in Baumgartner's terminology [B]). If $a \in A$, then $a|\xi \in P_\xi$ for every $\xi < \alpha$ and so $a(\xi) \in V_\kappa$. Hence $a \in V_\kappa$, and so $A \subset V_\kappa$. Since P_α contains all limits with support from I_α, $P_\alpha \subset A$, thus $P_\alpha \subset V_\kappa$ and so (30.6) holds.

To show that P_α for $\alpha \le \kappa$, α limit, is separative, consider $p, q \in P_\alpha$ so that $p \not\le q$. Since $(\forall \xi < \alpha)(p|\xi \le q|\xi)$ implies that $p \le q$, there is $\xi < \alpha$ so that $p|\xi \not\le q|\xi$. Take such ξ. Since P_ξ is separative by the induction hypothesis (30.5), there is $t \in P_\xi$ so that $t \le p|\xi$ and $t \supset\subset q|\xi$ in P_ξ. Define r by $r(\eta) = t(\eta)$ for all $\eta < \xi$, and $r(\eta) = p(\eta)$ for all $\xi \le \eta < \alpha$. Then $supp(r) \subset supp(t) \cup supp(p)$, hence $supp(r) \in I_\alpha$, and so $r \in P_\alpha$. Clearly $r \le p$, and since $r|\xi = t$, $r \supset\subset q$ in P_α. Thus (30.5) holds.

To prove (30.12) : let $p \in P_\alpha$. Then $(p{\upharpoonright}V_\alpha)(0) = p(0) \cap V_\alpha \in V_\alpha$ by (28.1). If $p(\xi) = \emptyset$, then $(p{\upharpoonright}V_\alpha)(\xi) = \emptyset \in V_\alpha$. On the other hand if $p(\xi) \ne \emptyset$, then ξ is inacc. and $\xi < \alpha$. Then $(p{\upharpoonright}V_\alpha)(\xi) \in V_\alpha^{P_\xi{\upharpoonright}V_\xi}$. Thus $(p{\upharpoonright}V_\alpha) \subset V_\alpha$. Since α is inacc., $|supp(p{\upharpoonright}V_\alpha)| < \alpha$ and so $(p{\upharpoonright}V_\alpha)|supp(p{\upharpoonright}V_\alpha) \in V_\alpha$.

We have proven everything but (30.7) - (30.11). So let's assume that $\alpha \le \kappa$ is inacc. We shall discuss it in three steps, (A), (B), and (C). Let β be inacc. so that $\alpha \le \beta \le \kappa$.

(A) Case that α is the least inacc. (and so $\alpha < \kappa$).

$p{\in}P_\alpha$ iff $p(0){\in}C(\omega_0,\kappa)$ and $p(\xi)=\emptyset$ for all $0<\xi<\alpha$.

$p{\in}P_\alpha{\upharpoonright}V_\beta$ iff $p(0){\in}C(\omega_0,\beta)$ and $p(\xi)=\emptyset$ for all $0<\xi<\alpha$ (it follows from (28.2)).

Verify (30.7): follows from $C(\omega_0,\beta){\subset}C(\omega_0,\kappa)$, which follows from (28.3).

Verify (30.8): follows from (28.6).

Verify (30.9): follows from (28.5).

Verify (30.10): follows from (28.4).

Verify (30.11): follows from (28.3).

(B) Case that α is a successor inacc., i.e. α has an immediate inacc. predecessor γ (and so $\alpha<\kappa$). Let $\mathring{C}\in V^{P_\gamma{\upharpoonright}V_\gamma}$ be a name for $C(\gamma,\kappa)$ as defined in $V^{P_\gamma{\upharpoonright}V_\gamma}$. Let $\mathring{C}_\beta\in V^{P_\gamma{\upharpoonright}V_\gamma}$ be a name for $C(\gamma,\beta)$ as defined in $V^{P_\gamma{\upharpoonright}V_\gamma}$.

$p{\in}P_\alpha$ iff $supp(p){\subset}\gamma$ & $p|\gamma{\in}P_\gamma$, or $\gamma{\in}supp(p){\subset}\gamma{+}1$ & $p|\gamma{\in}P_\gamma$ & $p(\gamma){\in}V_\kappa^{P_\gamma{\upharpoonright}V_\gamma}$ & $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}$".

(*) $p{\in}P_\alpha{\upharpoonright}V_\beta$ iff $supp(p){\subset}\gamma$ & $p|\gamma{\in}P_\gamma{\upharpoonright}V_\beta$, or $\gamma{\in}supp(p){\subset}\gamma{+}1$ & $p|\gamma{\in}P_\gamma{\upharpoonright}V_\beta$ & $p(\gamma){\in}V_{\beta+1}^{P_\gamma{\upharpoonright}V_\gamma}$ & $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$".

The direction from right to left is easy, as any p satisfying the right hand side must be in P_α, and since $p{\upharpoonright}V_\beta=p$, p must be in $P_\alpha{\upharpoonright}V_\beta$. Now, the opposite direction. Let $p=q{\upharpoonright}V_\beta$ for some $q{\in}P_\alpha$. There are two possibilities:

(i) $supp(q){\subset}\gamma$. Then $supp(p){\subset}\gamma$ as well. $p|\gamma=(q{\upharpoonright}V_\beta)|\gamma=(q|\gamma){\upharpoonright}V_\beta$. Thus the first part of the right hand side condition is satisfied.

(ii) $\gamma{\in}supp(q){\subset}\gamma{+}1$. Then $q|\gamma\in P_\gamma$, $q(\gamma){\in}V_\kappa^{P_\gamma{\upharpoonright}V_\gamma}$ and $q|\gamma \Vdash_{P_\gamma}$ "$q(\gamma){\in}\mathring{C}$". $p(\gamma)=q(\gamma){\cap}V_\beta \in V_{\beta+1}^{P_\gamma{\upharpoonright}V_\gamma}$. By Lemma 19, using (30.10), $(q|\gamma){\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$q(\gamma){\in}\mathring{C}$". $(q|\gamma){\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$p(\gamma)=q(\gamma){\cap}V_\beta^{P_\gamma{\upharpoonright}V_\gamma}$ & $q(\gamma){\in}\mathring{C}$". By (28.2), $(q|\gamma){\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$", as β is inacc. in $V^{P_\gamma{\upharpoonright}V_\gamma}$ since $|P_\gamma{\upharpoonright}V_\gamma|\le\gamma$, and $\gamma<\beta$, and β is inacc. in V. By Lemma 17, $(q|\gamma){\upharpoonright}V_\gamma \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$". Using (30.11), $q|\gamma \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$". By Lemma 19, using (30.10), $(q|\gamma){\upharpoonright}V_\beta \Vdash_{P_\gamma{\upharpoonright}V_\beta}$ "$p(\gamma){\in}\mathring{C}_\beta$". By Lemma 17, $(q|\gamma){\upharpoonright}V_\beta \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$". $p|\gamma=(q{\upharpoonright}V_\beta)|\gamma=(q|\gamma){\upharpoonright}V_\beta$ by (1*), hence $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma){\in}\mathring{C}_\beta$".

Verify (30.7): follows immediately from (*).

Verify (30.8): Let $p,q{\in}P_\alpha{\upharpoonright}V_\beta$ so that $p\not\le q$. There are two possible cases:

(i) $p|\gamma\not\le q|\gamma$. Then there is $t{\in}P_\gamma{\upharpoonright}V_\beta$ so that $t\le p|\gamma$ and $t{\supset\subset}q|\gamma$, by (30.8). Define s so that $s|\gamma=t$, $s(\xi)=p(\xi)$ for all $\gamma\le\xi<\alpha$. Then (by (*)) $s{\in}P_\alpha{\upharpoonright}V_\beta$, and $s\le p$, and $s{\supset\subset}q$.

(ii) $p|\gamma\le q|\gamma$. Then $\gamma{\in}supp(p){\subset}\gamma{+}1$, and $p|\gamma,q|\gamma{\in}P_\gamma{\upharpoonright}V_\beta$, $p(\gamma),q(\gamma){\in}V_{\beta+1}^{P_\gamma{\upharpoonright}V_\gamma}$. Clearly, $p|\gamma \not\Vdash_{P_\gamma}$ "$p(\gamma)\le q(\gamma)$ *in* $\mathring{C}_\beta$". So there is $t{\in}P_\gamma$, $t\le p|\gamma$ so that $t \Vdash_{P_\gamma}$ "$p(\gamma)\not\le q(\gamma)$ *in* $\mathring{C}_\beta$". By Lemma 19, $t{\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$p(\gamma)\not\le q(\gamma)$ *in* $\mathring{C}_\beta$". By (28.6) $t{\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$(\exists s{\in}\mathring{C}_\beta)(s\le p(\gamma)$ & $s{\supset\subset}q(\gamma))$". Thus there is $\mathring{s}\in V_{\beta+1}^{P_\gamma{\upharpoonright}V_\gamma}$ so that $t{\upharpoonright}V_\gamma \Vdash_{P_\gamma{\upharpoonright}V_\gamma}$ "$\mathring{s}\in\mathring{C}_\beta$ & $\mathring{s}\le p(\gamma)$ & $\mathring{s}{\supset\subset}q(\gamma)$". By Lemma 17, $t{\upharpoonright}V_\gamma \Vdash_{P_\gamma}$ "$\mathring{s}\in\mathring{C}_\beta$ & $\mathring{s}\le p(\gamma)$ & $\mathring{s}{\supset\subset}q(\gamma)$". By (5*), Lemma 15, using (30.8), $t{\upharpoonright}V_\gamma\le t{\upharpoonright}V_\beta\le p|\gamma$. By (3*) $t{\upharpoonright}V_\gamma\in P_\gamma{\upharpoonright}V_\gamma{\subset\subset}P_\gamma{\upharpoonright}V_\beta$. Define r so that $r|\gamma=t{\upharpoonright}V_\gamma$, $r(\gamma)=\mathring{s}$, and $s(\xi)=\emptyset$ for all $\gamma<\xi<\alpha$. Then $r{\in}P_\alpha{\upharpoonright}V_\beta$, $r\le p$ and $r{\supset\subset}q$.

Verify (30.9): it suffices to show it from right to left. Let $p,q{\in}P_\alpha{\upharpoonright}V_\beta$, so that $p\not\supset\subset q$ in P_α. Then for some $r{\in}P_\alpha$ $r\le p,q$ in P_α. There are two possibilities:

(i) $supp(r){\subset}\gamma$. Then $supp(p),supp(q){\subset}\gamma$ as well. $r|\gamma\le p|\gamma,q|\gamma$ in P_γ. By (5*) $(r|\gamma){\upharpoonright}V_\beta\le(p|\gamma){\upharpoonright}V_\beta=p|\gamma$, $(q|\gamma){\upharpoonright}V_\beta=q|\gamma$, hence $(r{\upharpoonright}V_\beta)|\gamma\le p|\gamma,q|\gamma$ in $P_\gamma{\upharpoonright}V_\beta$. Since $supp(r){\subset}\gamma$, $supp(r{\upharpoonright}V_\beta){\subset}\gamma$, and so $r{\upharpoonright}V_\beta\le p,q$ in $P_\alpha{\upharpoonright}V_\beta$.

(ii) $\gamma\in supp(r)\subset\gamma+1$. Then $r|\gamma\in P_\gamma$, $r(\gamma)\in V_\kappa^{P_\gamma\uparrow V_\gamma}$ so that $r|\gamma \Vdash_{P_\gamma}$ "$r(\gamma)\in\overset{\circ}{C}$". Thus $r|\gamma \Vdash_{P_\gamma}$ "$r(\gamma)\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$". By (*), $r|\gamma \Vdash_{P_\gamma}$ "$p(\gamma),q(\gamma)\in\overset{\circ}{C}_\beta$". By Lemma 19, using (30.10), $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$r(\gamma)\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$ & $p(\gamma),q(\gamma)\in\overset{\circ}{C}_\beta$". By (28.2) and (28.3), $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$(\exists t\in\overset{\circ}{C}_\beta)(t\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C})$". Thus there is $\overset{\circ}{t}\in V_{\beta+1}^{P_\gamma\uparrow V_\gamma}$ so that $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}_\beta$ & $\overset{\circ}{t}\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$". By Lemma 17, $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}_\beta$ & $\overset{\circ}{t}\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$". By (30.11), $r|\gamma \Vdash_{P_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}_\beta$ & $\overset{\circ}{t}\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$". Since $r|\gamma\leq p|\gamma,q|\gamma$, by Lemma 15, using (30.8) $(r|\gamma)\uparrow V_\beta\leq p|\gamma,q|\gamma$. By Lemma 19, $(r|\gamma)\uparrow V_\beta \Vdash_{P_\gamma\uparrow V_\beta}$ "$\overset{\circ}{t}\in\overset{\circ}{C}_\beta$ & $\overset{\circ}{t}\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$", and so by Lemma 17, using (30.11), $(r|\gamma)\uparrow V_\beta \Vdash_{P_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}_\beta$ & $\overset{\circ}{t}\leq p(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$". Define s so that $s|\gamma=(r|\gamma)\uparrow V_\beta$, $s(\gamma)=\overset{\circ}{t}$, and $s(\xi)=\emptyset$ for all $\gamma<\xi<\alpha$. Then $s\in P_\alpha\uparrow V_\beta$ by (*), $s\leq p,q$. Thus $p\sqsupset\!\sqsubset q$ in $P_\alpha\uparrow V_\beta$.

Verify (30.10): Let $q=p\uparrow V_\beta$. Let $p'\in P_\alpha\uparrow V_\beta$ so that $p'\sqsupset\!\sqsubset q$ in $P_\alpha\uparrow V_\beta$. There is $r\in P_\alpha\uparrow V_\beta$ such that $r\leq p',q$. There are two possibilities:

(i) $supp(r)\subset\gamma$. Then $supp(p'),supp(q)\subset\gamma$. $r|\gamma\leq p'|\gamma,q|\gamma$ in $P_\gamma\uparrow V_\beta$. $q|\gamma=(p\uparrow V_\beta)|\gamma=(p|\gamma)\uparrow V_\beta$, so $p'|\gamma\sqsupset\!\sqsubset(p|\gamma)\uparrow V_\beta$ in $P_\gamma\uparrow V_\beta$. By (30.10) $p'|\gamma\sqsupset\!\sqsubset p|\gamma$ in P_γ, and so $p'\sqsupset\!\sqsubset p$ in P_α.

(ii) $\gamma\in supp(r)\subset\gamma+1$. Then $r|\gamma,p'|\gamma,q|\gamma\in P_\gamma\uparrow V_\beta$, $r(\gamma),p'(\gamma),q(\gamma)\in V_{\beta+1}^{P_\gamma\uparrow V_\gamma}$, and $r|\gamma\leq p'|\gamma,q|\gamma$, and $r|\gamma \Vdash_{P_\gamma}$ "$r(\gamma)\leq p'(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$ & $r(\gamma),p'(\gamma),q(\gamma)\in\overset{\circ}{C}_\beta$". By Lemma 19, using (30.10), $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$r(\gamma)\leq p'(\gamma),q(\gamma)$ *in* $\overset{\circ}{C}$ & $r(\gamma),p'(\gamma),q(\gamma)\in\overset{\circ}{C}_\beta$". Since $q(\gamma)=p(\gamma)\cap V_\beta$, $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$r(\gamma)\leq p'(\gamma),p(\gamma)\cap V_\beta^{P_\gamma\uparrow V_\gamma}$ *in* $\overset{\circ}{C}$ & $r(\gamma),p'(\gamma)\in\overset{\circ}{C}_\beta$". By (28.5), $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p'(\gamma)\sqsupset\!\sqsubset p(\gamma)$ *in* $\overset{\circ}{C}$". Hence $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$(\exists t\in\overset{\circ}{C})(t\leq p'(\gamma),p(\gamma))$". Thus there is $\overset{\circ}{t}\in V_\kappa^{P_\gamma\uparrow V_\gamma}$ so that $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}$ & $\overset{\circ}{t}\leq p'(\gamma),p(\gamma)$". By Lemma 17, $(r|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}$ & $\overset{\circ}{t}\leq p'(\gamma),p(\gamma)$". By (30.11), $r|\gamma \Vdash_{P_\gamma}$ "$\overset{\circ}{t}\in\overset{\circ}{C}$ & $\overset{\circ}{t}\leq p'(\gamma),p(\gamma)$". $r|\gamma\leq p'|\gamma,q|\gamma$ in $P_\gamma\uparrow V_\beta$, $q|\gamma=(p\uparrow V_\beta)|\gamma=(p|\gamma)\uparrow V_\beta$. Thus $r|\gamma\sqsupset\!\sqsubset(p|\gamma)\uparrow V_\beta$ in $P_\gamma\uparrow V_\beta$, and so by (30.10), $r|\gamma\sqsupset\!\sqsubset p|\gamma$ in P_γ. Let $t\in P_\gamma$ so that $t\leq r|\gamma,p|\gamma$. Define s so that $s|\gamma=t$, $s(\gamma)=\overset{\circ}{t}$, $s(\xi)=\emptyset$ for all $\gamma<\xi<\alpha$. Then $s\leq p,p'$ and so $p'\sqsupset\!\sqsubset p$ in P_α.

Verify (30.11): Let $p\in P_\alpha$. There are two possibilities:

(i) $supp(p)\subset\gamma$. $p|\gamma\in P_\gamma$, by (30.11) $p|\gamma\leq(p|\gamma)\uparrow V_\beta=(p\uparrow V_\beta)|\gamma$, thus $p\leq p\uparrow V_\beta$.

$\gamma\in supp(p)\subset\gamma+1$. Then $p|\gamma\in P_\gamma$, $p(\gamma)\in V_\kappa^{P_\gamma\uparrow V_\gamma}$ so that $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma)\in\overset{\circ}{C}$". By Lemma 19, using (30.10), $(p|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p(\gamma)\in\overset{\circ}{C}$". By (28.2), $(p|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p(\gamma)\leq p(\gamma)\cap V_\beta^{P_\gamma\uparrow V_\gamma}$ *in* $\overset{\circ}{C}$". So, $(p|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p(\gamma)\leq q(\gamma)$ *in* $\overset{\circ}{C}$", since $q(\gamma)=p(\gamma)\cap V_\beta$. By Lemma 17, $(p|\gamma)\uparrow V_\gamma \Vdash_{P_\gamma}$ "$p(\gamma)\leq q(\gamma)$ *in* $\overset{\circ}{C}$". Using (30.11), $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma)\leq q(\gamma)$ *in* $\overset{\circ}{C}$". Again by (30.11), $p|\gamma\leq(p|\gamma)\uparrow V_\beta=(p\uparrow V_\beta)|\gamma=q|\gamma$. Thus $p\leq q=p\uparrow V_\beta$.

(C) Case $\alpha\leq\kappa$, is a limit inacc., i.e. there is a cofinal sequence of inacc. cardinals bellow α.
By (29.3), P_α contains only direct limits.

Verify (30.7): Let $p\in P_\alpha$. Then there is an inacc. $\gamma<\alpha$ so that $supp(p)\subset\gamma$. Then $p|\gamma\in P_\gamma$ and $(p\uparrow V_\beta)|\gamma=(p|\gamma)\uparrow V_\beta\in P_\gamma\uparrow V_\beta\subset\subset P_\gamma$ by (3*). Thus $(p\uparrow V_\beta)|\gamma\in P_\gamma$, and since $supp(p\uparrow V_\beta)\subset supp(p)\subset\gamma$, $p\uparrow V_\beta\in P_\alpha$. Thus $P_\alpha\uparrow V_\beta\subset P_\alpha$.

Verify (30.8): Let $p,q\in P_\alpha\uparrow V_\beta$ so that $p\not\leq q$. There is an inacc. $\gamma<\alpha$ so that $supp(p),supp(q)\subset\gamma$. Then $p|\gamma\not\leq q|\gamma$ in $P_\gamma\uparrow V_\beta$. By the induction hypothesis (30.8), there is $t\in P_\gamma\uparrow V_\beta$ so that $t\leq p|\gamma$ and $t\supset\!\subset q|\gamma$. Define s so that $s|\gamma=t$, $s(\xi)=\emptyset$ for all $\gamma\leq\alpha$. Then $s\in P_\alpha\uparrow V_\beta$, $s\leq p$, and $s\supset\!\subset q$ in $P_\alpha\uparrow V_\beta$.

Verify (30.9): It suffices to prove right-to-left direction. Let $p,q \in P_\alpha \upharpoonright V_\beta$ so that $p \asymp q$ in P_α. There is $t \in P_\alpha$ so that $t \leq p,q$ in P_α. Then there is an inacc. $\gamma < \alpha$ so that $supp(t), supp(p), supp(q) \subset \gamma$. Hence $t|\gamma \leq p|\gamma, q|\gamma$ in P_γ. Since $p|\gamma, q|\gamma \in P_\gamma \upharpoonright V_\beta$, by the induction hypothesis (30.9)$p|\gamma \asymp q|\gamma$ in P_γ. So there is $r \in P_\gamma$ so that $r \leq p|\gamma, q|\gamma$. Define s so that $s|\gamma = r$, $s(\xi) = \emptyset$ for all $\gamma \leq \xi < \alpha$. Then $s \in P_\alpha$ and $s \leq p,q$.
Verify (30.10): Let $p \in P_\alpha$. There is an inacc. $\gamma < \alpha$ so that $supp(p) \subset \gamma$. Then $p|\gamma \in P_\gamma$. By the induction hypothesis (30.10) $(p|\gamma)\upharpoonright V_\beta \prec_{P_\alpha \upharpoonright V_\beta} p|\gamma$. Let $p' \in P_\alpha \upharpoonright V_\beta$ so that $p' \asymp p \upharpoonright V_\beta$ in $P_\alpha \upharpoonright V_\beta$. Since $(p \upharpoonright V_\beta)|\gamma = (p|\gamma)\upharpoonright V_\beta$, $p'|\gamma \asymp (p \upharpoonright V_\beta)|\gamma$ in $P_\gamma \upharpoonright V_\beta$, and so $p'|\gamma \asymp p|\gamma$ in P_γ. Since $supp(p) \subset \gamma$, $p' \asymp p$ in P_α.
Verify (30.11): Let $p \in P_\alpha$, there is an inacc. $\gamma < \alpha$ so that $supp(p) \subset \gamma$. Then $p|\gamma \in P_\gamma$ and by the induction hypothesis (30.11) $p|\gamma \leq (p|\gamma)\upharpoonright V_\beta$ in P_γ. Then $p|\gamma \leq (p \upharpoonright V_\beta)|\gamma$ in P_γ. Since $supp(p), supp(p \upharpoonright V_\beta) \subset \gamma$, $p \leq p \upharpoonright V_\beta$ in P_α. $\square$

Def. 31: If C, I, κ, and $\langle P_\alpha : \alpha \leq \kappa \rangle$ are as in Lemma 30, we shall call $\langle P_\alpha : \alpha \leq \kappa \rangle$ the (I,κ)-iteration of C in V.

Properties 32: Let $C(\gamma,\delta)$ define a poset in V ($\gamma \leq \delta$ cardinals). Let λ be cardinal.
(32.1) For every transitive $M \subset V$ such that $M_\alpha = V_\alpha$ for all $\alpha \leq \lambda$, $C(\gamma,\delta)^V = C(\gamma,\delta)^M$ whenever $\delta \leq \lambda$, δ inacc. in V as well as in M.

Properties 33: Let $I = \langle I_\alpha : \alpha \leq \kappa,\ \alpha \text{ limit} \rangle$. Let $j{:}V \to M$ be an elementary embedding with critical point κ. Let $j(I) = \langle \hat{I}_\alpha : \alpha \leq j(\kappa),\ \alpha \text{ limit} \rangle$.
(33.1) $I_\alpha = \hat{I}_\alpha$ for all $\alpha \leq \kappa$, α limit.

Lemma 34: Let $\langle P_\alpha : \alpha \leq \kappa \rangle$ be the (I,κ)-iteration of C in V. Let $j{:}V \to M$ be a huge elementary embedding with critical point κ. Let $I = \langle I_\alpha : \alpha \leq \kappa,\ \alpha \text{ limit} \rangle$ satisfy (33.1) with respect to j. Let C satisfy (32.1) with respect to $j(\kappa)$. Let $\langle \hat{P}_\alpha : \alpha \leq j(\kappa) \rangle = j(\langle P_\alpha : \alpha \leq \kappa \rangle$. Then
$\langle \hat{P}_\alpha : \alpha \leq j(\kappa) \rangle$ is the $(j(I), j(\kappa))$-iteration of C in V, and $P_\kappa * C(\kappa, j(\kappa))^{P_\kappa} \subset\subset j(P_\kappa)$ in V.

Proof: Let $\langle R_\alpha : \alpha \leq j(\kappa) \rangle$ be the $(j(I), j(\kappa))$-iteration of C in V. By induction we shall prove that
(1) $R_\alpha = \hat{P}_\alpha$ for every $\alpha \leq j(\kappa)$;
(2) $P_\alpha \upharpoonright V_\beta = R_\alpha \upharpoonright V_\beta = \hat{P}_\alpha \upharpoonright M_\beta$, for every inacc. $\alpha \leq \kappa$, and every inacc. β so that $\alpha \leq \beta \leq \kappa$.

Let $\alpha = 0$. $R_0 = C(\omega_0, j(\kappa))^V$. By (32.1) $C(\omega_0, j(\kappa))^V = C(\omega_0, j(\kappa))^M = \hat{P}_0$. Thus $R_0 = \hat{P}_0$.

Assume that $\alpha < j(\kappa)$ is not inacc. in V (and hence in M, by Lemma 4). Then $R_{\alpha+1} = R_\alpha * \{\emptyset\}$, while $\hat{P}_{\alpha+1} = \hat{P}_\alpha * \{\emptyset\}$. By the induction hypothesis (1) $R_\alpha = \hat{P}_\alpha$, hence $R_{\alpha+1} = \hat{P}_{\alpha+1}$.

Let $\alpha < j(\kappa)$ be inacc. in V (and hence in M, by Lemma 4). Let $\mathring{C} \in V^{R_\alpha \upharpoonright V_\alpha}$ be a name for $C(\alpha, j(\kappa))$ as defined in $V^{R_\alpha \upharpoonright V_\alpha}$. Let Let $\mathring{D} \in M^{\hat{P}_\alpha \upharpoonright M_\alpha}$ be a name for $C(\alpha, j(\kappa))$ as defined in $M^{\hat{P}_\alpha \upharpoonright M_\alpha}$. Then $R_{\alpha+1}$ $= R_\alpha \otimes_{R_\alpha \upharpoonright V_\alpha} \mathring{C}$ as defined in V, and $\hat{P}_{\alpha+1} = \hat{P}_\alpha \otimes_{\hat{P}_\alpha \upharpoonright M_\alpha} \mathring{D}$ as defined in M. Thus
$\langle p, \mathring{q} \rangle \in R_{\alpha+1}$ iff $p \in R_\alpha$ & $\mathring{q} \in V_{j(\kappa)}^{R_\alpha \upharpoonright V_\alpha}$ & $p \Vdash^{V}_{R_\alpha}$ "$\mathring{q} \in \mathring{C}$".
$\langle p, \mathring{q} \rangle \in \hat{P}_{\alpha+1}$ iff $p \in \hat{P}_\alpha$ & $\mathring{q} \in M_{j(\kappa)}^{\hat{P}_\alpha \upharpoonright M_\alpha}$ & $p \Vdash^{M}_{\hat{P}_\alpha}$ "$\mathring{q} \in \mathring{D}$".
Let $\langle p, \mathring{q} \rangle \in R_{\alpha+1}$.
Then $p \Vdash^{V}_{R_\alpha}$ "$\mathring{q} \in \mathring{C}$", so $p \upharpoonright V_\alpha \Vdash^{V}_{R_\alpha \upharpoonright V_\alpha}$ "$\mathring{q} \in \mathring{C}$", by Lemma 19, using (30.10). Let $p_1 \leq p \upharpoonright V_\alpha$ in $R_\alpha \upharpoonright V_\alpha = \hat{P}_\alpha \upharpoonright M_\alpha$. $p_1 \Vdash^{V}_{R_\alpha \upharpoonright V_\alpha}$ "$\mathring{q} \in \mathring{C}$". Let G be $R_\alpha \upharpoonright V_\alpha$-generic over V (and hence $\hat{P}_\alpha \upharpoonright M_\alpha$-generic over M by Lemma 25, as $|R_\alpha \upharpoonright V_\alpha| = |\hat{P}_\alpha \upharpoonright M_\alpha| \leq \alpha < \kappa$) so that $p_1 \in G$. Then $V[G] \models$ "$\mathring{q}^G \in C(\alpha, j(\kappa))$", and so $\mathring{q}^G \in C(\alpha, j(\kappa))^{V[G]}$. By (26.3) $V[G]_\xi = M[G]_\xi$ for all $\xi \leq j(\kappa)$, hence by (32.1), $\mathring{q}^G \in C(\alpha, j(\kappa))^{M[G]}$. So $M[G] \models$ "$\mathring{q}^G \in C(\alpha, j(\kappa))$", and so for some $p_2 \in G$,

$p_2 \Vdash^{M}_{\hat{P}_\alpha \upharpoonright M_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". Since $p_2 \asymp p_1$ in $\hat{P}_\alpha \upharpoonright M_\alpha$, there is p_3 so that $p_3 \le p_1$ in $\hat{P}_\alpha \upharpoonright M_\alpha$ and $p_3 \Vdash^{M}_{\hat{P}_\alpha \upharpoonright M_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". Thus, $p \upharpoonright V_\alpha \Vdash^{M}_{\hat{P}_\alpha \upharpoonright M_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". $p \upharpoonright V_\alpha = p \upharpoonright M_\alpha$, hence by Lemma 17, using (30.11), $p \Vdash^{M}_{\hat{P}_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". Now it follows that $\langle p, \overset{\circ}{q} \rangle \in \hat{P}_{\alpha+1}$.

Let $\langle p, \overset{\circ}{q} \rangle \in \hat{P}_{\alpha+1}$.

Then $p \Vdash^{M}_{\hat{P}_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". By Lemma 19, using (30.10), $p \upharpoonright M_\alpha \Vdash^{M}_{\hat{P}_\alpha \upharpoonright M_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". Let $p_1 \le p \upharpoonright M_\alpha$ in $\hat{P}_\alpha \upharpoonright M_\alpha = R_\alpha \upharpoonright V_\alpha$. Then $p_1 \Vdash^{M}_{\hat{P}_\alpha \upharpoonright M_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{\hat{D}}$". Let G be $\hat{P}_\alpha \upharpoonright M_\alpha$-generic over M (and hence $R_\alpha \upharpoonright V_\alpha$-generic over V by Lemma 25, as $|R_\alpha \upharpoonright V_\alpha| = |\hat{P}_\alpha \upharpoonright M_\alpha| \le \alpha < \kappa$) so that $p_1 \in G$. Then $M[G] \models$ "$\overset{\circ}{q}^{G} \in C(\gamma, j(\kappa))$", and so $\overset{\circ}{q}^{G} \in C(\gamma, j(\kappa))^M[G]$. By Lemma 26 $V[G]_\xi = M[G]_\xi$ for all $\xi \le j(\kappa)$,and so by (32.1) $C(\gamma, j(\kappa))^M[G] = C(\gamma, j(\kappa))^V[G]$. Hence $\overset{\circ}{q}^{G} \in C(\gamma, j(\kappa))^V[G]$ and so $V[G] \models$ "$\overset{\circ}{q}^{G} \in C(\gamma, j(\kappa))$". Therefore there is $p_2 \in G$ so that $p_2 \Vdash^{V}_{R_\alpha \upharpoonright V_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{C}$". Since $p_2 \asymp p_1$ in $R_\alpha \upharpoonright V_\alpha$, there is $p_3 \le p_1, p_2$ in $R_\alpha \upharpoonright V_\alpha$ and so $p_3 \Vdash^{V}_{R_\alpha \upharpoonright V_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{C}$". Thus $p \upharpoonright M_\alpha \Vdash^{V}_{R_\alpha \upharpoonright V_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{C}$", and thus $p \Vdash^{V}_{R_\alpha}$ "$\overset{\circ}{q} \in \overset{\circ}{C}$", by Lemma 17, using (30.11) and the fact that $p \upharpoonright M_\alpha = p \upharpoonright V_\alpha$. It follows that $\langle p, \overset{\circ}{q} \rangle \in R_{\alpha+1}$.

Let $\alpha \le j(\kappa)$ be limit.

Let's prove (1) first.

$p \in R_\alpha$ iff $supp(p) \in \hat{I}_\alpha$ & $p(\xi) = \emptyset$ if $\xi < \alpha$ not inacc. in V, and $p(\xi) \in V_{j(\kappa)}^{R_\xi \upharpoonright V_\xi}$ if $\xi < \alpha$ is inacc. in V, and $p|\xi \in R_\xi$ for every $\xi < \alpha$.

$p \in \hat{P}_\alpha$ iff $supp(p) \in \hat{I}_\alpha$ & $p(\xi) = \emptyset$ if $\xi < \alpha$ not inacc. in M, and $p(\xi) \in M_{j(\kappa)}^{\hat{P}_\xi \upharpoonright M_\xi}$ if $\xi < \alpha$ is inacc. in M, and $p|\xi \in \hat{P}_\xi$ for every $\xi < \alpha$.

Since $\xi < \alpha$ is inacc. in V iff $\xi < \alpha$ is inacc. in M (by Lemma 4), and since $M_{j(\kappa)}^{\hat{P}_\xi \upharpoonright M_\xi} = V_{j(\kappa)}^{R_\xi \upharpoonright V_\xi}$ for every inacc. $\xi < \alpha$ (as $\hat{P}_\xi \upharpoonright M_\xi = R_\xi \upharpoonright V_\xi$ by the induction hypothesis (2)), and since $R_\xi = \hat{P}_\xi$ for every $\xi < \alpha$ by the induction hypothesis (1), then $R_\alpha = \hat{P}_\alpha$.

Let's prove (2) for inacc. $\alpha \le \kappa$, inacc. β so that $\alpha \le \beta \le \kappa$.

(A) α is the least inacc. (and so $\alpha < \kappa$). Then
$p \in P_\alpha \upharpoonright V_\beta$ iff $supp(p) = \{0\}$ & $p(0) \in C(\omega_0, \beta)^V$ (by (28.2)),
$p \in R_\alpha \upharpoonright V_\beta$ iff $supp(p) = \{0\}$ & $p(0) \in C(\omega_0, \beta)^V$ (by (28.2)),
hence $P_\alpha \upharpoonright V_\beta = R_\alpha \upharpoonright V_\beta$. By (1) $R_\alpha \upharpoonright V_\beta = \hat{P}\alpha \upharpoonright M_\beta$, since $V_\beta = M_\beta$.

(B) α has an immediate inacc. predecessor γ (and so $\alpha < \kappa$).
Let $\overset{\circ}{C}_\beta \in V^{P_\gamma \upharpoonright V_\gamma} = V^{R_\gamma \upharpoonright V_\gamma}$ be a name for $C(\gamma, \beta)$ as defined in $V^{R_\gamma \upharpoonright V_\gamma}$. Let $p \in P_\alpha \upharpoonright V_\beta$. Then there are two possibilities (see (*) in the proof of Lemma 30):
(i) $supp(p) \subset \gamma$ and $p|\gamma \in P_\gamma \upharpoonright V_\beta$. Since $P_\gamma \upharpoonright V_\beta = R_\gamma \upharpoonright V_\beta$ by the induction hypothesis (1), $p \in R_\alpha \upharpoonright V_\beta$.
(ii) $\gamma \in supp(p) \subset \gamma{+}1$, $p|\gamma \in P_\gamma \upharpoonright V_\beta$, $p(\gamma) \in V_{\beta+1}^{P_\gamma \upharpoonright V_\gamma}$ so that $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma) \in \overset{\circ}{C}_\beta$". By Lemma 19, using (30.10), $(p|\gamma) \upharpoonright V_\gamma \Vdash_{P_\gamma \upharpoonright V_\gamma}$, "$p(\gamma) \in \overset{\circ}{C}_\beta$". Thus $(p|\gamma) \upharpoonright V_\gamma \Vdash_{R_\gamma \upharpoonright V_\gamma}$, "$p(\gamma) \in \overset{\circ}{C}_\beta$", as $R_\gamma \upharpoonright V_\gamma = P_\gamma \upharpoonright V_\gamma$ by the induction hypothesis. By Lemma 17, using (30.11), $p|\gamma \Vdash_{R_\gamma \upharpoonright V_\gamma}$ "$p(\gamma) \in \overset{\circ}{C}_\beta$". Hence $p \in R_\alpha \upharpoonright V_\beta$ (by (*) in the proof of Lemma 30).
On the other hand, let $p \in R_\alpha \upharpoonright V_\beta$. Then there are two possibilities (see (*) in the proof of Lemma 30):
(i) $supp(p) \subset \gamma$ and $p|\gamma \in R_\gamma \upharpoonright V_\beta$. Since $R_\gamma \upharpoonright V_\beta = P_\gamma \upharpoonright V_\beta$ by the induction hypothesis (1), $p \in P_\alpha \upharpoonright V_\beta$.
(ii) $\gamma \in supp(p) \subset \gamma{+}1$, $p|\gamma \in R_\gamma \upharpoonright V_\beta$, $p(\gamma) \in V_{\beta+1}^{R_\gamma \upharpoonright V_\gamma}$ so that $p|\gamma \Vdash_{R_\gamma}$ "$p(\gamma) \in \overset{\circ}{C}_\beta$". By Lemma 19, using (30.10), $(p|\gamma) \upharpoonright V_\gamma \Vdash_{R_\gamma \upharpoonright V_\gamma}$, "$p(\gamma) \in \overset{\circ}{C}_\beta$". Thus $(p|\gamma) \upharpoonright V_\gamma \Vdash_{P_\gamma \upharpoonright V_\gamma}$, "$p(\gamma) \in \overset{\circ}{C}_\beta$", as $P_\gamma \upharpoonright V_\gamma = R_\gamma \upharpoonright V_\gamma$ by the induction hypothesis. By Lemma 17, using (30.11), $p|\gamma \Vdash_{P_\gamma \upharpoonright V_\gamma}$

"$p(\gamma)\in\overset{\circ}{C}_\beta$". Hence $p\in P_\alpha\restriction V_\beta$ (by (*) in the proof of Lemma 30).

Thus $P_\alpha\restriction V_\beta = R_\alpha\restriction V_\beta$. Since $V_\beta = M_\beta$, by (1) $R_\alpha\restriction V_\beta = \hat{P}_\alpha\restriction M_\beta$.

(C) $\alpha \le \kappa$ has a cofinal sequence of inacc. cardinals below.

Then $M\models$ "$(\forall X\in M_\kappa)(X\in\hat{I}_\alpha \Rightarrow |X| < \alpha)$". Since $M_\kappa = V_\kappa$, and using Lemma 4, $V\models$ "$(\forall X\in V_\kappa)(X\in\hat{I}_\alpha \Rightarrow |X| < \alpha)$". Hence both, P_α and R_α contain only direct limits with the same support.

Let $p\in P_\alpha\restriction V_\beta$. Then for some inacc. $\gamma < \alpha$ $supp(p)\subset\gamma$ and $p|\gamma\in P_\gamma\restriction V_\beta$. Hence $p|\gamma\in R_\gamma\restriction V_\beta$ by the induction hypothesis, and so $p\in R_\alpha\restriction V_\beta$.

The proof that $p\in R_\alpha\restriction V_\beta \Rightarrow p\in P_\alpha\restriction V_\beta$ is identical.

Thus (1) and (2) are proven.

By (30.6), if $p\in P_\kappa$, then $p(\xi)\in V_\kappa$ for every $\xi < \kappa$, and so $p\restriction V_\kappa = p$. Hence $P_\kappa\restriction V_\kappa = P_\kappa$, and by (1) and (2), $R_\kappa\restriction V_\kappa = \hat{P}_\kappa\restriction V_\kappa = P_\kappa$.

$(R_\kappa\restriction V_\kappa)\otimes_{R_\kappa\restriction V_\kappa} C(\kappa,j(\kappa))^{R_\kappa\restriction V_\kappa} \subset\subset R_\kappa\otimes_{R_\kappa\restriction V_\kappa} C(\kappa,j(\kappa))^{R_\kappa\restriction V_\kappa} = R_{\kappa+1}$ by Lemma 22 as R_κ is separative (by (30.5)), and $R_\kappa\restriction V_\kappa \subset\subset R_\kappa$ by (3*) in the proof of Lemma 30. As we have proven (see above) that $R_\kappa\restriction V_\kappa = P_\kappa$, it follows that $(R_\kappa\restriction V_\kappa)\otimes_{R_\kappa\restriction V_\kappa} C(\kappa,j(\kappa))^{R_\kappa\restriction V_\kappa} = P_\kappa\otimes_{P_\kappa} C(\kappa,j(\kappa))^{P_\kappa} = P_\kappa * C(\kappa,j(\kappa))^{P_\kappa}$. Since $j(P_\kappa) = \hat{P}_{j(\kappa)} = R_{j(\kappa)}$ and contains only direct limits (by (29.3)) as $j(\kappa)$ is inacc. in both, M and V by Lemma 4). $R_{\kappa+1}\subset\subset R_{j(\kappa)}$. Hence $P_\kappa * C(\kappa,j(\kappa))^{P_\kappa}\subset\subset j(P_\kappa)$.□

To simplify the notation, we shall fix it for the rest of this chapter.

Let $j{:}V\to M$, κ, $I = \langle I_\alpha : \alpha \text{ limit} \le \kappa\rangle$, $\hat{I} = j(I) = \langle \hat{I}_\alpha : \alpha \text{ limit} \le j(\kappa)\rangle$, $\mathcal{P} = \langle P_\alpha : \omega \le \alpha \le \kappa\rangle$, $\hat{\mathcal{P}} = j(\mathcal{P}) = \langle \hat{P}_\alpha : \alpha \le j(\kappa)\rangle$, and C be as in Lemma 34.

Let P denote P_κ. Then $|P| \le \kappa$. Let G_1 be P-generic over V. Let Q denote $C(\alpha, j(\kappa))^{V[G_1]}$. Let $\overset{\circ}{Q}$ be a V^P-term for Q. Let B denote $P*\overset{\circ}{Q}$. Then by Lemma 34, B can be completely embedded in $j(P)$ and so $j(P) = B * j(P)/B$. Let G_2 be Q-generic over $V[G_1]$, and G_3 $j(P)/B$-generic over $V[G]$ ($G = G_1 * G_2$), then $H_1 = G * G_3$ is $j(P)$-generic over V (and hence over M). If $p\in P$, then $supp(p)\in I_\kappa$ and so $supp(p) = supp(j(p))$. Thus $j(p)(\alpha) = p(\alpha)$ for $\omega \le \alpha \le \kappa$, and $j(p)(\alpha) = 1_{P_\alpha}$ for $\kappa < \alpha \le j(\kappa)$. Hence $p\in G_1$ iff $j(p)\in H_1$. By Lemma 26 there is an elementary $\hat{\jmath}{:}V[G_1]\to M[H_1]$ definable in $V[H_1]$ and extending j, so that $(V[H_1])_\alpha = (M[H_1])_\alpha$ for every $\alpha \le j(\kappa)$. Then $\hat{\jmath}(Q) = C(j(\kappa), j(j(\kappa)))^{M[H_1]}$.

Lemma 35: assume that

(35.1) j is huge;

(35.2) P satisfies the κ-c.c. in V;

(35.3) $V[G_1]\models$ "$|Q| \le j(\kappa)$";

(35.4) for every directed $A\subset\hat{\jmath}''Q$ of size $\le j(\kappa)$ and $A\in M[H_1]$, there is a $q\in\hat{\jmath}(Q)$ so that $q << A$.

Then there is a so-called master condition $q_m\in\hat{\jmath}(Q)$ so that if H_2 is $\hat{\jmath}(Q)$-generic over $V[H_1]$ and $q_m\in H_2$, then $j(p)\in H = H_1 * H_2$ whenever $p\in G$. Therefore, there is an elementary embedding $i{:}V[G]\to M[H]$ definable in $V[H]$ extending $\hat{\jmath}$ so that if $V[G_1]\models$ "Q satisfies the $j(\kappa)$-c.c.", then if $X\in V$, $Y\in V[H]$, $Y\subset X$, $|Y| \le j(\kappa)$, $Y\subset M[H]$, then $Y\in M[H]$.

Proof: $G_2\in V[H_1]$ and $|G_2| \le j(\kappa)$ by (35.3). Let $G_2 = \{e_\alpha : \alpha < j(\kappa)\}$. Since $\hat{\jmath}$ is definable in $V[H_1]$, $\hat{\jmath}''G_2 = \{\hat{\jmath}(e_\alpha) : \alpha < j(\kappa)\}\in V[H_1]$, and $\hat{\jmath}''G_2\subset\hat{\jmath}''Q$. Since ${}^{j(\kappa)}M\subset M$, and since P satisfies the κ-c.c. in V by (35.2), $j(P)$ satisfies the $j(\kappa)$-c.c. in M, and by hugeness of j, in V as well, by (26.2) $\hat{\jmath}''G_2\in M[H_1]$. Since $\hat{\jmath}''G_2$ is directed, there is a $q_m\in\hat{\jmath}(Q)$ so that $q_m << \hat{\jmath}''G_2$ by (35.4). Let H_2 be $\hat{\jmath}(Q)$-generic over $V[H_1]$ (and hence also over $M[H_1]$) so that $q_m\in H_2$. Then, if $\langle p,q\rangle\in G = G_1 * G_2$, $\hat{\jmath}(\langle p,q\rangle) = \langle\hat{\jmath}(p),\hat{\jmath}(q)\rangle$, and $\hat{\jmath}(p)\in H_1$ and $\hat{\jmath}(q) \ge q_m$, and so $\hat{\jmath}(q)\in H_2$. Thus $\hat{\jmath}(\langle p,q\rangle)\in H$. By Lemma 26 there is an elementary embedding $i{:}V[G]\to M[H]$ definable in $V[H]$ extending j (and also $\hat{\jmath}$). If $V[G_1]\models$ "Q satisfies the $j(\kappa)$-c.c.", then B satisfies the $j(\kappa)$-c.c., and so if $X\in V$, $Y\in V[H]$, $|Y| \le j(\kappa)$, $Y\subset X$, and $Y\subset M[H]$, then $Y\in M[H]$ by (26.2). □

Lemma 36: Assume that

(36.1) j is huge;

(36.2) P satisfies the κ-c.c. in V;

(36.3) $V[G_1]\models$ "$|Q| \leq j(\kappa)$";

(36.4) $V[G_1]\models$ "Q is κ-closed";

(36.5) $V[G]\models$ "$|\wp(\kappa)| = \kappa^+$";

(36.6) for every directed $A \subset \hat{\jmath}''Q$ of size $\leq j(\kappa)$ and so that $A \in M[H_1]$, there is a $q \in \hat{\jmath}(Q)$ so that $q << A$.

Then $V[H_1]\models$ "$(\exists \mathcal{U})(\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa))$".
(In fact $\Vdash_{j(P)/B}^{V[G]}$ "$(\exists \mathcal{U})(\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa))$", since G_3 was chosen arbitrarily.).

Proof: Apply Lemma 35 to obtain a master condition $q_m \in \hat{\jmath}(Q)$, and H_2 $\hat{\jmath}(Q)$-generic over $V[H_1]$ so that $q_m \in H_2$, and an elementary $i{:}V[G] \to M[H]$ definable in $V[H]$ and extending $\hat{\jmath}$ (where $H = H_1 * H_2$). In $V[H]$ define for $X \in V[G] \cap \wp(j(\kappa))$: $X \in \mathcal{W}$ iff $\bigcup(i''j(\kappa)) \in i(X)$.
It is easy to check that $\mathcal{W}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa)$ in $V[H]$. Hence $q_m \Vdash_{\hat{\jmath}(Q)}^{V[H_1]}$ "$(\exists \mathcal{W})(\mathcal{W}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa))$". Let $\overset{\circ}{\mathcal{W}} \in V[H_1]^{\hat{\jmath}(Q)}$ so that $q_m \Vdash_{\hat{\jmath}(Q)}^{V[H_1]}$ "$\overset{\circ}{\mathcal{W}}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa)$". Now, $V[G]\models$ "$|\wp(\kappa)| = \kappa^+$", so $M[H]\models$ "$|\wp(j(\kappa))| = j(\kappa)^+$" by the elementarity of i. Hence $M[H]\models$ "$|\wp(j(\kappa)) \cap V[G]| \leq j(\kappa)^+$", and so $V[H]\models$ "$|\wp(j(\kappa)) \cap V[G]| \leq j(\kappa)^+$". Since Q is κ-closed, $\hat{\jmath}(Q)$ is $j(\kappa)$-closed, and so $(j(\kappa)^+)^{V[H]} = (j(\kappa)^+)^{V[H_1]}$. Thus $V[H_1]\models$ "$|\wp(j(\kappa)) \cap V[G]| \leq j(\kappa)^+$". Let $\{K_\alpha : \alpha < j(\kappa)^+\} = \wp(j(\kappa)) \cap V[G]$ in $V[H_1]$. In $V[H_1]$ let $\langle s_\alpha : \alpha < j(\kappa)^+\rangle$ be a descending sequence of elements of $\hat{\jmath}(Q)$ so that each s_α decides "$K_\alpha \in \overset{\circ}{\mathcal{W}}$". In $V[H_1]$ define $\mathcal{U}$ by:
if $X \in \wp(j(\kappa)) \cap V[G]$, then $X \in \mathcal{U}$ iff $(\exists \alpha < j(\kappa)^+)(s_\alpha \Vdash_{\hat{\jmath}(Q)}^{V[H_1]}$ "$X \in \overset{\circ}{\mathcal{W}}$)".
It is left to the reader to verify that $\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $j(\kappa)$ in $V[H_1]$. $\square$

Lemma 37: Assume that

(37.1) j is huge;

(37.2) P satisfies the κ-c.c. in V;

(37.3) $V[G_1]\models$ "$|Q| \leq j(\kappa)$";

(37.4) $V[G_1]\models$ "Q is $<\kappa$-closed";

(37.5) $V[G]\models$ "$|\wp(\kappa)| = j(\kappa)$";

(37.6) for every directed $A \subset \hat{\jmath}''Q$ of size $\leq j(\kappa)$ and so that $A \in M[H_1]$, there is a $q \in \hat{\jmath}(Q)$ so that $q << A$.

Then $V[H_1]\models$ "$(\exists \mathcal{U})(\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $\kappa)$".
(In fact $\Vdash_{j(P)/B}^{V[G]}$ "$(\exists \mathcal{U})(\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $\kappa)$", since G_3 was chosen arbitrarily.).

Proof: So similar to the proof of Lemma 36, that it is left to the reader. $\square$

Lemma 38: If $\Vdash_{j(P)/B}^{V[G]}$ "$(\exists \mathcal{U})(\mathcal{U}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over $\lambda)$", then $V[G]\models$ "$(\exists \mathcal{I})(\mathcal{I}$ is a κ-complete ideal over λ so that $\wp(\lambda)/\mathcal{I}$ can be embedded into $Comp(j(P)/B))$".

Proof: Let $\overset{\circ}{\mathcal{U}}$ be a $V[G]^{j(P)/B}$-term so that $\Vdash_{j(P)/B}^{V[G]}$ "$\overset{\circ}{\mathcal{U}}$ is a non-principal $V[G]-\kappa$-complete $V[G]$-ultrafilter over λ". Define $\mathcal{I}$ in $V[G]$ by:
if $X \subset \lambda$, then $X \in \mathcal{I}$ iff for no $p \in j(P)/B$, $p \Vdash_{j(P)/B}^{V[G]}$ "$X \in \overset{\circ}{\mathcal{U}}$".

(1) Let $X \subset Y \subset \lambda$, and $Y \in \mathcal{I}$.
By the way of contradiction assume that $X \notin \mathcal{I}$. Hence for some $p \in j(P)/B$, $p \Vdash_{j(P)/B}^{V[G]}$

"$X\in\mathring{\mathcal{U}}$". Also $p \Vdash_{j(P)/B}^{V[G]}$ "$X\subset Y$". Then $p \Vdash_{j(P)/B}^{V[G]}$ "$Y\in\mathring{\mathcal{U}}$", and so $Y\notin\mathcal{I}$, a contradiction.

(2) Let $\{X_\alpha : \alpha < \xi\}\subset\mathcal{I}$, $\xi < \kappa$.
By the way of contradiction assume that $\bigcup\{X_\alpha : \alpha < \xi\}\in\mathcal{I}$. Then for $p\in j(P)/B$, $p \Vdash_{j(P)/B}^{V[G]}$ "$\bigcup\{X_\alpha : \alpha < \xi\}\in\mathring{\mathcal{U}}$". But then $p \Vdash_{j(P)/B}^{V[G]}$ "$(\exists\alpha<\xi)(X_\alpha\in\mathring{\mathcal{U}})$". Hence there are $q\in j(P)/B$ and $\alpha < \xi$ so that $q \le p$ and $q \Vdash_{j(P)/B}^{V[G]}$ "$X_\alpha\in\mathring{\mathcal{U}})$", hence $X_\alpha\notin\mathcal{I}$, a contradiction.

(3) $\emptyset\in\mathcal{I}$, for no p can force "$\emptyset\in\mathring{\mathcal{U}}$".

(4) $\lambda\notin\mathcal{I}$, for $\Vdash_{j(P)/B}^{V[G]}$ "$\lambda\in\mathring{\mathcal{U}}$".

(5) if $\alpha\in\lambda$, then $\{\alpha\}\in\mathcal{I}$, for no p can force "$\{\alpha\}\in\mathring{\mathcal{U}}$".

To embed $\wp(\lambda)/\mathcal{I}$ into $D = Comp(j(P)/B)$, notice that

(i) $X\in\mathcal{I}$ iff $\|X\in\mathring{\mathcal{U}}\|_D = O_D$, and

(ii) $X, Y\notin\mathcal{I}$ and $X = Y\ (mod\ \mathcal{I})$, then $\|X\in\mathring{\mathcal{U}}\|_D = \|Y\in\mathring{\mathcal{U}}\|_D$. For $(X-Y), (Y-X)\in\mathcal{I}$ and so then $\|(X-Y)\in\mathring{\mathcal{U}}\|_D = \|(Y-X)\in\mathring{\mathcal{U}}\|_D$, thus $\|X\in\mathring{\mathcal{U}}\|_D = \|(X\cap Y)\in\mathring{\mathcal{U}}\|_D = \|Y\in\mathring{\mathcal{U}}\|_D$.

For $[X]\in\wp(\lambda)/\mathcal{I}$ define $h([X]) = \|X\in\mathring{\mathcal{U}}\|_D$. By (i) and (ii), this is a well-defined mapping from $\wp(\lambda)/\mathcal{I}$ into D. Let $[X] \le [Y]$. Then $(X-Y)\in\mathcal{I}$ and so $\|X\in\mathring{\mathcal{U}}\|_D \le \|Y\in\mathring{\mathcal{U}}\|_D$, hence $h([X]) \le h([Y])$. Also, if $[X] \ne [Y]$, then $\|X\in\mathring{\mathcal{U}}\|_D \ne \|Y\in\mathring{\mathcal{U}}\|_D$, so $h([X]) \ne h([Y])$. Thus h is an embedding. $\square$

Chapter 2.

Model I.

A model with an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1, and which satisfies Chang's conjecture.

(Kunen's model, see [K_2].)

We shall start with a huge embedding $j{:}V\to M$ with critical point κ. We shall do a (finite support,κ)-iteration of Silver's collapse S.

Def. 39: Let γ, δ be regular cardinals, $\gamma < \delta$. Silver's collapse of δ to γ^+ is a poset $S(\gamma, \delta)$ defined by: $s\in S(\gamma, \delta)$ iff

(39.1) $s\subset\delta\times\wp(\gamma\times\delta)$ is a function with $dom(s)\subset\delta$;

(39.2) $|s| \le \gamma$;

(39.3) there is $\beta\in\gamma$ so that for every $\alpha\in dom(s)$, $s(\alpha)\subset\beta\times\alpha$ is a function with $dom(s(\alpha))\subset\beta$;

(39.4) if $s, t\in S(\gamma, \delta)$, then $s \le t$ iff $dom(t)\subset dom(s)$ and for every $\alpha\in dom(t)$, $t(\alpha)\subset s(\alpha)$.

Note: If δ is inacc., then $S(\gamma, \delta)$ is a $<\gamma$-closed δ-c.c. poset and $\Vdash_{S(\gamma,\delta)}$ "$2^\gamma = \gamma^+ = \delta$" (see [J],[$K_1$]).

Lemma 39: Silver's collapse satisfies (28.1) - (28.6), and also (32.1).

Proof: Left to the reader. $\square$

Lemma 40: Let $I_\alpha = [\alpha]^{<\omega}$ for every limit $\alpha \le \kappa$. Then $I = \langle I_\alpha : \alpha \le \kappa\rangle$ satisfies (29.1) - (29.3), and (33.1).

Proof: Easy, and hence left to the reader. $\square$

Let $\mathcal{P} = \langle P_\alpha : \alpha \le \kappa\rangle$ be the (I, κ)-iteration of S in V (see Lemma 30). Then $\hat{\mathcal{P}} = j(\mathcal{P}) = \langle \hat{P}_\alpha : \alpha \le j(\kappa)\rangle$ is the $(j(I), j(\kappa))$-iteration of S in V by Lemma 34.

Lemma 41: P_κ satisfies the κ-c.c. in V. (and so $j(P_\kappa)$ satisfies $j(\kappa)$-c.c. in V.)

Proof: A sketch:

It is carried by induction in the usual way. The limit case is standard, for we are using finite support iteration (e.g. see [K$_1$], [B]). Thus we shall prove that each P_α for α a successor satisfies the κ-c.c. For P_0 it is known. Consider $P_{\alpha+1}$. If α is not inacc., then $P_{\alpha+1} = P_\alpha * \{\emptyset\}$, and so satisfies the κ-c.c. as P_α does. For α inacc. it is a bit harder.
Let $\{\langle p_\xi, \mathring{q}_\xi\rangle : \xi \in \kappa\}$ be an antichain in $P_{\alpha+1}$. Let $\mathring{C} \in V^{P_\alpha \upharpoonright V_\alpha}$ be a name for $S(\alpha,\kappa)$ as defined in $V^{P_\alpha \upharpoonright V_\alpha}$. Since $|P_\alpha \upharpoonright V_\alpha| \leq \alpha < \kappa$, by pigeon-hole argument there exists $X \in [\kappa]^\kappa$ and $p \in P_\alpha \upharpoonright V_\alpha$ so that $p_\xi \upharpoonright V_\alpha = p$ for every $\xi \in X$. Then $p_\xi \Vdash_{P_\alpha}$ "$\mathring{q}_\xi \in \mathring{C}$" for any $\xi \in X$. By Lemma 19, using 30.10, $p_\xi \upharpoonright V_\alpha \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$\mathring{q}_\xi \in \mathring{C}$" for any $\xi \in X$, and so $p \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$\mathring{q}_\xi \in \mathring{C}$" for any $\xi \in X$, and hence $p \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$|\mathring{q}_\xi| \leq \alpha$". By Lemma 12, we can assume WLOG that $|\mathring{q}_\xi| \leq \gamma$ for some $\alpha < \gamma < \kappa$, as $|P_\alpha \upharpoonright V_\alpha| \leq \alpha < \gamma$ and hence satisfies the γ-c.c. By the $\triangle$-system lemma (see e.g. [K$_1$]), there must be $Y \in [X]^\kappa$, $\mathring{q} \in V^{P_\alpha \upharpoonright V_\alpha}$, $|\mathring{q}| \leq \gamma$, so that $dom(\mathring{q}_\xi) \cap dom(\mathring{q}_\rho) = dom(\mathring{q})$ whenever $\xi, \rho \in Y$. Hence for every $\xi, \rho \in Y$, $p \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$\mathring{q}_\xi \cap \mathring{q}_\rho = \mathring{q}$", and so $p \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$\mathring{q}_\xi \supset\subset \mathring{q}_\rho$". Then $p_\xi \upharpoonright V_\alpha \Vdash_{P_\alpha \upharpoonright V_\alpha}$ "$\mathring{q}_\xi \supset\subset \mathring{q}_\rho$", and by Lemma 17, using (30.11), $p_\xi \Vdash_{P_\alpha}$ "$\mathring{q}_\xi \supset\subset \mathring{q}_\rho$". Since $\langle p_\xi, \mathring{q}_\xi\rangle \supset\subset \langle p_\rho, \mathring{q}_\rho\rangle$, it follows that $p_\xi \supset\subset p_\rho$. Therefero $\{p_\xi : \xi \in Y\}$ is an antichain of size κ in P_α, which contradicts the induction hypothesis.$\square$

Let G_1 be P-generic over V. Let Q denote $S(\kappa, j(\kappa))$ as defined in $V[G_1]$. Let $\mathring{Q}$ be a V^P-term so that $(\mathring{Q})^{G_1} = Q$. Let $B = P * \mathring{Q}$. Let G_2 be Q-generic over $V[G_1]$. Let $G = G_1 * G_2$. By Lemma 34, B can be regularly embedded into $j(P)$. By Lemma 26 there is an elementary embedding $\hat{\jmath}: V[G_1] \to M[H_1]$ extending j and definable in $V[H_1]$. Since $\hat{\jmath}''S(\kappa, j(\kappa))^{V[G_1]} \subset S(j(\kappa), j(j(\kappa)))^{M[H_1]}$, for every $A \subset \hat{\jmath}''S(\kappa, j(\kappa))^{V[G_1]}$, A directed, $|A| \leq j(\kappa)$, and $A \in M[H_1]$, there is $s \in S(j(\kappa), j(j(\kappa)))^{M[H_1]}$ so that $s << A$ (s is the set union of A; note that for every $s \in S(\kappa, j(\kappa))$, the β [see (39.3)] is not moved by j, hence $j(s)$ has the same β as s, and that 's why the union of A is a condition from $S(j(\kappa), jj(\kappa))$). Therefore, by Lemma 37, there exists a non-principal $V[G]$-κ-complete $V[G]$-ultrafilter over κ in $V[H_1]$. By Lemma 38 there is a κ-complete ideal $\mathcal{I}$ over κ so that $\wp(\kappa)/\mathcal{I}$ can be embedded into $Comp(j(P)/B)$. Since $j(P)$ satisfies the $j(\kappa)$-c.c. in V, $j(P)/B$ satisfies the $j(\kappa)$-c.c. in $V[G]$. Hence $Comp(j(P)/B)$ satisfies the $j(\kappa)$-c.c. in $V[G]$, and so $\wp(\kappa)/\mathcal{I}$ satisfies the $j(\kappa)$-c.c. in $V[G]$ as well; and so, $V[G] \models$ "$\mathcal{I}$ is $j(\kappa)$-saturated". Since $V[G] \models$ "$\aleph_1 = \kappa$ and $\aleph_2 = j(\kappa)$", $V[G] \models$ "$\mathcal{I}$ is an $\aleph_1$-complete, $\aleph_2$-saturated ideal over ω_1".
Now to show that Chang's conjecture holds in $V[G]$: by Lemma 35 there are H $j(B)$-generic over V and an elementary embedding $i: V[G] \to M[H]$ definable in $V[H]$ and extending $\hat{\jmath}$ so that if $X \in V$, $Y \in V[H]$, $Y \subset X$, $|Y| \leq j(\kappa)$, and $Y \subset M[H]$, then $Y \in M[H]$. Let $\mathcal{A}$ be a structure of type $(\aleph_1, \aleph_2)$ (i.e. of type $(\kappa, j(\kappa))$) in $V[G]$. WLOG assume that its universe is $j(\kappa)$. Then $i(\mathcal{A})$ is a structure of type $(\aleph_1, \aleph_2)$ in $M[H]$. Since $i''\mathcal{A} \subset M[H]$, $i''\mathcal{A} \in V[H]$ and has size $\leq j(\kappa)$, $i''\mathcal{A} \in M[H]$ and it is not hard to prove that $M[H] \models$ "$i''\mathcal{A}$ is a structure of type $(\kappa, j(\kappa))$, it is an elementary substructure of $i(\mathcal{A})$, $|\kappa| = \aleph_0$ and $|j(\kappa)| = \aleph_1$". Hence $M[H] \models$ "$i(\mathcal{A})$ has an elementary substructure of type $(\aleph_0, \aleph_1)$". By the elementarity of i, $V[G] \models$ "$\mathcal{A}$ has an elementary substructure of type $(\aleph_0, \aleph_1)$". $\square$

Note: If GCH holds in V, then GCH also holds in $V[G]$.

Model II.

A model with an $\aleph_1$-complete $\aleph_3$-saturated ideal over ω_3.

(Magidor's model - see [M].)

We shall start with a huge embedding $j: V \to M$ with critical point κ. We shall do a (finite support,κ)-iteration of Magidor's collapse D.

Def. 42: Let γ, δ be regular cardinals, $\gamma < \delta$. <u>Magidor's γ, δ collapse</u> is a poset $D(\gamma, \delta)$ defined by:
$D(\gamma,\delta) = \begin{cases} S(\omega_0, \delta), & \text{if } \gamma = \omega_0; \\ S(\gamma^+, \delta), & \text{otherwise,} \end{cases}$

where S is Silver's collapse (see Def. 39).

Note: if δ is inacc. and γ regular so that $\omega < \gamma < \delta$, then $D(\gamma, \delta)$ is a γ-closed, δ-c.c. poset.

Lemma 43: Magidor's collapse satisfies (28.1)-(28.6), (32.1).

Proof: See Lemma 39. $\square$

For every α limit so that $\omega < \alpha \leq \kappa$ define $I_\alpha = [\alpha]^{<\omega}$. Then $I = \langle I_\alpha : \text{limit } \alpha \leq \kappa \rangle$ satisfies (29.1) - (29.3), (33.1) (see Lemma 40). Let $\mathcal{P} = \langle P_\alpha : \alpha \leq \kappa \rangle$ be the (I, κ)-iteration of D in V. Then $\hat{\mathcal{P}} = j(\mathcal{P}) = \langle \hat{P}_\alpha : \alpha \leq j(\kappa) \rangle$ is the $(j(I), j(\kappa))$-iteration of M in V by Lemma 34. By induction in the usual way (see Lemma 41) it is easy to show that P_κ satisfies the κ-c.c. in V, and so $j(P_\kappa) = \hat{P}_{j(\kappa)}$ satisfies the $j(\kappa)$-c.c. in V. Let P denote P_κ. Let G_1 be P-generic over V. Let Q denote $D(\kappa, j(\kappa)) = S(\kappa^+, j(\kappa))$ as defined in $V[G_1]$. Let $\mathring{Q}$ be a V^P-term so that $(\mathring{Q})^{G_1} = Q$. Let $B = P * \mathring{Q}$. Let G_2 be Q-generic over $V[G_1]$. Let $G = G_1 * G_2$. By Lemma 34, B can be regularly embedded into $j(P)$. Since $j(P)$ satisfies the $j(\kappa)$-c.c. in V, by Lemma 26 there is an elementary embedding $\hat{\jmath}{:}V[G_1] \to M[H_1]$ extending j, definable in $V[H_1]$. Since $\hat{\jmath}\,\text{''}\, D(\kappa, j(\kappa))^{V[G_1]} \subset D(j(\kappa), j(j(\kappa)))^{M[H_1]}$, for every $A \subset \hat{\jmath}\,\text{''}\, D(\kappa, j(\kappa))^{V[G_1]}$, A directed, $|A| \leq j(\kappa)$, and $A \in M[H_1]$, there is $s \in D(j(\kappa), j(j(\kappa)))^{M[H_1]}$ so that $s << A$; s is the set union of A. Since Q is κ-closed, by Lemma 36 there is a non-principal $V[G]$-κ-complete $V[G]$-ultrafilter over $j(\kappa)$ in $V[H_1]$. Therefore by Lemma 38 there is a κ-complete ideal $\mathcal{I}$ over $j(\kappa)$ so that $\wp(j(\kappa))/\mathcal{I}$ can be embedded into $Comp(j(P)/B)$. Since $j(P)$ satisfies the $j(\kappa)$-c.c. in V, $j(P)/B$ satisfies the $j(\kappa)$-c.c. in $V[G]$. Hence $Comp(j(P)/B)$ satisfies the $j(\kappa)$-c.c. in $V[G]$, and so $\wp(j(\kappa))/\mathcal{I}$ satisfies the $j(\kappa)$-c.c. in $V[G]$. In other words, $V[G] \models$ "$\mathcal{I}$ is $j(\kappa)$-saturated". Since $V[G] \models$ "$\aleph_1 = \kappa$ and $\aleph_3 = j(\kappa)$", $V[G] \models$ "$\mathcal{I}$ is an $\aleph_1$-complete, $\aleph_3$-saturated ideal over ω_3". $\square$

Note: If GCH holds in V, then GCH also holds in $V[G]$.

Model III.

A model with an $\aleph_1$-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal over ω_1, and which satisfies Chang's conjecture.

(Laver's model, see [L].)

We shall start with a huge embedding $j{:}V \to M$ with critical point κ. We shall do an (ω-Easton,κ)-iteration of Easton's collapse E.

Def. 44: Let γ, δ be regular cardinals, $\gamma < \delta$. Easton's collapse of δ to γ^+ is a poset $E(\gamma, \delta)$ defined by: $s \in E(\gamma, \delta)$ iff

(44.1) $s \subset \delta \times \wp(\gamma \times \delta)$ is a function with $dom(s) \subset \delta$;
(44.2) $dom(s)$ is a γ-Easton subset of δ;
(44.3) there is $\beta \in \gamma$ so that for every $\alpha \in dom(s)$, $s(\alpha) \subset \beta \times \alpha$ is a function with $dom(s(\alpha)) \subset \beta$;
(44.4) if $s, t \in s(\gamma, \delta)$, then $s \leq t$ iff $dom(t) \subset dom(s)$ and for every $\alpha \in dom(t)$, $t(\alpha) \subset s(\alpha)$.

Note: If δ is Mahlo, then $E(\gamma, \delta)$ is a $<\gamma$-closed δ-c.c. poset and $\Vdash_{E(\gamma,\delta)}$ "$2^\gamma = \gamma^+ = \delta$" (see [L]).

Lemma 45: Let δ be Mahlo and let γ be regular so that $\gamma < \delta$. Let $\mathcal{A}$ be a family of γ-Easton subsets of δ so that $|\mathcal{A}| \geq \delta$. Then there is a family $\mathcal{B} \subset \mathcal{A}$, $|\mathcal{B}| \geq \delta$ so that $\mathcal{B}$ forms a $\triangle$-system with root $\triangle \subset \sigma$ for some $\sigma < \delta$.

Proof: WLOG assume that $|\mathcal{A}| = \delta$. Let $A = \{\beta \in \delta : \beta \text{ regular}\}$, and let $\mathcal{A} = \{X_\beta : \beta \in A\}$. By Lemma 4, for each X_β there is some σ_β so that $\gamma \leq \sigma_\beta < \delta$ and $X_\beta \subset \sigma_\beta$. Let $B = \{\beta \in A : \sigma_\beta \leq \beta\}$.

(a) Assume that B is stationary in δ.

For every $\beta \in B-\gamma$ define $f(\beta) =$ "the least τ so that $X_\beta \cap \beta \subset \tau$". Since $|X_\beta \cap \beta| < \beta$ and β is regular, f is regressive. By Fodor's theorem there are stationary $C \subset B-\gamma$ and $\sigma < \delta$ so that $f''C = \{\sigma\}$. Thus, if $\beta \in C$, $X_\beta \cap \beta \subset \sigma$ and $X_\beta \subset \sigma_\beta \subset \beta$, so $X_\beta \subset \sigma$.

Thus $\mathcal{D} = \{X \in \mathcal{A} : X \subset \sigma\}$ has size δ. Now apply the $\triangle$-system lemma to $\mathcal{D}$ to obtain a $\triangle$-system $\mathcal{B} \subset \mathcal{D}$ of size δ. Then $\triangle$, the root of $\mathcal{B}$, is a subset of σ.

(b) Assume that B is not stationary in δ.

Then there is a cub C in δ so that $B \cap C = \emptyset$. $D = A \cap C$ is stationary and if $\beta \in D$, then $\beta \notin B$ and so $\beta < \sigma_\beta$. Define a regressive function f on $D-\gamma$ by $f(\beta) =$ "the least τ so that $X_\beta \cap \beta \subset \tau$". By Fodor's theorem there are a stationary $E \subset D-\gamma$ and $\sigma < \delta$ so that $f''E = \{\sigma\}$. So for all $\beta \in E$, $X_\beta \cap \beta \subset \sigma$ and $\beta < \sigma_\beta$. By induction choose a sequence $\langle \beta_\alpha : \alpha < \delta \rangle \subset E$ so that $\sigma_{\beta_\alpha} < \beta_{\alpha+1}$ for all $\alpha < \delta$. Let $\mu < \nu < \delta$ and let $\xi \in X_{\beta_\mu} \cap X_{\beta_\nu}$. Then $\xi \in X_{\beta_\mu} \subset \sigma_{\beta_\mu} \subset \beta_\nu$, so $\xi \in X_{\beta_\nu} \cap \beta_\nu \subset \sigma$. Thus $X_{\beta_\mu} \cap X_{\beta_\nu} \subset \sigma$ whenever $\mu, \nu < \sigma$. Now apply $\triangle$-system lemma to $\{X_{\beta_\alpha} \cap \sigma : \alpha < \delta\}$. So there is $F \in [\delta]^\delta$ so that $\{X_{\beta_\alpha} \cap \sigma : \alpha \in F\}$ is a $\triangle$-system with root $\triangle \subset \sigma$. Then $\mathcal{B} = \{X_{\beta_\alpha} : \alpha \in F\}$ is also a $\triangle$-system with the same root $\triangle$. $\square$

Lemma 46: Easton's collapse satisfies (28.1) - (28.6), (32.1), if one replaces "inacc." by "Mahlo".

Proof: Left to the reader. $\square$

Lemma 47: Let I_α is the ideal of ω-Easton subsets of α, for every limit $\alpha \leq \kappa$. Then $I = \langle I_\alpha : \text{limit } \alpha \leq \kappa \rangle$ satisfies (29.1) - (29.3), and (33.1) with respect to j, if one replaces "inacc." by "Mahlo".

Proof: Left to the reader. $\square$

Let $\mathcal{P} = \langle P_\alpha : \alpha \leq \kappa \rangle$ be the (I, κ)-iteration of E in V as described in Lemma 30 with "inacc." replaced by "Mahlo". One can check that (30.7) - (30.11) still hold true with this replacement. Using the fact that κ is huge (in fact for this measurability suffices), the set of Mahlo cardinals bellow κ is stationary in κ. Since all Mahlo cardinals $\leq j(\kappa)$ in M are the same as in V (see Lemma 4), conclusions of Lemma 34 still hold. Hence $j(\mathcal{P}) = \langle \hat{P}_\alpha : \alpha \leq j(\kappa) \rangle$ is the $(j(I), j(\kappa))$-iteration of E in V, and $P_\kappa * E(\kappa, j(\kappa))^{P_\kappa}$ can be regularly embedded into $j(P_\kappa)$ in V.

Lemma 48: P_κ satisfies the $(\kappa, \kappa, {<}\kappa)$-c.c..

Proof: By Induction.

(a) Let α be the least Mahlo. We shall show that P_α satisfies the (κ, κ, σ)-c.c.. WLOG assume that $\alpha < \sigma$.

Since α is the least Mahlo, P_α is isomorphic to $P_0 = E(\omega, \kappa)$. Let $X \in [E(\omega, \kappa)]^\kappa$. By Lemma 45 there is $X_1 \in [X]^\kappa$ so that $\{dom(p) : p \in X_1\}$ is a $\triangle$-system with root $\triangle \subset \nu < \kappa$. WLOG assume $\sigma < \nu$. Since κ is inacc., there are less than κ possibilities for $p|\nu$. Thus, by pigeon-hole argument, there is $Y \in [X_1]^\kappa$ so that if $p_1 \neq p_2 \in Y$, then $p_1|\nu = p_2|\nu$. Let $Z \in [Y]^\nu$. Define $q \in E(\omega, \kappa)$ by $dom(q) = \bigcup\{dom(p) : p \in Z\}$ and $q(\alpha) = p(\alpha)$ for any $p \in Z$ so that $\alpha \in dom(p)$. Then $q \leq p$ for all $p \in Z$, since a union of ν ω-Easton sets is ν-Easton, hence $dom(q)$ is ν-Easton and so $dom(q)-\nu$ is ω-Easton. But $dom(q) \cap \nu = dom(p) \cap \nu$ for any $p \in Z$ and hence ω-Easton. So $dom(q)$ is ω-Easton. Therefore $q \in E(\omega, \kappa)$ and $q << Z$.

(b) Assume that α has an immediate Mahlo predecessor γ. We are going to show that P_α satisfies the (κ, κ, σ)-c.c. for $\sigma < \kappa$. WLOG assume that $\alpha < \sigma$.

Let $\mathring{E} \in V^{P_\gamma \uparrow V_\gamma}$ be a name for $E(\gamma, \kappa)$ as defined in $V^{P_\gamma \uparrow V_\gamma}$. $p \in P_\alpha$ iff $supp(p) \subset \gamma+1$, $p|\gamma \in P_\gamma$, $p(\gamma) \in V^{P_\gamma \uparrow V_\gamma}$, and $p|\gamma \Vdash_{P_\gamma}$ "$p(\gamma) \in \mathring{E}$". Since $|P_\gamma \uparrow V_\gamma| \leq \gamma < \kappa$, there are $p \in P_\gamma \uparrow V_\gamma$, and $X_1 \in [\kappa]^\kappa$ so that $p_\xi \uparrow V_\gamma = p$ for every $\xi \in X_1$. Since each $p_\xi|\gamma \Vdash_{P_\gamma}$ "$p_\xi(\gamma) \in \mathring{E}$", be Lemma

19, using (30.10), $(p_\xi|\gamma)\!\uparrow\! V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p_\xi(\gamma)\in\mathring{E}$", and so $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$p_\xi(\gamma)\in\mathring{E}$" for all $\xi\in X_1$. Define $A_\xi = \{\delta\in\kappa : p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\delta\in dom(p_\xi(\gamma))$"$\}$, for $\xi\in X_1$. Let $B_\xi = A_\xi - \sigma$. Then each B_ξ is an ω-Easton subset of κ:

If not, then for some regular $\tau\geq\omega$, $|B_\xi\cap\tau| = \tau$ (so $\tau>\sigma>\alpha>\gamma$). Let $B_\xi\cap\tau = \{\delta_\eta : \eta<\tau\}$. Then each $\delta_\eta\in\tau$. For every $\eta<\tau$, $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\delta_\eta\in dom(p_\xi(\gamma))$". Since $P_\gamma\!\uparrow\! V_\gamma$ preserves τ, $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$|dom(p_\xi(\gamma))\cap\tau| = \tau$", which is a contradiction.

By Lemma 45 there is $X_2\in[X]^\kappa$ so that $\{B_\xi : \xi\in X_2\}$ form a $\triangle$-system with the root $\triangle\subset\nu$, for some $\nu<\kappa$. WLOG assume that $\sigma<\nu$. By smallness of V_ν, there is $X_3\in[X_2]^\kappa$ so that $p_\xi\cap V_\nu = p_\rho\cap V_\nu$ whenever $\xi,\rho\in X_3$.

Let $Y\in[X_3]^\sigma$. Since P_γ satisfies the (κ,κ,σ)-c.c., as by the induction hypothesis it satisfies the $(\kappa,\kappa,{<}\kappa)$-c.c., there is $r\in P_\gamma$ so that $r\leq p_\xi|\gamma$ for every $\xi\in Y$. Since the root of $\{B_\xi : \xi\in X_2\}$ is a subset of ν, for every $\delta>\nu$, at most one $\xi\in Y$ satisfies that $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\delta\in dom(p_\xi(\gamma))$". So, $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\bigcup\{dom(p_\xi(\gamma)) : \xi\in Y\}$ *is a σ-Easton subset of* κ". It follows that (for $\sigma<\nu$) $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\bigcup\{dom(p_\xi(\gamma)) : \xi\in Y\}$ *is a γ-Easton subset of* κ", and so $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\bigcup\{p_\xi(\gamma) : \xi\in Y\}\in\mathring{E}$". Let $\mathring{t}\in V^{P_\gamma\uparrow V_\gamma}$ be so that $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\mathring{t} = \bigcup\{p_\xi(\gamma) : \xi\in Y\}$". Then for every $\xi\in Y$, $p \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\mathring{t}\leq p_\xi$ *in* $\mathring{E}$". By (5*) from the proof of Lemma 30, $r\!\uparrow\! V_\gamma\leq(p_\xi|\gamma)\!\uparrow\! V_\gamma = p$, so $r\!\uparrow\! V_\gamma \Vdash_{P_\gamma\uparrow V_\gamma}$ "$\mathring{t}\leq p_\xi$ *in* $\mathring{E}$", for every $\xi\in Y$. By Lemma 17, using (30.11), $r \Vdash_{P_\gamma}$ "$\mathring{t}\leq p_\xi$ *in* $\mathring{E}$", for every $\xi\in Y$. Now define t so that $t|\gamma = r$, $t(\gamma) = \mathring{t}$, and $t(\xi) = \emptyset$ for all $\gamma<\xi<\alpha$. $t\in P_\alpha$, and $t\leq p_\xi$ for every $\xi\in Y$.

(c) Assume that α has a cofinal sequence of smaller Mahlo cardinals. Then the support is (by Lemma 3) is of size smaller than α, and in fact direct limits are taken. The proof now continues along standard lines, using Lemma 45 to obtain the required $\triangle$-system of supports (see e.q. [K$_1$], [B]).□

Since $j(\mathcal{P})$ is the same kind of iteration in V, and since $j(\kappa)$ is Mahlo in V, we also proved that $j(P_\kappa)$ satisfies the $(j(\kappa), j(\kappa), {<}j(\kappa))$-c.c. in V.

As before, let P denote P_κ, let Q denote $E(\kappa, j(\kappa))$ as defined in V^P, and let B denote $P*Q$.

Lemma 49: $\Vdash_B$ "$j(P)/B$ satisfies the $(j(\kappa), j(\kappa), {<}\kappa)$-c.c."

Proof: Let $\Vdash_B$ "$j(P)/B = \{s_\alpha : \alpha<j(\kappa)\}$". Let $\mathring{X}\in V^B$ and $b_0\in B$ so that $b_0 \Vdash_B$ "$\mathring{X}\in [j(\kappa)]^{j(\kappa)}$". There is $Y_0\in[j(\kappa)]^{j(\kappa)}$ so that for any $\alpha\in Y_0$ there is $b\in B$, $b\leq b_0$ and $b \Vdash_B$ "$\alpha\in\mathring{X}$". For each $\alpha\in Y_0$ choose one such b and denote it b_α. Since $B = P*Q$, for each $\alpha\in Y_0$ there are $p_\alpha\in P$ and $q_\alpha\in Q$ so that $b_\alpha = \langle p_\alpha, q_\alpha\rangle$. Hence, for every $\alpha\in Y_0$, $\langle p_\alpha, q_\alpha\rangle \Vdash_B$ "$\alpha\in\mathring{X}$". Since $|P|<j(\kappa)$, there are $Y_1\in[Y_0]^{j(\kappa)}$ and $p\in P$ so that $\langle p_\alpha, q_\alpha\rangle = \langle p, q_\alpha\rangle$ whenever $\alpha\in Y_1$. Thus, for every $\alpha\in Y_1$, $\langle p, q_\alpha\rangle \Vdash_B$ "$\alpha\in\mathring{X}$". For any $\alpha\in Y_1$, $\langle p, q_\alpha, s_\alpha\rangle\in j(P)$. By the proof of Lemma 48,

(*) there is $Y_2\in[Y_1]^{j(\kappa)}$ so that the coordinatewise union of $\{\langle p, q_\alpha, s_\alpha\rangle : \alpha\in C\}$ is a condition from $j(P)$ whenever $C\in[Y_2]^{<j(\kappa)}$.

Let G_1 be P-generic over V so that $p\in G_1$. Now switch to $V[G_1]$. Since $j(\kappa)$ still is Mahlo here (as $|P|<j(\kappa)$), there are $Y_3\in[Y_2]^{j(\kappa)}$ and $\sigma<j(\kappa)$ so that $\{dom(q_\alpha) : \alpha\in Y_3\}$ form a $\triangle$-system with root $\triangle\subset\sigma$. By pigeon-hole argument there are $Y_4\in[Y_3]^{j(\kappa)}$ and $q\in Q$ so that $q_\alpha|\sigma = q$ for every $\alpha\in Y_4$. Hence

(**) $\{dom(q_\alpha) : \alpha\in Y_4\}$ form a $\triangle$-system with root $\triangle\subset\sigma$, and $q_\alpha|\sigma = q$ for every $\alpha\in Y_4$.

Let $\mathring{Y}_5 \in V[G_1]^Q$ so that $\Vdash_{Q}^{V[G_1]}$ "$\alpha \in \mathring{Y}_5$ iff $\alpha \in Y_4$ & $q_\alpha \in \underline{G}_2$", where $\underline{G}_2$ is the canonical name for the Q-generic filter over $V[G_1]$. Let $\mathring{Y} \in V[G_1]^Q$ so that $\Vdash_{Q}^{V[G_1]}$ "$\alpha \in \mathring{Y}$ iff $\alpha \in \mathring{X} \cap \mathring{Y}_5$". Let G_2 be Q-generic over $V[G_1]$ so that $q \in G_2$. Let $G = G_1 * G_2$. Then G is B-generic over V.

(***) $V[G] \models$ "$|Y| = j(\kappa)$", where $Y = (\mathring{Y})^{G_2}$.

It will suffice to prove that $q \Vdash_{Q}^{V[G_1]}$ "$|\mathring{Y}| = j(\kappa)$", since $q \in G_2$. Let us assume by the way of contradiction that $q \not\Vdash_{Q}^{V[G_1]}$ "$|\mathring{Y}| = j(\kappa)$". There are $\bar{q} \leq q$ and $\nu < j(\kappa)$ (WLOG assume that $\sigma \leq \nu$) so that $\bar{q} \Vdash_{Q}^{V[G_1]}$ "$\mathring{Y} \subset \nu$". Since $\bar{q} \leq q$, by (**) there is $\tau \in Y_4 - \nu$ so that $(dom(\bar{q}) \cap dom(q_\tau)) - \sigma = \emptyset$. Thus $\bar{q}$ and q_τ are compatible, i.e. there is $q' \leq \bar{q}, q_\tau$. Since $q_\tau \Vdash_{Q}^{V[G_1]}$ "$q_\tau \in \underline{G}_2$", $q_\tau \Vdash_{Q}^{V[G_1]}$ "$\tau \in \mathring{Y}_5$". Hence $q' \Vdash_{Q}^{V[G_1]}$ "$\tau \in \mathring{Y}_5$". Since $q_\tau \Vdash_{Q}^{V[G_1]}$ "$\tau \in \mathring{X}$", $q' \Vdash_{Q}^{V[G_1]}$ "$\tau \in \mathring{X}$". Therefore $q' \Vdash_{Q}^{V[G_1]}$ "$\tau \in \mathring{Y}$". On the other hand, $\bar{q} \Vdash_{Q}^{V[G_1]}$ "$\mathring{Y} \subset \nu$", and so $q' \Vdash_{Q}^{V[G_1]}$ "$\mathring{Y} \subset \nu$", a contradiction as $\tau \geq \nu$.

(****) $V[G_1] \models$ "$(\forall D \in [Y_3]^{<\kappa})(\exists t \in Q * j(P)/B)(t = \langle \bigcup\{q_\alpha : \alpha \in D\}, s\rangle << \{\langle q_\alpha, s_\alpha\rangle : \alpha \in D\})$"

Let $V[G_1] \models$ "$D \in [Y_3]^{<\kappa}$". Then $V[G_1] \models$ "$D \in [Y_2]^{<\kappa}$". Since P satisfies the κ-c.c. in V, and since $Y_2 \in V$, there is $C \in [Y_3]^{<\kappa}$ so that $V[G_1] \models$ "$D \subset C$". By (*), the coordinatewise union of $\{\langle p, q_\alpha, s_\alpha\rangle : \alpha \in C\}$ is a condition from $j(P)$. Since $p \in G_1$, in $V[G_1]$, the coordinatewise union of $\{\langle q_\alpha, s_\alpha\rangle : \alpha \in C\}$ is a condition from $Q * j(P)/B$. Since $D \subset C$, the coordinatewise union of of $\{\langle q_\alpha, s_\alpha\rangle : \alpha \in D\}$ is a condition t from $Q * j(P)/B$. The condition t has the form $\langle \bigcup\{q_\alpha : \alpha \in D\}, s\rangle$ for some V^B-term s so that $\Vdash_{B}$ "$s \in j(P)/B$", and clearly $t << \{\langle q_\alpha, s_\alpha\rangle : \alpha \in D\}$.

(*****) $V[G] \models$ "$(\forall Z \in [Y]^{<\kappa})(\exists s \in j(P)/B)(s << \{s_\alpha : \alpha \in Z\})$.

If $V[G] \models$ "$Z \in [Y]^{<\kappa}$", then $V[G] \models$ "$Z \in [Y_3]^{<\kappa}$". Since $Y_3 \in V[G_1]$, and since Q is κ-closed, $Z \in V[G_1]$. By (****), in $V[G_1]$ there is a condition $t = \langle \bigcup\{q_\alpha : \alpha \in Z\}, s\rangle \in Q * j(P)/B$ so that $t << \{\langle q_\alpha, s_\alpha\rangle : \alpha \in Z\}$. Since $Z \subset Y$, and so $Z \subset (\mathring{Y}_5)^{G_2}$, each q_α, $\alpha \in Z$, is in G_2. Since $\{q_\alpha : \alpha \in Z\} \in V[G_1]$, and G_2 is Q-generic over $V[G_1]$, $\bigcup\{q_\alpha : \alpha \in Z\} \in G_2$. Thus $V[G] \models$ "$s << \{s_\alpha : \alpha \in Z\}$".

If $\alpha \in Y$, then $b_0 \geq \langle p, q_\alpha\rangle$ and $\langle p, q_\alpha\rangle \in G$, hence $b_0 \in G$. Thus there is $b_1 \leq b_0$ so that $b_1 \Vdash_{B}$ "$(\forall Z \in [\mathring{Y}]^{<\kappa})(\exists s \in j(P)/B)(s << \{s_\alpha : \alpha \in Z\})$. $\square$

Let G_1 be P-generic over V, let G_2 be Q-generic over $V[G_1]$. Then $G = G_1 * G_2$ is B-generic over V. Let G_3 be $j(P)/B$-generic over $V[G]$ (possible by Lemma 34). Then $H_1 = G_1 * G_2 * G_3$ is $j(P)$-generic over V. By Lemma 26 there is an elementary embedding $\hat{\jmath}: V[G_1] \to M[H_1]$ definable in $V[H_1]$ and extending j. Similarly as in Model I, for every directed $A \subset \hat{\jmath}``E(\kappa, j(\kappa))^{V[G_1]}$, $|A| < j(\kappa)$, and $A \in V[H_1]$, there is $q \in E(j(\kappa), j(j(\kappa)))^{V[H_1]}$ so that $q << A$; the set union of A. Thus, by Lemma 37 there is a non-principal $V[G]$-κ-complete $V[G]$-ultrafilter over κ in $V[H_1]$. By Lemma 38 there is a κ-complete ideal $\mathcal{I}$ over κ in $V[G]$, so that $\wp(\kappa)/\mathcal{I}$ can be embedded into $Comp(j(P)/B)$. By Lemma 49, $j(P)/B$ satisfies the $(j(\kappa), j(\kappa), {<}\kappa)$-c.c. in $V[G]$. Since $\kappa = \aleph_1$ and $j(\kappa) = \aleph_2$ in $V[G]$, $\mathcal{I}$ is ω_1-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal over ω_1.

Note:

(1) If GCH holds in V, then it also holds in $V[G]$.

(2) $V[G]$ also satisfies Chang's conjecture (see Model I).

(3) Laver showed (see [L]) that from the existence of an ω_1-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal over ω_1 follows that $\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_0}{\aleph_1}^{1,1}_2$. Juhazs and Hajnal (private communication to Laver) showed that adding $\aleph_1$ Cohen reals destroys the partition relation, hence destroys all ω_1-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideals over ω_1. Since adding $\aleph_1$ Cohen reals is a σ,finite-c.c. forcing, it preserves (see [BT]) ω_1-complete $\aleph_2$-saturated ideals over ω_1. Hence if N is a model with an ω_1-complete $\aleph_2$-saturated ideal over ω_1, and if P is a forcing notion for adding $\aleph_1$ Cohen reals,

than any generic extension of N via P is a model with an ω_1-complete $\aleph_2$-saturated ideal over ω_1, and with no ω_1-complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal over ω_1.

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