

Chapter 13

Randomized Algorithms



Slides by Kevin Wayne.
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Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- **Randomization.**

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

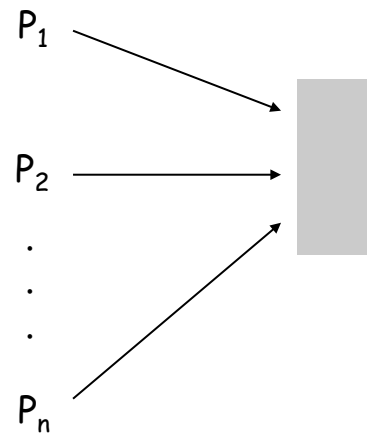
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes P_1, \dots, P_n , each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need **symmetry-breaking** paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability $p = 1/n$.

Claim. Let $S[i, t]$ = event that process i succeeds in accessing the database at time t . Then $1/(e \cdot n) \leq \Pr[S(i, t)] \leq 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p (1-p)^{n-1}$.

process i requests access \nearrow \nwarrow none of remaining $n-1$ processes request access

- Setting $p = 1/n$, we have $\Pr[S(i, t)] = 1/n \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪
value that maximizes $\Pr[S(i, t)]$

Useful facts from calculus. As n increases from 2, the function:

- $(1 - 1/n)^n$ converges monotonically from $1/4$ up to $1/e$
- $(1 - 1/n)^{n-1}$ converges monotonically from $1/2$ down to $1/e$.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in en rounds is at most $1/e$. After $e \cdot n(c \ln n)$ rounds, the probability is at most n^{-c} .

Pf. Let $F[i, t]$ = event that process i fails to access database in rounds 1 through t . By independence and previous claim, we have $\Pr[F(i, t)] \leq (1 - 1/(en))^t$.

- Choose $t = \lceil e \cdot n \rceil$: $\Pr[F(i, t)] \leq \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $\Pr[F(i, t)] \leq \left(\frac{1}{e}\right)^{c \ln n} = n^{-c}$

Contention Resolution: Randomized Protocol

Claim. The probability that **all** processes succeed within $2e \cdot n \ln n$ rounds is at least $1 - 1/n$.

Pf. Let $F[t]$ = event that at least one of the n processes fails to access database in any of the rounds 1 through t .

$$\Pr[F[t]] = \Pr\left[\bigcup_{i=1}^n [E_i, t]\right] \leq \sum_{i=1}^n \Pr[E_i, t] \leq n \cdot (1 - \frac{1}{2e})^t$$

↑ union bound
 ↑ previous slide

- Choosing $t = 2 \lceil en \rceil \lceil c \ln n \rceil$ yields $\Pr[F[t]] \leq n \cdot n^{-2} = 1/n$. ■

Union bound. Given events E_1, \dots, E_n , $\Pr\left[\bigcup_{i=1}^n E_i\right] \leq \sum_{i=1}^n \Pr[E_i]$

13.2 Global Minimum Cut

Global Minimum Cut

Global min cut. Given a connected, undirected graph $G = (V, E)$ find a cut (A, B) of minimum cardinality.

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

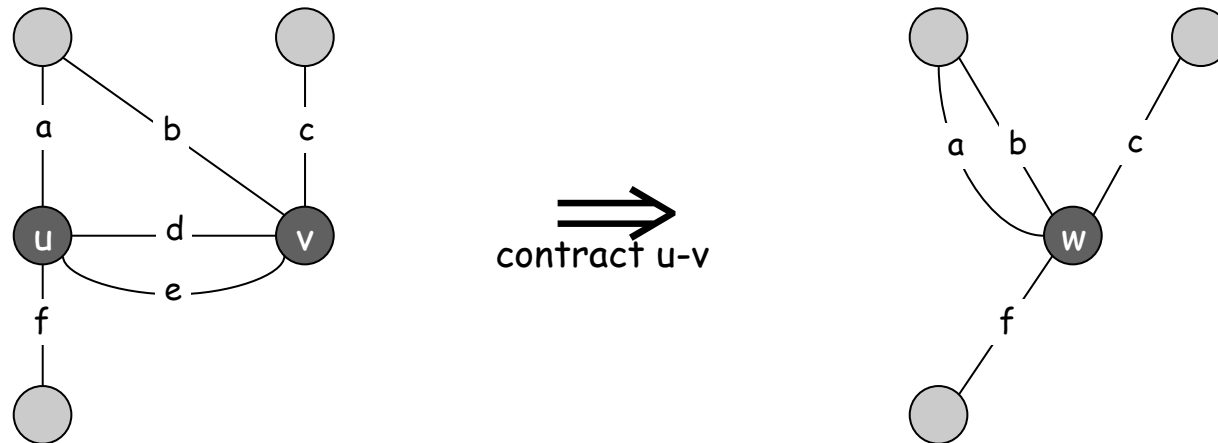
- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u) .
- Pick some vertex s and compute min s - v cut separating s from each other vertex $v \in V$.

False intuition. Global min-cut is harder than min s - t cut.

Contraction Algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge $e = (u, v)$ uniformly at random.
- **Contract** edge e .
 - replace u and v by single new super-node w
 - preserve edges, updating endpoints of u and v to w
 - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes v_1 and v_2 .
- Return the cut (all nodes that were contracted to form v_1).

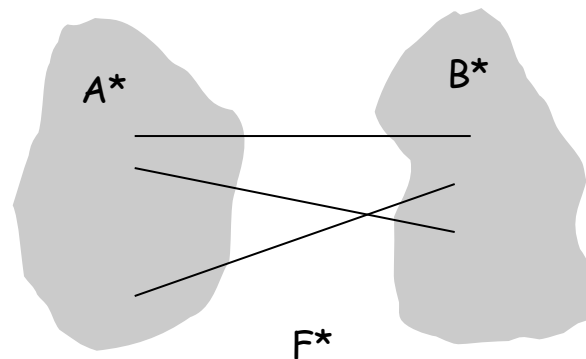


Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G . Let F^* be edges with one endpoint in A^* and the other in B^* . Let $k = |F^*| =$ size of min cut.

- In first step, algorithm contracts an edge in F^* probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be min-cut. $\Rightarrow |E| \geq \frac{1}{2}kn$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n$.



Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G . Let F^* be edges with one endpoint in A^* and the other in B^* . Let $k = |F^*| =$ size of min cut.

- Let G' be graph after j iterations. There are $n' = n - j$ supernodes.
- Suppose no edge in F^* has been contracted. The min-cut in G' is still k .
- Since value of min-cut is k , $|E'| \geq \frac{1}{2}kn'$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n'$.

- Let $E_j =$ event that an edge in F^* is not contracted in iteration j .

$$\begin{aligned}
 \Pr[E_1 \cap E_2 \cdots \cap E_{-2}] &= \Pr[E_1] \times \Pr[E_2 | E_1] \times \cdots \times \Pr[E_{-2} | E_1 \cap E_2 \cdots \cap E_{-3}] \\
 &\geq \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{3}\right) \\
 &= \left(\frac{-2}{-1}\right) \left(\frac{-3}{-1}\right) \cdots \left(\frac{2}{4}\right) \left(\frac{1}{3}\right) \\
 &= \frac{2}{(-1)} \\
 &\geq \frac{2}{n}
 \end{aligned}$$

Contraction Algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times with independent random choices, the probability of failing to find the global min-cut is at most $1/n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1 - \frac{2}{n}\right)^{2 \ln n} = \left[\left(1 - \frac{2}{n}\right)^{\frac{1}{2}} \right]^{2 \ln n} \leq \left(1 - \frac{1}{n}\right)^{2 \ln n} = \frac{1}{n^2}$$

\uparrow
 $(1 - 1/x)^x \leq 1/e$

Global Min Cut: Context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm **twice** on resulting graph, and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

↖ faster than best known max flow algorithm or deterministic global min cut algorithm

13.3 Linearity of Expectation

Expectation: Two Properties

Useful property. If X is a 0/1 random variable, $E[X] = \Pr[X = 1]$.

Pf.
$$E[X] = \sum_{i=0}^{\infty} i \cdot \Pr[X = i] = \sum_{i=0}^1 i \cdot \Pr[X = i] = \Pr[X = 1]$$

Linearity of expectation. Given two random variables X and Y defined over the same probability space, $E[X + Y] = E[X] + E[Y]$.
not necessarily independent

Decouples a complex calculation into simpler pieces.

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf. (surprisingly effortless using linearity of expectation)

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let $X =$ number of correct guesses $= X_1 + \dots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1/n$.
- $E[X] = E[X_1] + \dots + E[X_n] = 1/n + \dots + 1/n = 1$. ▪

↑
linearity of expectation

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let $X =$ number of correct guesses $= X_1 + \dots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - i - 1)$.
- $E[X] = E[X_1] + \dots + E[X_n] = 1/n + \dots + 1/2 + 1/1 = H(n)$. ▪

↑
linearity of expectation

↑
 $\ln(n+1) < H(n) < 1 + \ln n$

Coupon Collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is $\Theta(n \log n)$.

Pf.

- Phase j = time between j and $j+1$ distinct coupons.
- Let X_j = number of steps you spend in phase j .
- Let X = number of steps in total = $X_0 + X_1 + \dots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{1}{\text{prob of success}} = \sum_{j=1}^n \frac{1}{j} = \Theta(n \log n) \quad ()$$

prob of success = $(n-j)/n$
 \Rightarrow expected waiting time = $n/(n-j)$

13.4 MAX 3-SAT

Maximum 3-Satisfiability

↙ exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$\begin{aligned} 1 &= 2 \vee \overline{3} \vee \overline{4} \\ 2 &= 2 \vee 3 \vee \overline{4} \\ 3 &= \overline{1} \vee 2 \vee 4 \\ 4 &= \overline{1} \vee \overline{2} \vee 3 \\ 5 &= 1 \vee \overline{2} \vee \overline{4} \end{aligned}$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the **expected number** of clauses satisfied by a random assignment is $7k/8$.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let $Z =$ weight of clauses satisfied by assignment Z_j .

$$\begin{aligned}
 \mathbb{E}[Z] &= \sum_{j=1}^k \mathbb{E}[Z_j] \\
 \text{linearity of expectation} &= \sum_{j=1}^k \Pr[\text{clause } j \text{ is satisfied}] \\
 &= \frac{7}{8}
 \end{aligned}$$

The Probabilistic Method

Corollary. For any instance of 3-SAT, **there exists** a truth assignment that satisfies at least a $7/8$ fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ■

Probabilistic method. We showed the existence of a non-obvious property of 3-SAT by showing that a random construction produces it with positive probability!

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a $7/8$ -approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies $\geq 7k/8$ clauses is at least $1/(8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k/8$ clauses are satisfied.

$$\begin{aligned}
 \frac{7}{8} &= \left[\frac{7k}{8} \right] = \sum_{j \geq 0} p_j \frac{j}{k} \\
 &= \sum_{j < 7k/8} p_j \frac{j}{k} + \sum_{j \geq 7k/8} p_j \frac{j}{k} \\
 &\leq \left(\frac{7}{8} - \frac{1}{8} \right) \sum_{j < 7k/8} p_j + \sum_{j \geq 7k/8} p_j \\
 &\leq \left(\frac{7}{8} - \frac{1}{8} \right) \cdot 1 + p
 \end{aligned}$$

Rearranging terms yields $p \geq 1 / (8k)$. ■

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

Theorem. Johnson's algorithm is a $7/8$ -approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least $1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ■

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max **weighted** set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a $7/8$ -approximation algorithm for version of MAX-3SAT where each clause has **at most** 3 literals.

Theorem. [Håstad 1997] Unless $P = NP$, no ρ -approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

↑
very unlikely to improve over simple randomized algorithm for MAX-3SAT

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed to run in poly-time, likely to find correct answer.

Ex: Contraction algorithm for global min cut.

Las Vegas algorithm. Guaranteed to find correct answer, likely to run in poly-time.

Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

stop algorithm after a certain point



Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with **one-sided error** in poly-time.

One-sided error.

- If the correct answer is **no**, always return **no**.
- If the correct answer is **yes**, return **yes** with probability $\geq \frac{1}{2}$.

Can decrease probability of false negative to 2^{-100} by 100 independent repetitions



ZPP. [Las Vegas] Decision problems solvable in **expected** poly-time.



running time can be unbounded, but on average it is fast

Theorem. $P \subseteq ZPP \subseteq RP \subseteq NP$.

Fundamental open questions. To what extent does randomization help?

Does $P = ZPP$? Does $ZPP = RP$? Does $RP = NP$?

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that **inserting**, deleting, and **searching** in S is efficient.

Dictionary interface.

- **Create ():** Initialize a dictionary with $S = \phi$.
- **Insert (u):** Add element $u \in U$ to S .
- **Delete (u):** Delete u from S , if u is currently in S .
- **Lookup (u):** Determine whether u is in S .

Challenge. Universe U can be extremely large so defining an array of size $|U|$ is infeasible.

Applications. File systems, databases, Google, compilers, checksums
P2P networks, associative arrays, cryptography, web caching, etc.

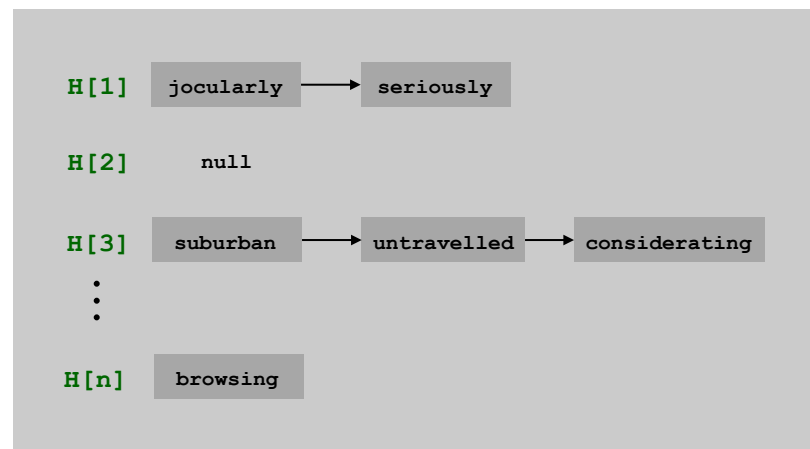
Hashing

Hash function. $h : U \rightarrow \{ 0, 1, \dots, n-1 \}$.

Hashing. Create an array H of size n . When processing element u , access array element $H[h(u)]$.

Collision. When $h(u) = h(v)$ but $u \neq v$.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions. This phenomenon is known as the "birthday paradox."
- Separate chaining: $H[i]$ stores linked list of elements u with $h(u) = i$.



Ad Hoc Hash Function

Ad hoc hash function.

```
int h(String s, int n) {  
    int hash = 0;  
    for (int i = 0; i < s.length(); i++)  
        hash = (31 * hash) + s[i];  
    return hash % n;  
}
```

hash function ala Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h , there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

Q. But isn't ad hoc hash function good enough in practice?

Algorithmic Complexity Attacks

When can't we live with ad hoc hash function?

- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary learns **your** ad hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements **uniformly at random** to n hash slots.

- Running time depends on length of chains.
- Average length of chain = $\alpha = m / n$.
- Choose $n \approx m \Rightarrow$ on average $O(1)$ per insert, lookup, or delete.

Challenge. Achieve idealized randomized guarantees, but with a hash function where you can easily find items where you put them.

Approach. Use randomization in the choice of h .

↑
adversary knows the randomized algorithm you're using,
but doesn't know random choices that the algorithm makes

Universal Hashing

Universal class of hash functions. [Carter-Wegman 1980s]

- For any pair of elements $u, v \in U$, $\Pr_{h \in H} [h(u) = h(v)] \leq 1/n$
- Can select random h efficiently. ← chosen uniformly at random
- Can compute $h(u)$ efficiently.

Ex. $U = \{a, b, c, d, e, f\}$, $n = 6$.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$$H = \{h_1, h_2\}$$

$$\Pr_{h \in H} [h(a) = h(b)] = 1/2$$

$$\Pr_{h \in H} [h(a) = h(c)] = 1$$

$$\Pr_{h \in H} [h(a) = h(d)] = 0$$

...

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$$H = \{h_1, h_2, h_3, h_4\}$$

$$\Pr_{h \in H} [h(a) = h(b)] = 1/2$$

$$\Pr_{h \in H} [h(a) = h(c)] = 1/2$$

$$\Pr_{h \in H} [h(a) = h(d)] = 1/2$$

$$\Pr_{h \in H} [h(a) = h(e)] = 1/2$$

$$\Pr_{h \in H} [h(a) = h(f)] = 0$$

...

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let $h \in H$ be chosen uniformly at random from H ; and let $u \in U$. For any subset $S \subseteq U$ of size at most n , the expected number of items in S that collide with u is at most 1.

Pf. For any element $s \in S$, define indicator random variable $X_s = 1$ if $h(s) = h(u)$ and 0 otherwise. Let X be a random variable counting the total number of collisions with u .

$$\begin{aligned}
 \mathbb{E}[X] &= \mathbb{E}\left[\sum_{s \in S} X_s\right] = \sum_{s \in S} \mathbb{E}[X_s] = \sum_{s \in S} \Pr[X_s = 1] \leq \sum_{s \in S} \frac{1}{|U|} = |S| \frac{1}{|U|} \leq 1
 \end{aligned}$$

linearity of expectation
 X_s is a 0-1 random variable
universal
(assumes $u \notin S$)

Designing a Universal Family of Hash Functions

Theorem. [Chebyshev 1850] There exists a prime between n and $2n$.

Modulus. Choose a prime number $p \approx n$. ← no need for randomness here

Integer encoding. Identify each element $u \in U$ with a base- p integer of r digits: $x = (x_1, x_2, \dots, x_r)$.

Hash function. Let $A =$ set of all r -digit, base- p integers. For each $a = (a_1, a_2, \dots, a_r)$ where $0 \leq a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^r x_i a_i \right) \bmod p$$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a Universal Class of Hash Functions

Theorem. $H = \{ h_a : a \in A \}$ is a universal class of hash functions.

Pf. Let $x = (x_1, x_2, \dots, x_r)$ and $y = (y_1, y_2, \dots, y_r)$ be two distinct elements of U . We need to show that $\Pr[h_a(x) = h_a(y)] \leq 1/n$.

- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_z = \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_m \pmod{p}$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \neq j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \neq j$.
- Since p is prime, $a_j z = m \pmod{p}$ has at most one solution among p possibilities. ← see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] = 1/p \leq 1/n$. ▪

Number Theory Facts

Fact. Let p be prime, and let $z \not\equiv 0 \pmod{p}$. Then $\alpha z = m \pmod{p}$ has at most one solution $0 \leq \alpha < p$.

Pf.

- Suppose α and β are two different solutions.
- Then $(\alpha - \beta)z = 0 \pmod{p}$; hence $(\alpha - \beta)z$ is divisible by p .
- Since $z \not\equiv 0 \pmod{p}$, we know that z is not divisible by p ; it follows that $(\alpha - \beta)$ is divisible by p .
- This implies $\alpha = \beta$. ▪

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Euclid's algorithm.

13.9 Chernoff Bounds

Chernoff Bounds (above mean)

Pf. (cont)

- Let $p_i = \Pr[X_i = 1]$. Then,

$$E[e^{tX_i}] = p_i e^t + (1 - p_i)e^0 = 1 + p_i(e^t - 1) \leq e^{p_i(e^t - 1)}$$

\uparrow
 for any $\alpha \geq 0, 1 + \alpha \leq e^\alpha$

- Combining everything:

$$\Pr[X > (1 + \delta)\mu] \leq e^{-(1 + \delta)\mu} \prod [\] \leq e^{-(1 + \delta)\mu} \prod (e^{-1}) \leq e^{-(1 + \delta)\mu} \mu^{-1}$$

\uparrow previous slide \uparrow inequality above \uparrow $\sum_i p_i = E[X] \leq \mu$

- Finally, choose $t = \ln(1 + \delta)$. ■

Chernoff Bounds (below mean)

Theorem. Suppose X_1, \dots, X_n are independent 0-1 random variables. Let $X = X_1 + \dots + X_n$. Then for any $\mu \leq E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1 - \delta)\mu] < e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load Balancing

Analysis.

- Let X_i = number of jobs assigned to processor i .
- Let $Y_{ij} = 1$ if job j assigned to processor i , and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \frac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e \gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{(n)^{1/e}}\right)^{e \gamma(n)} < \left(\frac{1}{(n)^{1/e}}\right)^{2 \gamma(n)} = \frac{1}{n^2}$$

- Union bound \Rightarrow with probability $\geq 1 - 1/n$ no processor receives more than $e \gamma(n) = \Theta(\log n / \log \log n)$ jobs.

Fact: this bound is asymptotically tight: with high probability, some processor receives $\Theta(\log n / \log \log n)$

Load Balancing: Many Jobs

Theorem. Suppose the number of jobs $m = 16n \ln n$. Then on average, each of the n processors handles $\mu = 16 \ln n$ jobs. With high probability every processor will have between half and twice the average load.

Pf.

- Let X_i, Y_{ij} be as before.
- Applying Chernoff bounds with $\delta = 1$ yields

$$\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16n \ln n} < \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n^2} \quad \Pr[X_i < \frac{1}{2}\mu] < e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2(16 \ln n)} = \frac{1}{n^2}$$

- Union bound \Rightarrow every processor has load between half and twice the average with probability $\geq 1 - 2/n$. ▪

Extra Slides

13.5 Randomized Divide-and-Conquer

Quicksort

Sorting. Given a set of n distinct elements S , rearrange them in ascending order.

```
RandomizedQuicksort(S) {
  if |S| = 0 return

  choose a splitter  $a_i \in S$  uniformly at random
  foreach (a  $\in S$ ) {
    if (a <  $a_i$ ) put a in  $S^-$ 
    else if (a >  $a_i$ ) put a in  $S^+$ 
  }
  RandomizedQuicksort( $S^-$ )
  output  $a_i$ 
  RandomizedQuicksort( $S^+$ )
}
```

Remark. Can implement in-place.

↑
 $O(\log n)$ extra space

Quicksort

Running time.

- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

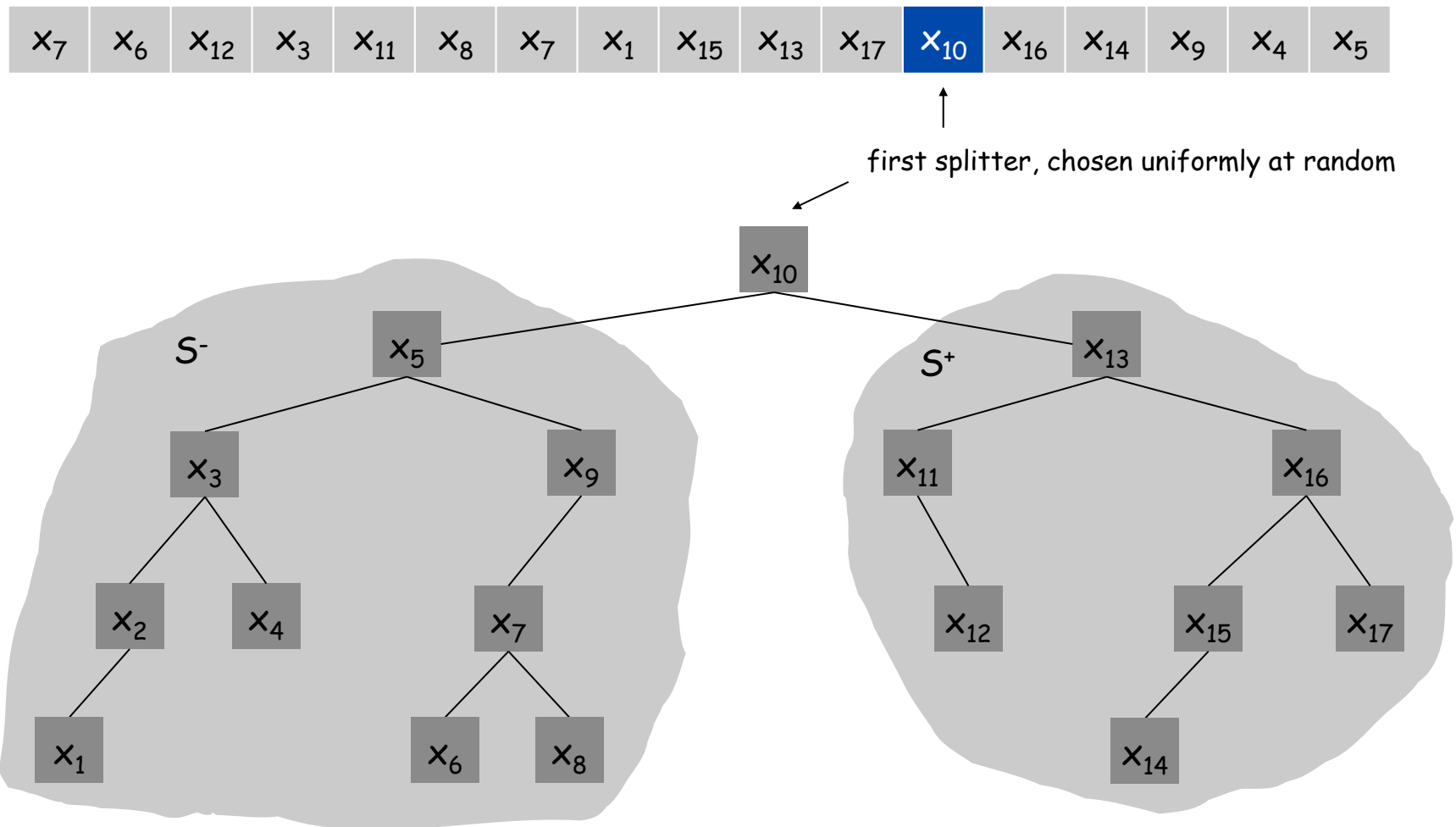
Randomize. Protect against worst case by choosing splitter at **random**.

Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \dots < x_n$.

Quicksort: BST Representation of Splitters

BST representation. Draw recursive BST of splitters.

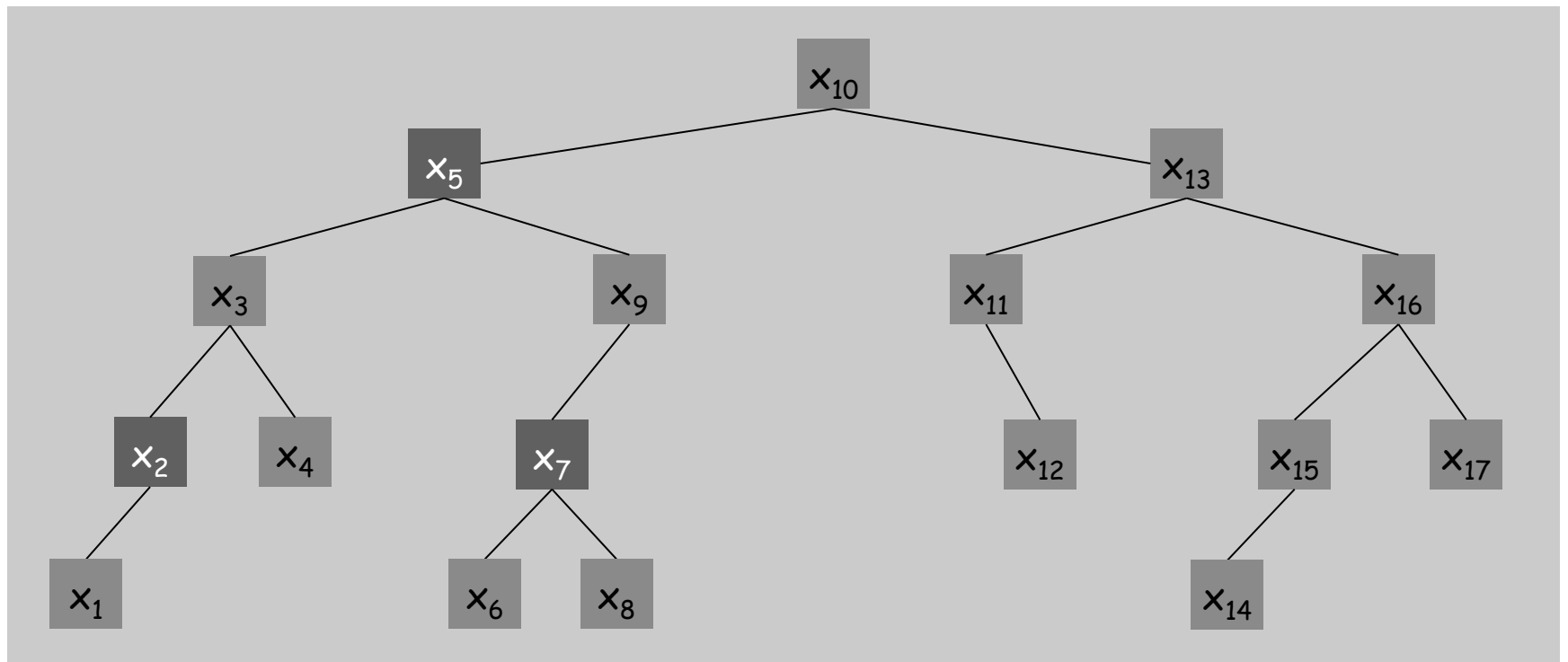


Quicksort: BST Representation of Splitters

Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7 .
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6 .

Claim. $\Pr[x_i \text{ and } x_j \text{ are compared}] = 2 / |j - i + 1|$.



Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is $O(n \log n)$.

Pf.

$$\sum_{1 \leq i < j \leq n} \frac{2}{i+j-1} = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{1}{i+j-1} \leq 2 \sum_{i=1}^{n-1} \frac{1}{i} \approx 2 \int_1^n \frac{1}{x} dx = 2 \ln n$$

\uparrow
 probability that i and j are compared

Theorem. [Knuth 1973] Stddev of number of comparisons is $\sim 0.65N$.

Ex. If $n = 1$ million, the probability that randomized quicksort takes less than $4n \ln n$ comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \geq k\delta] \leq 1 / k^2$.