Assignment 2 solutions

Exercise 3.2

(a) Let $R(x)$ be the polynomial-time reduction $L_2 \leq_P L_1$, i.e.,

$$x \in L_2 \Leftrightarrow R(x) \in L_1. \tag{1}$$

Since $L_1 \in NP$, there is a polynomial-time TM M such that

$$R(x) \in L_1 \Leftrightarrow \exists u : M(R(x), u) = 1. \tag{2}$$

(1) together with (2) imply that $x \in L_2 \Leftrightarrow \exists u : M(R(x), u) = 1$, where $M(R(x), u)$ runs in polynomial time, i.e., $L_2 \in NP$.

(b) Take any $L_2 \in SPACE(n^2) \setminus SPACE(n)$, which exists because of the Time Hierarchy theorem, and let M be the $O(n^2)$-space TM that decides L_2. Then the reduction that takes x and produces $x1^{|x|^2-|x|}$ runs in polynomial ($O(n^2)$) time. Let $L_1 = \{x1^{|x|^2-|x|} : x \in L_2\}$. $L_1 \in SPACE(n)$, since the TM M' that (i) takes input $y = x1^{|x|^2-|x|}$, (ii) strips away $1^{|x|^2-|x|}$, and (iii) runs $M(x)$, uses space $O(|x|^2 - |x| + |x|^2) = O(|x|^2) = O(|y|)$.

(c) Since NP is closed under polynomial reductions, but $SPACE(n)$ is not, we have $NP \neq SPACE(n)$.

Exercise 3.5

Let $L \in DTIME(n^2) \setminus DTIME(n)$, that exists because of the Time Hierarchy Theorem. The function $f(n) = n + L(n)$ takes values $n \leq f(n) \leq n+1$. Assume that $f(n)$ is time-constructible, i.e., there is TM $M(n) = f(n)$ that runs in $O(n)$ time. Then M uses $O(n)$ space, and TM M' that runs $M(n)$ and outputs $M(n) - n$ decides L in $O(n)$ space, i.e., $L \in SPACE(n)$, a contradiction.

Exercise 4.4

First, note that the language is in NL because the following algorithm is in NL: For every pair $u, v \in G$, if $PATH(G, u, v) = 0$ then return 0. After enumerating all pairs without termination, return 1. It is in NL because the enumeration of u, v needs $O(\log n)$ space, and $PATH \in NL$.

We reduce $PATH$ to the language. For any input G, u, v of $PATH$, we add to G edges (v, w) for all $w \in V$. This is done in log-space, since we just need to output 1 to any edge inquiry $EDGE?(w, x)$. Let G' be the new graph. Then it is easy to see that $PATH(G, u, v) = 1 \Leftrightarrow G'$ is strongly connected.

Exercise 4.7

(a) A polynomial-time $NDTM_{r-o}$ can simulate a polynomial-time NDTM by copying its read-once certificate onto its working tape, which can be done in polynomial time since the certificate is of polynomial length. Then Theorem 2.6 holds if NDTMs are replaced by $NDTM_{r-o}$'s.

(b) Since the NDTM M uses only $O(\log n)$ working-tape space and only $p(n)$ certificate-tape positions, each configuration for this NDTM uses only $O(\log n + \log p(n)) = O(\log n)$ space, i.e., there are at most $2^{O(\log n)} = O(q(n))$ configurations for some polynomial q. Therefore, there is a TM M' such that, given input x and a certificate u with $|u| = p(|x|)$, constructs in polynomial time the configuration graph of $M(x, u)$, and checks (in polynomial time) whether there is a path between the starting and the accepting configuration.

(c) Let $L \in NP$, then there is a NDTM M such that $x \in L \Leftrightarrow \exists u : M(x, u) = 1$. Following the Cook-Levin theorem proof, there is polynomial p such that the $p(|x|)$ computation steps of $M(x, u)$ correspond to tape, state, head(s) location snapshots, each of length $p(|x|)$. The correctness of each location (i, j) of this $p(|x|) \times p(|x|)$ of this matrix can be checked by checking whether locations $(i-1, j-1), (i-1, j), (i-1, j+1)$ produce (i, j). Only $O(\log |x|)$ bits are needed (for indexing) by a NDTM M' that takes this matrix as its certificate (of size $O(p^2(|X|)$, and checks whether it's a correct accepting computation of $M(x, u)$ for some u, by moving back-and-forth on this matrix.

Exercise 4.12

Savitch's theorem proof uses *polyL* space, but *exponential time*. *SC* is not the same as *polyL*$\cap P$ because $L \in$ *polyL*$\cap P$ if it has a polynomial-time solver M and a polylogarithmic-space solver M', but it may be the case $M \neq M'$.