Assignment 4

Do the following problems and exercises from the book. Note that the ordering reflects the order in which the relevant material is being covered by the course. Exercises are spread throughout a chapter, and problems can be found at the end of each chapter.

Exercise 7.4

Hint: What happens if we run $M(x)$ k independent times when $x \in L$ and when $x \notin L$?

Exercise 7.6

(a) *Hint:* Take exercise 7.2 as done (do it for fun, if you want). Essentially, the exercise asks you to show how to simulate the TM of Definition 7.7 (call it N) that may not halt with the TM M of the exercise and vice-versa, since then a language L that is decided by one can be decided by the other. If you have N, show how M is just the running of N for a finite (polynomial) number of steps, to get M's probabilities. For the other direction, show how N runs M repeatedly and fulfils the requirements of Definition 7.7.

(b) *Hint:* see lecture notes.

Exercise 7.8

(a) Show that if $\overline{3SAT} \leq_r 3SAT$ (i.e., $\overline{3SAT} \in BP \cdot NP$), and $M(x, r)$ is the probabilistic TM with random bits r in this randomized reduction, then there is a polynomial (in $|x|$)-size random string r_0 that works *for all* x of a given size $|x| = n$, i.e., $3SAT(M(x, r_0)) = \overline{3SAT}(x)$, $\forall x \in \{0, 1\}^n$. (*Hint:* This is exactly the same argument as the one in the beginning of the proof of Theorem 7.14.)

(b) Fix some x. Modify the proof in (a) to show how it works *for all* u if the input is (x, u), where $|u| = q(|x|)$ for a polynomial q. (*Hint:* The only change is that r_0 will now be of size polynomial in $q(|x|)$, which is still polynomial in $|x|$.)

(c) Note that in (b) you have shown that for a given x, *there is* a random string $|r_0| = s(|x|)$ (for polynomial s), such that *for all* polynomial-size strings u_1, u_2, u_3 we have $3SAT(M(x, r_0, u_1, u_2, u_3)) = \overline{3SAT}(x, u_1, u_2, u_3)$, i.e., $M()$ is a *deterministic* polynomial reduction of $\overline{3SAT}$ to $3SAT$.

Show that $\Sigma_3^p = \Sigma_4^p$, so PH collapses to level 3. (*Hint:* Σ_4^p languages have polynomial-time TM M' s.t.

$$\exists u_1 \forall u_2 \exists u_3 \forall u_4 : M'(x, u_1, u_2, u_3, u_4) = 1.$$

Can you incorporate r_0 into this formula to get the deterministic reduction of $\overline{3SAT}$ to $3SAT$ and then just follow the solution of exercise 5.3 in Assignment 3?)

Exercise 7.10

(*Hint:* Think a "Snakes and ladders" game, with no ladders and snakes all over the place! Use recursion to calculate the expected number of steps to reach t.)