# Chapter 2: *NP* and *NP*-completeness

### Definition 1

$L \in NP$ if there exists a polynomial-time TM $M$ (called the verifier for $L$) and polynomial $p : \mathbb{N} \to \mathbb{N}$ such that $\forall x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} : M(x,u) = 1.$$

**Note:** TM $M$ runs in polynomial time $q(|x| + |u|) = (|x| + |x|^c)^d = O(|x|^{cd})$ for $q(n) = n^d$ and $p(n) = n^c$.

**Note:** Equivalent definition via Not-Deterministic TM's (cf. 2.1.2).

Examples: INDSET, TSP, SUBSUM, LP, 0-1 IP, GRAPHISO, COMPSITES, FACTORING, CONNECTIVITY

### Definition 2 (alternative)

$L \in NP$ if $\exists$ poly-time non-deterministic TM $M(x) = 1 \Leftrightarrow x \in L$.

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### Definition 3

$EXP = \bigcup_{c \geq 0} DTIME(2^{n^c})$

Claim 1

$P \subseteq NP \subseteq EXP$

**Proof $P \subseteq NP$:** $L \in P \Rightarrow$ poly-time TM $M$ decides $L$
$\Rightarrow M'(x, 0 \text{ or } 1) := M(|x|)$ is a verifier for $L$ with $p(|x|) = |x|^0 = 1$

$NP \subseteq EXP$**:** $L \in NP \Rightarrow$ poly-time verifier $M(x, u)$ with $|u| = p(|x|)$
$\Rightarrow$ the following TM $M'$ decides $L$ in $O(2^{n^{p(|x|)}})$ time

**Algorithm** $M'(x)$

```
for each $u \in \{0,1\}^{p(|x|)}$ do
    if $M(x,u) = 1$ then
        return 1
return 0
```

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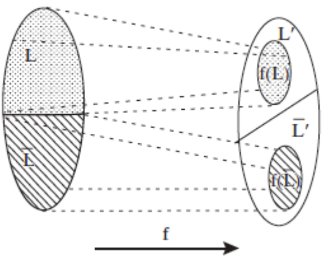
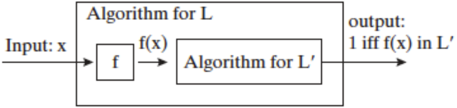
## Definition 4 (Reductions)

$L$ is poly-time Karp reducible to $L'$ ($L \leq_P L'$) if there is poly-time computable function $f$ s.t. $\forall x \in \{0,1\}^*$

$$x \in L \Leftrightarrow f(x) \in L'$$




Input: x

Algorithm for L

f

f(x)

Algorithm for L′

output:
1 iff f(x) in L′

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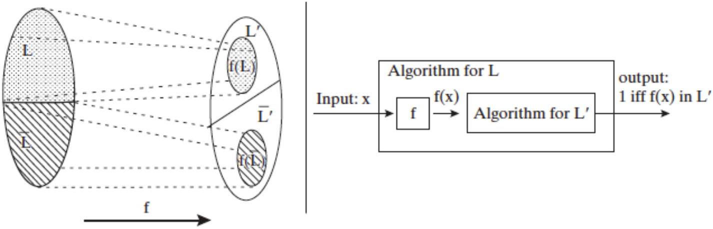

- Why $L \leq_P L'$?
  - $L' \in P \Rightarrow L \in P$, i.e., if $L'$ easy then $L$ easy
  - $L \notin P \Rightarrow L' \notin P$, i.e., if $L$ hard then $L'$ hard
- Karp vs. Turing reductions: In $L \leq_P^T L'$ we are allowed to use a polynomial number of calls to $L'$ (not just one).

Theorem 5 (Transitivity)

*If $L \leq_P L'$ and $L' \leq_P L''$, then $L \leq_P L''$.*

**Proof:**

- $x \rightarrow f_1(x)$ with $f_1(x)$ computable in $O(|x|^c)$ $(L \leq_P L')$
- $y \rightarrow f_2(y)$ with $f_2(y)$ computable in $O(|y|^d)$ $(L' \leq_P L'')$
- $x \rightarrow f_2(f_1(x))$ is $L \leq_P L''$, with $f_2(f_1(x))$ computable in $O((|x|^c)^d) = O(|x|^{cd})$.

since $x \in L \Leftrightarrow f_1(x) \in L' \Leftrightarrow f_2(f_1(x)) \in L''$. □

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## Definition 6 (*NP*-hardness)

$L$ is *NP*-hard if $L' \leq_P L$ for every $L' \in NP$.

## Definition 7 (*NP*-completeness)

$L$ is *NP*-complete if

1. $L$ is *NP*-hard, and
2. $L \in NP$.

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**Theorem 8**

1. *If L is NP-hard and $L \in P$, then $P = NP$.*
2. *If L is NP-complete, then $L \in P \Leftrightarrow P = NP$.*

**Proof:**

1. $L' \in NP \overset{L \text{ is } NP-\text{hard}}{\Rightarrow} L' \leq_P L \overset{L \in P}{\Rightarrow} L' \in P$
2. If $P = NP \Rightarrow L \in P$. If $L$ is NP-complete and $L \in P$ then $P = NP$ from (1).

□

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**Theorem 9**

*The following language is NP-complete:*

$$TMSAT = \{\langle a, x, 1^n, 1^t\rangle : \exists u \in \{0,1\}^n \textit{ s.t. } M_a(x,u) = 1 \textit{ within } t \textit{ steps}\}$$

**Proof:**

1. $TMSAT \in NP$ (easy)
2. $L \in NP \Rightarrow$ verifier $M(x,u)$ s.t.
   $x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|} : M(x,u) = 1$ in $q(|x| + p(|x|))$ steps
   $\Rightarrow x \to \langle \llcorner M \lrcorner, x, 1^{p(|x|)}, 1^{q(|x|+p(|x|))}\rangle$ gives $L' \leq_P TMSAT$.

□

...but too artificial, like rewriting the *NP*-completeness definition!

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### Definition 10 (CNF formula)

Given $n$ boolean variables $u_1, u_2, \ldots, u_n$

- $u_1, \bar{u}_1$ are the literals for variable $u_1$.
- A clause is an OR of literals, e.g., $(u_1 \vee \bar{u}_3 \vee u_4)$.
- A CNF formula is an AND of clauses, e.g., $(u_3) \wedge (\bar{u}_1 \vee u_3 \vee \bar{u}_4) \wedge (u_2 \vee \bar{u}_3)$.
- CNF formula is satisfiable if there is truth assignment to vars that makes formula true.

### Definition 11 (SAT)

Given a CNF formula with $n$ vars and $k$ clauses, is it satisfiable?

**Theorem 12 (Cook-Levin)**

*SAT is NP-complete.*

**Proof:**

1. $SAT \in NP$: easy
2. $\forall L \in NP : L \leq_P SAT$: Idea is like TMSAT, but for $L \in NP$ explicitly write the configuration of verifier $M(x, u)$ at every step as a big tableau of $O(q(|x| + |u|)) = O(q(|x| + p(|x|)))$ rows and $O(q(|x| + p(|x|)))$ columns
   $\Rightarrow$ encode certificate $u$ bits as vars, and correctness conditions of transition from step $i$ configuration to $i + 1$ configuration as clauses
   $\Rightarrow$ SAT formula $\phi_x(u)$ is satisfiable iff $\exists u$ to make $M(x, u)$ accept
   $\Rightarrow \phi_x(u) \in SAT$ iff $x \in L$

□

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### Levin-reductions

Reduction of Theorem 12 is Levin: One-to-one mapping between satisfying assignment for $\phi_x(u)$ and certificate for $x \in L$.

### Proving decision problem $L$ is $NP$-complete:

1. Prove that $L \in NP$.
2. Pick $NP$-complete problem $L'$. Show that $L' \leq_P L$.

### Example: 3SAT

SAT with all clauses with 3 literals.

Theorem 13

*3SAT is NP-complete.*

**Proof:**

1. $3SAT \in NP$: easy
2. $\forall L \in NP : L \leq_P SAT$: We show that $SAT \leq_P 3SAT$. Given a CNF formula $\phi(x)$ for SAT with $n$ vars $x_1, x_2, \ldots, x_n$ and $k$ clauses, we construct a 3SAT formula $\psi(y)$ s.t. $\phi(x) \in SAT \Leftrightarrow \psi(y) \in 3SAT$.
   - Keep all vars $x$
   - Let $C$ be a clause of $\phi(x)$. If $C$ has more than 3 literals break it into two clauses $C', C''$ using a new var $z_c$ as follows:

$$(x_1 \vee \bar{x}_2 \vee x_4 \vee \bar{x}_6 \vee \bar{x}_7) \rightarrow \begin{cases} (x_1 \vee \bar{x}_2 \vee z_c) \\ (x_4 \vee \bar{x}_6 \vee \bar{x}_7 \vee \bar{z}_c) \end{cases}$$

   - If $C$ fewer than 3: Repeat last literal

□

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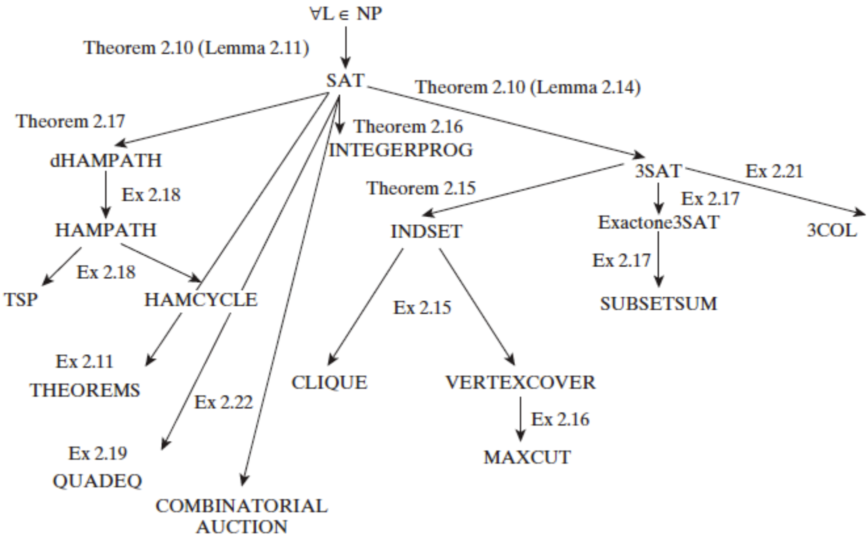

### Theorem 14

*If $P = NP$, then can also solve in poly-time the search version of SAT, i.e., compute a satisfying assignment.*

**Proof:** Let $A$ be a poly-time algorithm that decides SAT. Then the following algorithm $B$ computes a satisfying assignment for CNF $\phi(x)$:

**Algorithm** $B(\phi(x))$

```
$V[1..n]$ = truth assignment for $x_1, x_2, \ldots, x_n$
if $A(\phi(0, x_2, \ldots, x_n)) = 1$ then
    $V[1] = 0$
    $V[2, \ldots, n] = B(\phi(0, x_2, \ldots, x_n)$
    return $V$
else if $A(\phi(1, x_2, \ldots, x_n)) = 1$ then
    $V[1] = 1$
    $V[2, \ldots, n] = B(\phi(1, x_2, \ldots, x_n)$
    return $V$
return No
```

□

**Note:** SAT is self-reducible

### Theorem 15

*If $P = NP$, then can also compute in poly-time a certificate of any $L \in NP$.*

**Proof:**

The reduction $L \leq_P SAT$ is a Levin-reduction, i.e., if $x \in L$ then we can compute the certificate for $f(x) \in SAT$ in poly-time and from it the certificate for $x$.

□

### Definition 16

$coNP = \{L : \bar{L} \in NP\}$.

**Example:** $\overline{SAT} = \{\phi : \phi \text{ is unsatisfiable}\}$

**Note:** $L \in coNP$ has a certifier for its "No" instances. Does it have one for its "Yes" instances...?

### Definition 17

$L \in coNP$ if there exists a polynomial-time TM $M$ and polynomial $p : \mathbb{N} \to \mathbb{N}$ such that $\forall x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \forall u \in \{0,1\}^{p(|x|)} : M(x,u) = 1.$$

- Definitions 16 and 17 are equivalent (why?)
- Note that Definition 17 is exactly the same as our definition of NP *except* that $\forall u$ instead of $\exists u$.

**Theorem 18**

$TAUTOLOGY = \{\phi : \phi$ *is a tautology*$\}$ *is coNP-complete.*

**Proof:**

- $TAUTOLOGY \in co - NP$ from Definition 17.
- $L \in coNP \rightarrow \bar{L} \in NP \rightarrow \bar{L} \leq_P SAT$ (Cook-Levin)
  $\rightarrow x \notin \bar{L} \Leftrightarrow \phi_x \notin SAT$
  $\rightarrow x \in L \Leftrightarrow \neg\phi_x \in TAUTOLOGY \rightarrow L \leq_P TAUTOLOGY$ □

**Theorem 19**

$L \in coNP$*-complete* $\Leftrightarrow \bar{L} \in NP$*-complete.*

**Theorem 20**

$P = NP \Rightarrow P = NP = coNP.$

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**Definition 21**

$NEXP = \bigcup_{c \geq 0} NTIME(2^{n^c})$

**Theorem 22**

$EXP \neq NEXP \Rightarrow P \neq NP$ (or $P = NP \Rightarrow EXP = NEXP$)

**Proof:** Use input size to cheat! (padding)
Obviously $EXP \subseteq NEXP$. Show $NEXP \subseteq EXP$.
$L \in NTIME(2^{n^c}) \Rightarrow L_{pad} = \{\langle x, 1^{2^{|x|^c}} \rangle : x \in L\}$. Poly-time NDTM:

**Algorithm** $M_{L_{pad}}(y)$

```
if $y \neq \langle z, 1^{2^{|z|^c}} \rangle$ for some $z$ then
    return 0
return $M_L(z)$
```

$\Rightarrow L_{pad} \in NP \Rightarrow L_{pad} \in P \Rightarrow L \in EXP$ □

# Chapter 2: *NP* and *NP*-completeness

- The philosophical importance of *NP* (read 2.7.1)
- *NP* and (short) mathematical proofs:

  $$\mathit{THEOREMS}_{\mathcal{A}} = \{\langle \phi, 1^n \rangle : \phi \text{ has formal proof of} \leq n \text{ steps in system } \mathcal{A}\}$$

- Is there anything between *P* and *NP*-complete? (factoring, graph isomorphism, Nash equilibrium, Ladner's theorem)
- Coping with *NP*-hardness (approximation algorithms, average-case complexity)
- Read chapter notes & history!