Derivation of Equations of Motion for Inverted Pendulum Problem

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- It is defined as the work needed to accelerate a body of a given mass from rest to its stated velocity
- In classical mechanics, the kinetic energy E_k of a point object is defined by its mass m and velocity v:

$$E_k = \frac{1}{2}mv^2$$

Potential Energy

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The energy of an object or a system due to the position of the body or the arrangement of the particles of the system

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- The amount of gravitational potential energy possessed by an elevated object is equal to the work done against gravity in lifting it
- Thus, for an object at height h, the gravitational potential energy E_p is defined by its mass m, and the gravitational constant g:

$$E_p = mgh$$

Lagrangian Mechanics

- An analytical approach to the derivation of E.O.M. of a mechanical system

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- An analytical approach to the derivation of E.O.M. of a mechanical system
- Lagrange's equations employ a single scalar function, rather than vector components
- To derive the equations modeling an inverted pendulum all we need to know is how to take partial derivatives

Lagrangian

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In classical mechanics, the natural form of the Lagrangian is defined as $\mathcal{L} = E_k - E_p$

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In classical mechanics, the natural form of the Lagrangian is defined as $\mathcal{L} = E_k - E_p$

- E.O.M. can be directly derived by substitution using EulerLagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

Inverted Pendulum Problem

- The pendulum is a stiff bar of length L which is supported at one end by a frictionless pin
- The pin is given an oscillating vertical motion s defined by:

$$s(t) = A \sin \omega t$$

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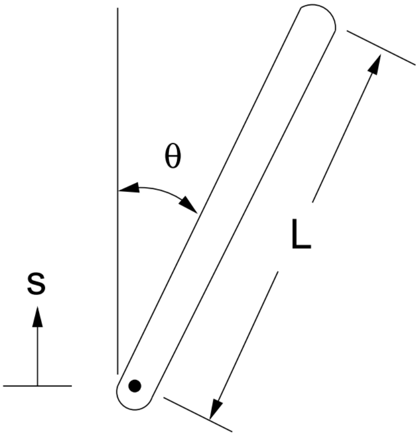
Problem

Our problem is to derive the E.O.M. which relates time with the acceleration of the angle from the vertical position

Visualization



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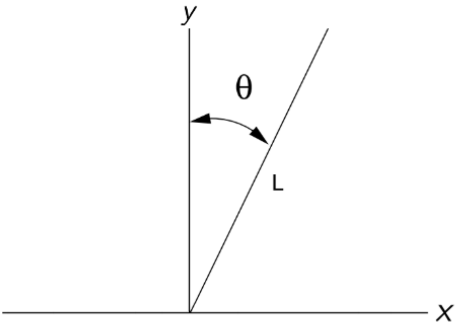

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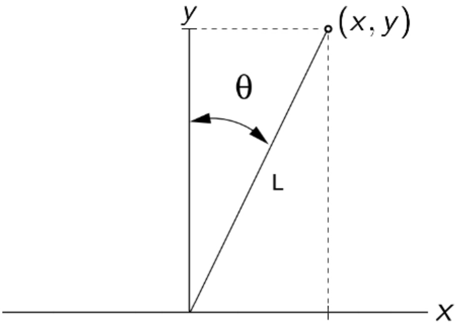

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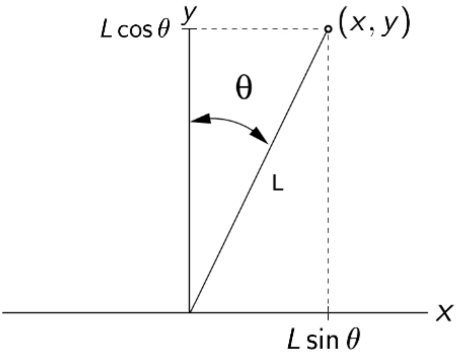

Setup

From the figure on previous page we know

$$x = L\sin\theta \qquad \dot{x} = L\cos(\theta)\dot{\theta}$$

$$y = L\cos\theta \qquad \dot{y} = -L\sin(\theta)\dot{\theta}$$

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Recall the definition of the Lagrangian

$$\mathcal{L} = E_k - E_p$$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgy$$

Setup Continued...

Velocity is a vector representing the change in position, hence

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= L^2\dot{\theta}^2\cos^2\theta + L^2\dot{\theta}^2\sin^2\theta \\ &= L^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \\ &= L^2\dot{\theta}^2 \end{aligned}$$

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$$\mathcal{L} = \frac{1}{2}mv^2 - mgy$$

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta$$

Setup Continued...

Recall the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

We shall now compute both sides of the equation and solve for $\ddot{\theta}$

Computing $\frac{\partial \mathcal{L}}{\partial \theta}$

$$\mathcal{L} = \frac{1}{2} m L^2 \dot{\theta}^2 - mgL\cos\theta$$

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$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta} &= 0 + mgL \sin\theta \\ &= mgL \sin\theta \end{aligned}$$

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We compute in two steps:

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Applying Euler-Lagrange Equation

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$mL^2\ddot{\theta} = mgL\sin\theta$$

$$\ddot{\theta} = \frac{g}{L}\sin\theta$$

Which is the equation presented in the assignment.

Setup

With the oscillator we must modify the equation for y

$$x = L \sin \theta \qquad \dot{x} = L \cos(\theta)\dot{\theta}$$

$$y = L \cos \theta + A \sin \omega t \qquad \dot{y} = -L \sin(\theta)\dot{\theta} + A\omega \cos \omega t$$

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Again, we use the definition of the Lagrangian

$$\mathcal{L} = E_k - E_p$$
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\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 \\
&= L^2\dot{\theta}^2 \cos^2\theta + L^2\dot{\theta}^2 \sin^2\theta - 2AL\omega \sin\theta \cos(\omega t)\dot{\theta} + A^2\omega^2\cos^2(\omega t) \\
&= L^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) - 2AL\omega \sin\theta \cos(\omega t)\dot{\theta} + A^2\omega^2\cos^2(\omega t) \\
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Substituting into the equation for the Lagrangian we get

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\begin{aligned}
\mathcal{L} &= \frac{1}{2}mv^2 - mgy \\
\mathcal{L} &= \frac{1}{2}mL^2\dot{\theta}^2 - mAL\omega\sin\theta\cos(\omega t)\dot{\theta} \\
&\quad + \frac{1}{2}mA^2\omega^2\cos^2(\omega t) - mgL\cos\theta - mgA\sin(\omega t)
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Computing $\frac{\partial \mathcal{L}}{\partial \theta}$

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$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 - mAL\omega\cos\theta\cos(\omega t)\dot{\theta} + 0 + mgL\sin\theta - 0 = -mAL\omega\cos\theta\cos(\omega t)\dot{\theta} + mgL\sin\theta$$

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We compute in two steps:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mL^2\dot{\theta} - mAL\omega \sin\theta\cos(\omega t) + 0 - 0 - 0 \\ &= mL^2\dot{\theta} - mAL\omega\sin\theta\cos(\omega t)\end{aligned}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} - mAL\omega\cos\theta\cos(\omega t)\dot{\theta} + mAL\omega^2\sin\theta\sin(\omega t)$$

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$$=$$
$$-mAL\omega \cos\theta \cos(\omega t)\dot{\theta} + mgL \sin\theta$$

$$L\ddot{\theta} + A\omega^2 \sin\theta \sin(\omega t) = g \sin\theta$$
$$L\ddot{\theta} = g \sin\theta - A\omega^2 \sin\theta \sin(\omega t)$$
$$\ddot{\theta} = \frac{1}{L}(g - A\omega^2 \sin(\omega t)) \sin\theta$$

References

- http://en.wikipedia.org/wiki/Euler-Lagrange_equation
- http://en.wikipedia.org/wiki/Lagrangian_mechanics
- http://en.wikipedia.org/wiki/Newton's_laws_of_motion