Adaptive Quadrature

Jingjing Huang

October 31, 2012

Motivation

Recall:

$$I(f) = \int_a^b f(x)dx$$

Partition

$$a = x_1 < x_2 < ... < x_{n+1} = b$$

and denote $h_i = x_{i+1} - x_i$, then

$$I(f) = \sum_{i=1}^{n} I_i$$

where,

$$I_i = \int_{x_i}^{x_{i+1}} f(x)dx$$

Error In Rectangle Rule

Recall: Taylor expantion $f(x)$ about the midpoint $y_i = \frac{x_i + x_{i+1}}{2}$ (in fact, we can expand it at any points within a small neighberhood, the midpoint is a special case.) :

$$f(x) = f(y_i) + \sum_{p=1}^{\infty} \frac{(x-y_i)^p}{p!} f^{(p)}(y_i).$$

Integrate the both sides and notes that

$$\int_{x_i}^{x_{i+1}} (x - y_i)^p dx = \begin{cases} \frac{h_i^{p+1}}{(p+1)2^p} & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

Why ?

Error In Rectangle Rule

Recall: $h_i = x_{i+1} - x_i$, then:

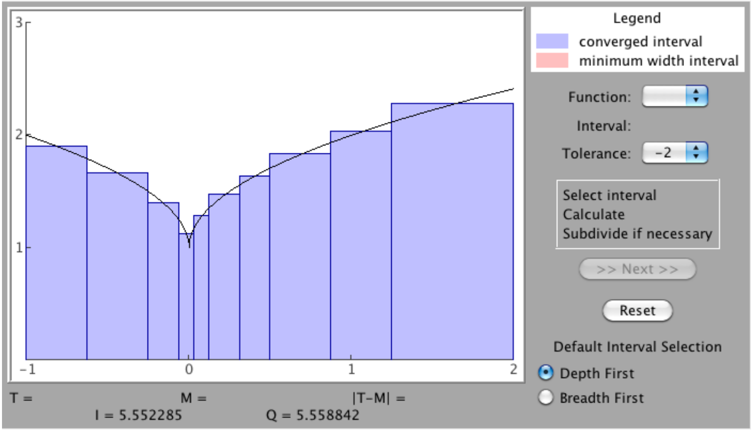
$$\int_{x_i}^{x_{i+1}} f(x)dx$$

$$= h_i f(y_i) + \underbrace{\frac{1}{24} h_i^3 f''(y_i) + \frac{1}{1920} h_i^5 f^{(4)}(y_i) + \cdots}_{\text{Error}}$$

Then

$$R_i(f) = I_i(f) - \frac{1}{24} h_i^3 f''(y_i) - \frac{1}{1920} h_i^5 f^{(4)}(y_i) + \cdots$$

- Rectangle Rule



Function evaluation: n

Error In Trapezoid Rule

Recall: $h_i = x_{i+1} - x_i$, then:

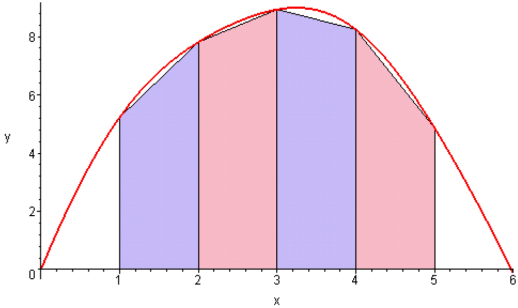
$$\int_{x_i}^{x_{i+1}} f(x)dx$$

$$= h_i \frac{f(x_i)+f(x_{i+1})}{2} \underbrace{- \frac{1}{12} h_i^3 f''(y_i) - \frac{1}{480} h_i^5 f^{(4)}(y_i) + \cdots}_{\text{Error}}$$

Then

$$T_i(f) = I_i(f) + \frac{1}{12} h_i^3 f''(y_i) + \frac{1}{480} h_i^5 f^{(4)}(y_i) + \cdots$$

- Trapezoid Rule



Function evaluation: n+1

Simpson's Rule

Recall: the Rectangle Rule and Trapezoid Rule

$$R_i(f) = I_i(f) - \frac{1}{24} h_i^3 f''(y_i) - \frac{1}{1920} h_i^5 f^{(4)}(y_i) + \cdots$$

$$T_i(f) = I_i(f) + \frac{1}{12} h_i^3 f''(y_i) + \frac{1}{480} h_i^5 f^{(4)}(y_i) + \cdots$$

Then a more accurate method by combing two together

$$S_i(f) = \frac{2}{3} R_i(f) + \frac{1}{3} T_i(f)$$

$$= I_i(f) + \underbrace{\frac{1}{2880} h_i^5 f^{(4)}(y_i) + \cdots}_{\text{Error}}$$

Simpson's Rule

In a few steps, we can get:

$$S_i(f) = \frac{2}{3}R_i(f) + \frac{1}{3}T_i(f)$$

$$= \frac{1}{6}h_i[f(x_i) + 4f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})]$$

Function evaluation: 2n + 1

Adaptive Quadrature

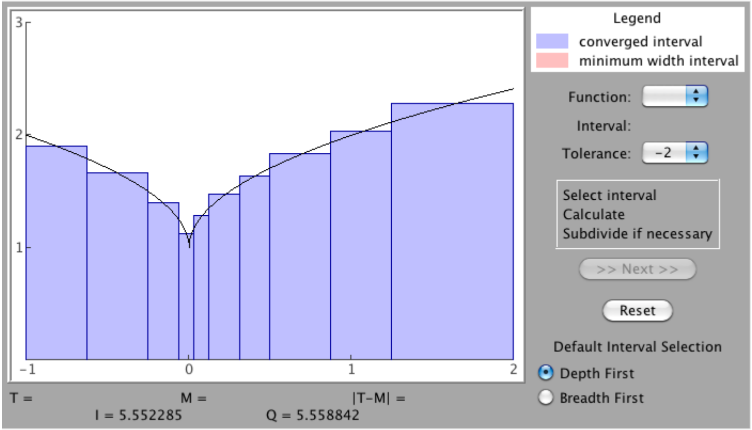
- What is an adaptive quadrature?

Definition

Given a predetermined tolerance ϵ, the algorithm automatically determines the panel size so that the computed approximation Q satifices

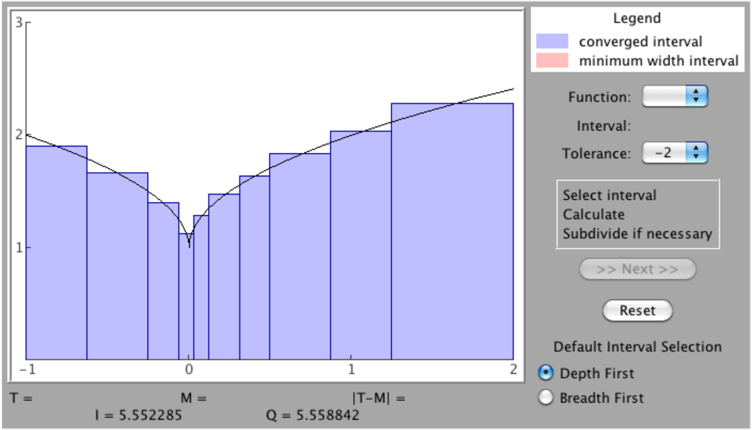
$$|Q - \int_a^b f(x)dx| < \epsilon$$

- Adaptive Quadrature using Rectangle Rule



- How to determine the tolerance ϵ_i in subinterval i ?

- Adaptive Quadrature using Rectangle Rule



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$$\epsilon_i = \frac{h_i}{b-a}\epsilon$$

- How to evaluate ϵ_i in the approximation function ?

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Error Estimation in Rectangle Rule

- $R_i(f) = I_i(f) - \frac{1}{24}h_i^3 f''(y_i) - \frac{1}{1920}h_i^5 f^{(4)}(y_i) + \cdots$, then

$$R_i(f) - I_i(f) \approx \frac{1}{24}h_i^3 f''(y_i)$$

When h_i is small.

- Are we going to calculate $f''(y_i)$?

Error Estimation in Rectangle Rule

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Error $\approx$ $1/3$ of deference between two iterations

When h_i is small and $f(x)$ is continuous.

Why ?

Error Estimation in Rectangle Rule

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Why ?

Error Estimation in Simpson's Rule

- $S_i(f) = I_i(f) + \frac{1}{2880} h_i^5 f^{(4)}(y_i) + \cdots$

$$R_i(f) - I_i(f) \approx \frac{1}{2880} h_i^5 f^{(4)}(y_i)$$

- Error Estimation

Error $\approx 1/15$ of deference between two iterations

When h_i is small and $f(x)$ is continuous.

Error Estimation in Simpson's Rule

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- Error Estimation

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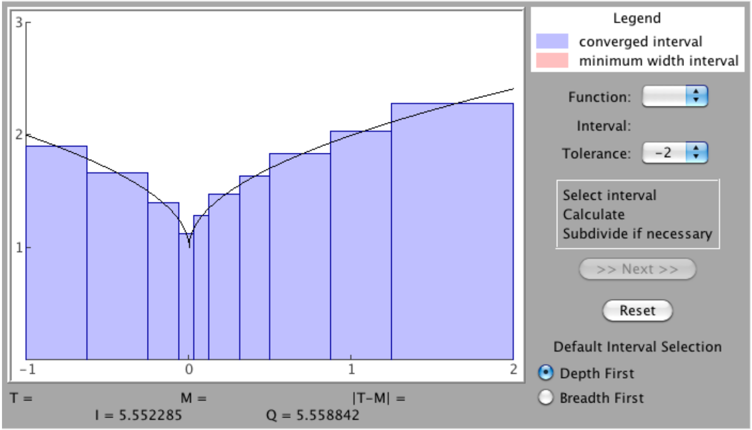
  When h_i is small and $f(x)$ is continuous.

- Adaptive Quadrature



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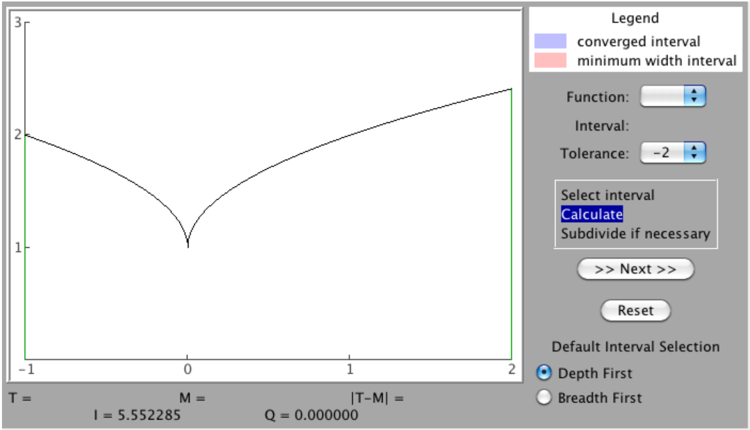

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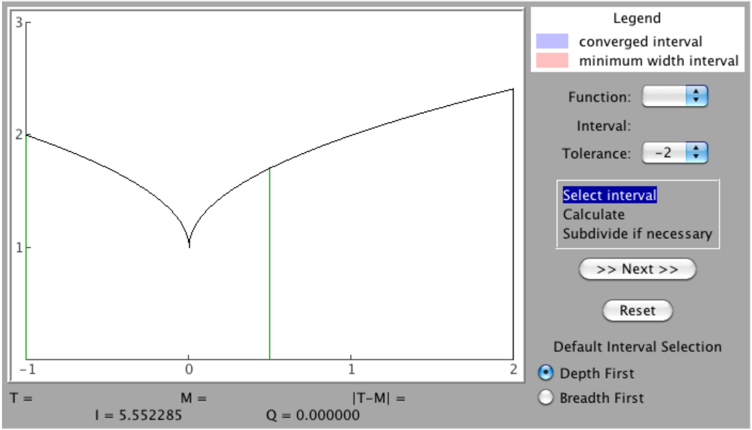

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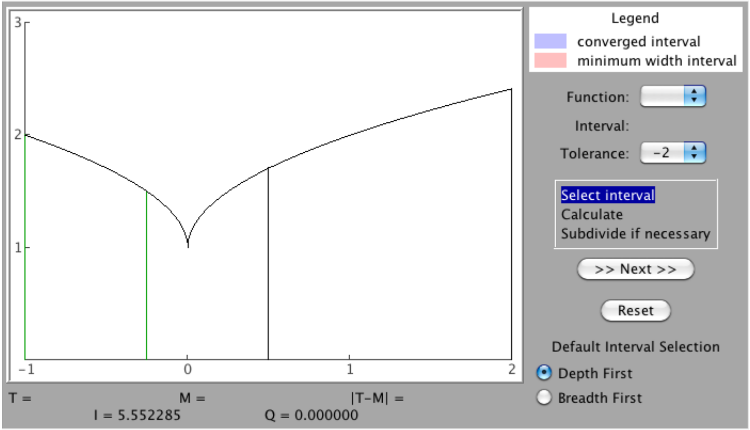

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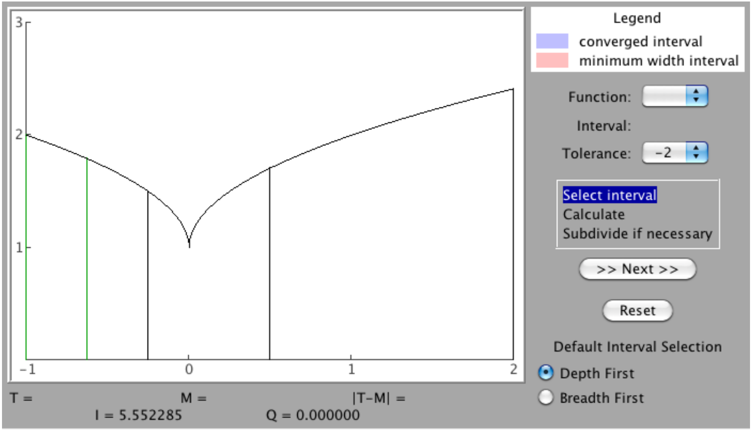

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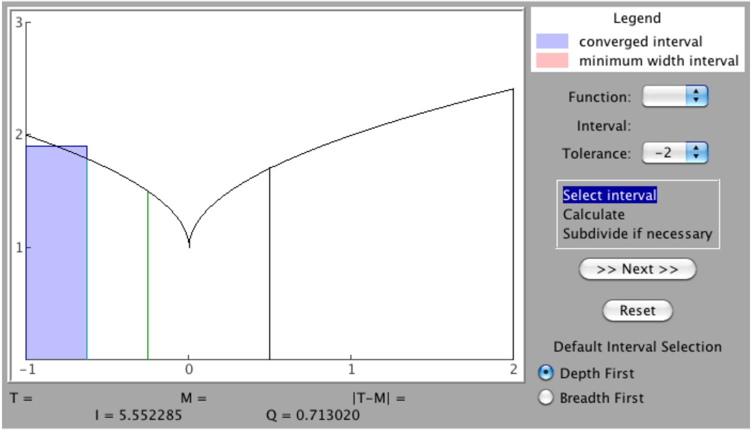

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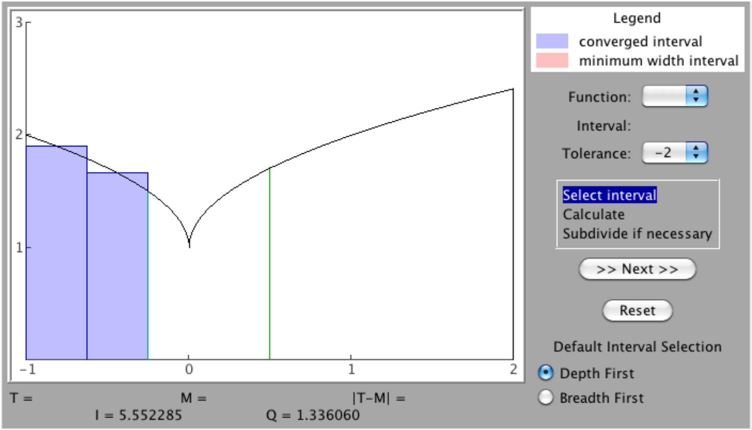

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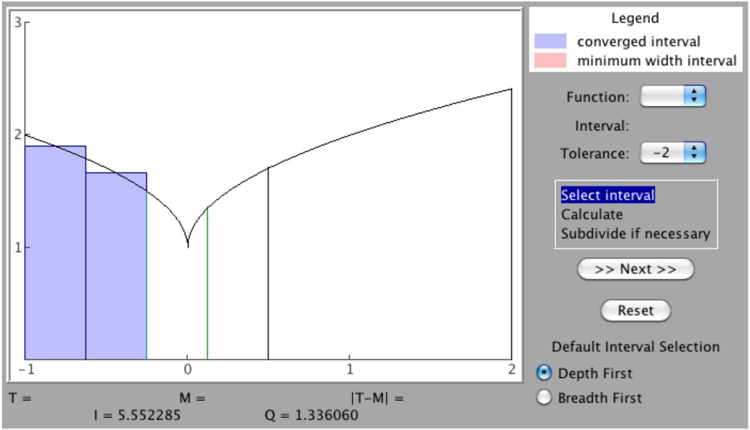

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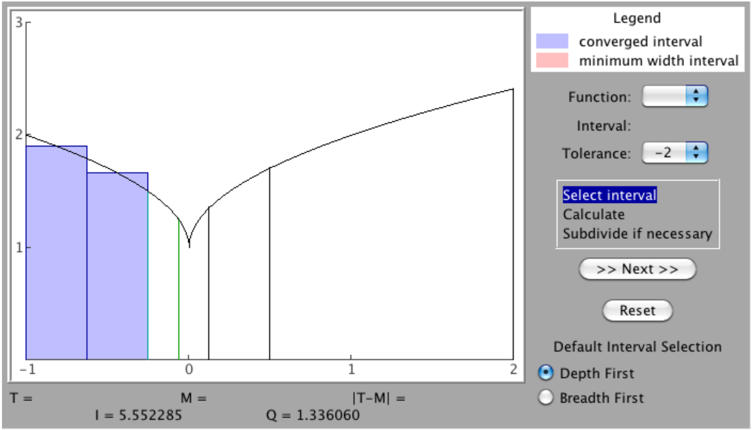

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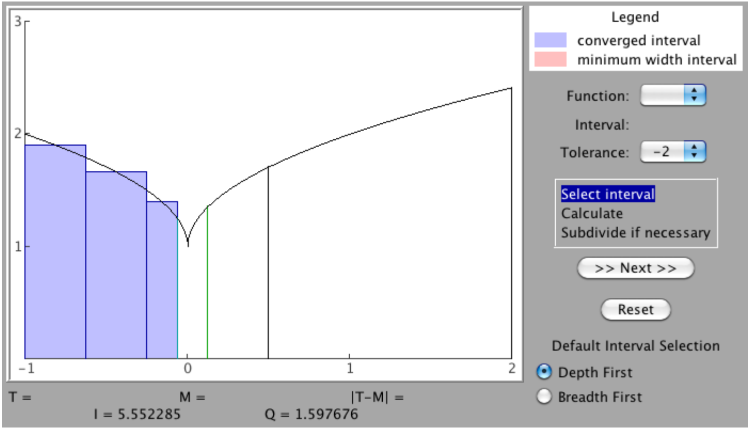

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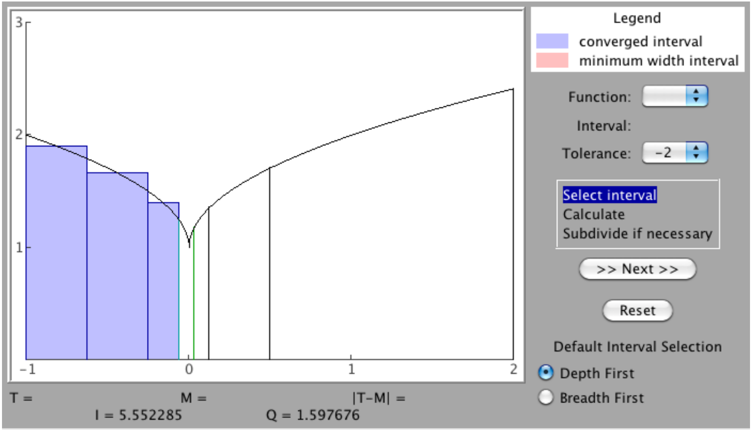

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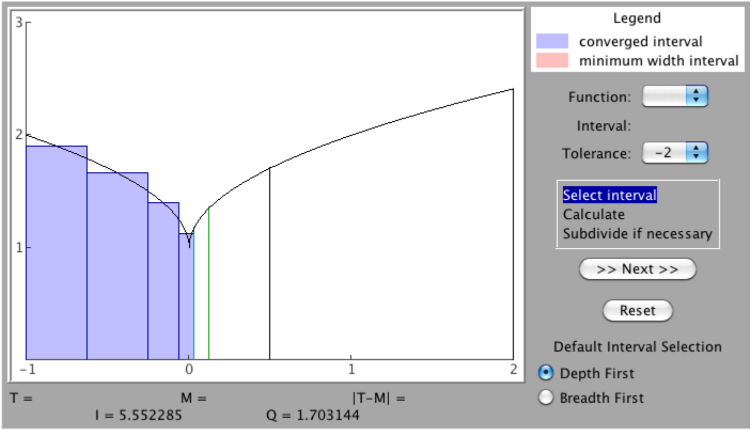

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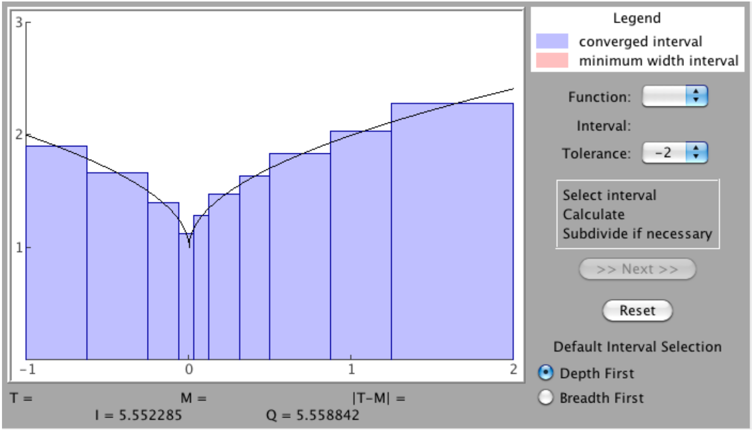
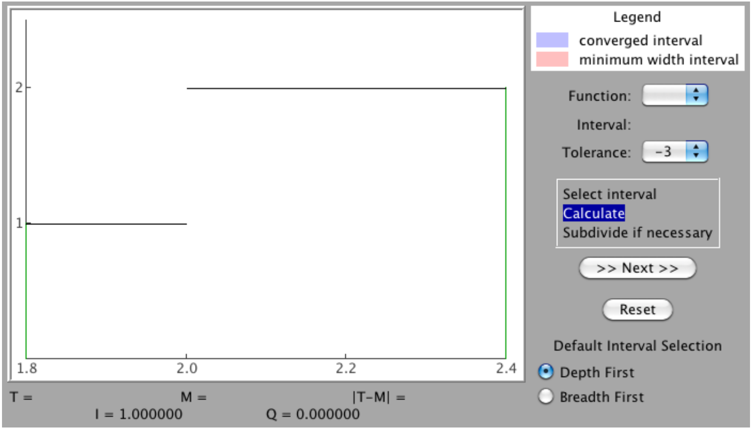


But...

- Adaptive Quadrature



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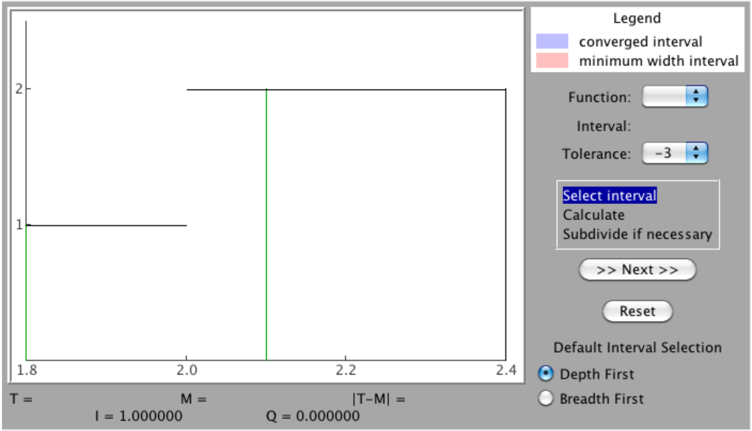

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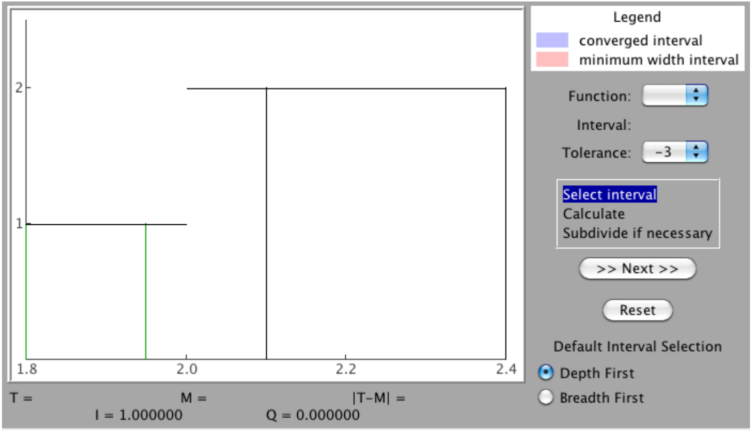

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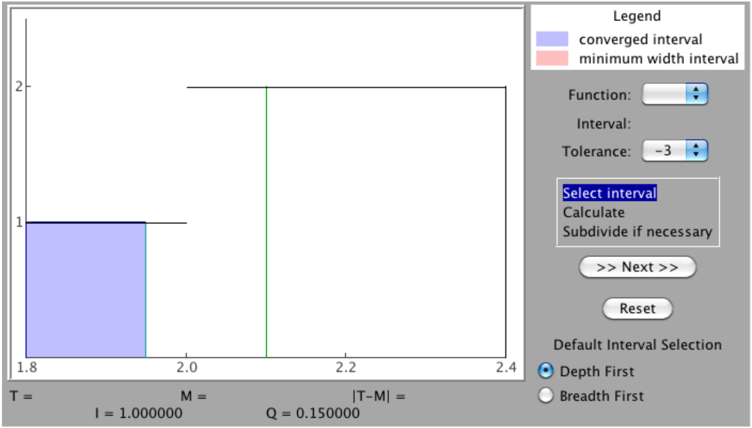

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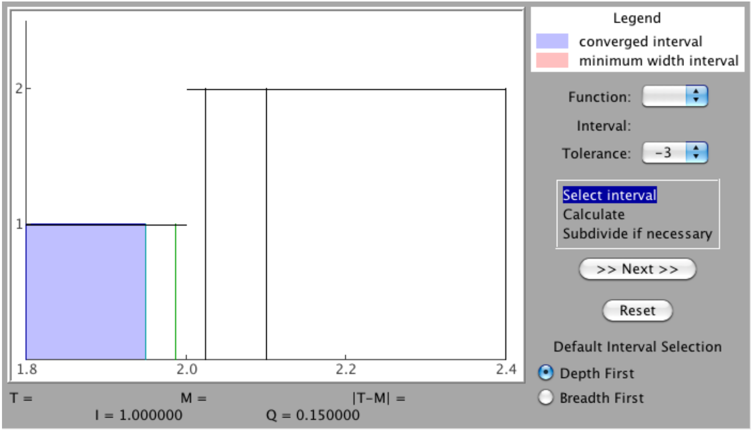

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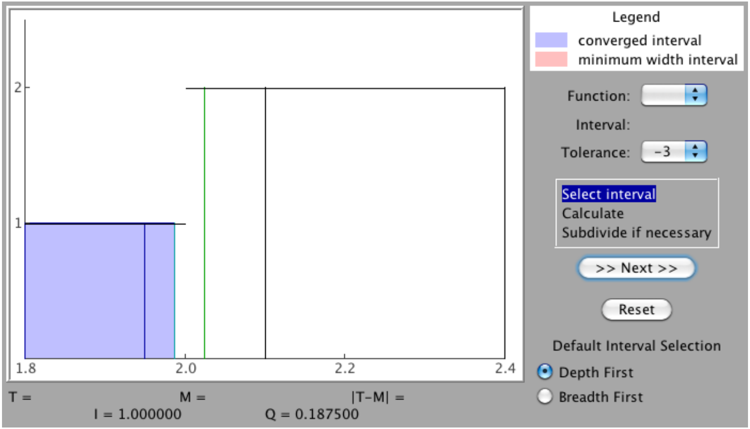

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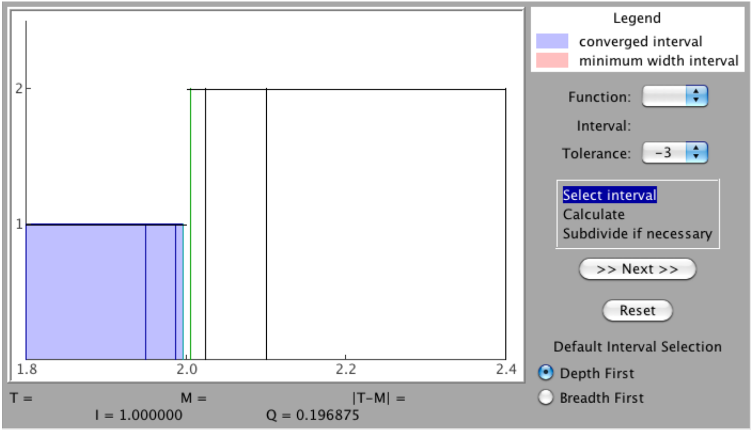

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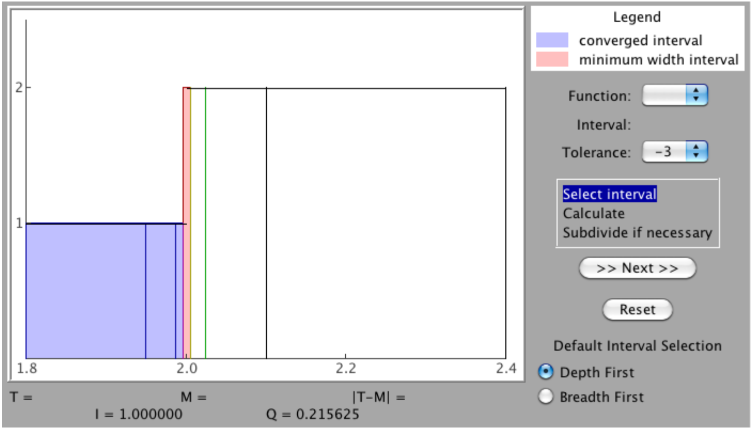

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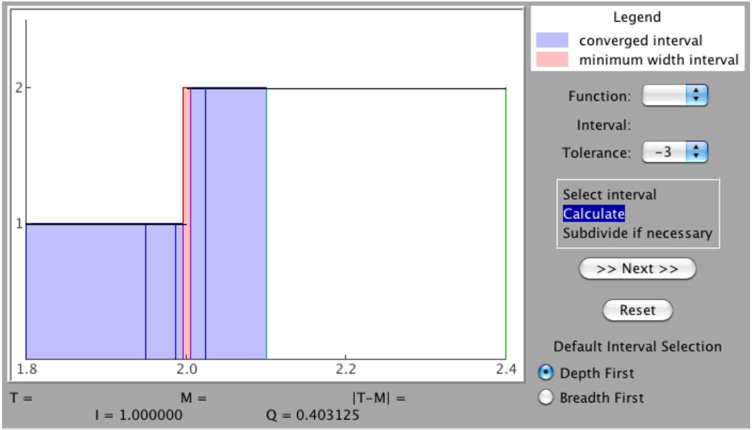

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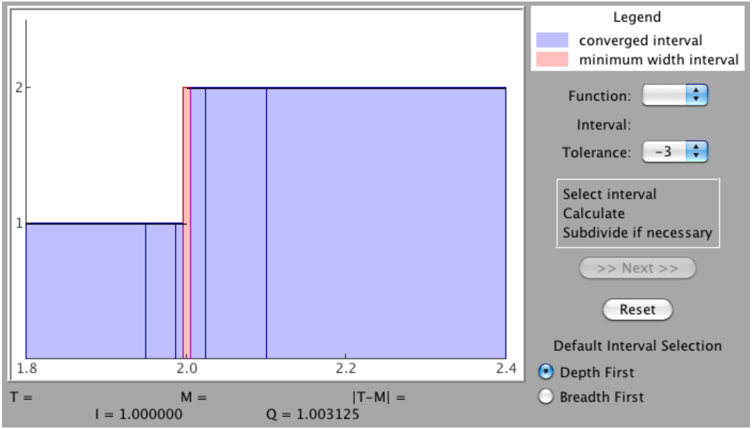

Thanks