Taylor expansion and Numerical Integration

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Taylor expansion

Definition

For a complex function $t(x)$ and a given point x_0, a power series can be used to estimate the value of the function:

$$t(x) = t(x_0) + \frac{t'(x_0)}{1!}(x - x_0) + \frac{t''(x_0)}{2!}(x - x_0)^2 + \ldots$$

where $x \in (x - x_0, x + x_0)$

Another form of Taylor expansion:

$$t(x) = \sum_{n=0}^{\infty} \frac{t^n(x_0)}{n!}(x - x_0)^n$$

- Brook Taylor

English mathematician (1685-1731)

To prove Taylor Expansion, we will use **L'Hopital's Rule**

L'Hopital's Rule

Given two functions $f(x)$, $g(x)$ and a point x_0,
if $\lim_{x \to x_0} f(x) = 0, \lim_{x \to x_0} g(x) = 0$, **then**

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

where $x \in (x - x_0, x + x_0)$

The type of limit like

$$\lim \frac{0}{0}$$

Example of L'Hopital's Rule

Suppose

$$f(x) = x,$$
$$g(x) = sin(x)$$

then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = ?$$

Proof of Taylor Expansion

Set $p(x) = \sum_{n=0}^{\mathbf{k}} \frac{t^n(x_0)}{n!}(x - x_0)^n$, for a large **k**.

To prove $p(x)$ is a good estimation of $t(x)$, it is sufficient to show

$$\lim_{x \to x_0} (p(x) - t(x)) = 0$$

We make it even stronger, prove:

$$\lim_{x \to x_0} \frac{(p(x) - t(x))}{(x - x_0)^n} = 0$$

Proof of Taylor expansion

Set

$$f(x) = p(x) - t(x),$$
$$g(x) = (x - x_0)^n$$

then

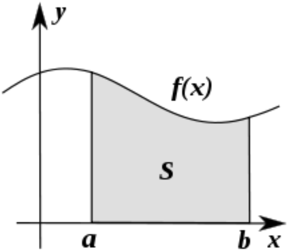
$$\lim_{x \to x_0} \frac{0}{0} = \cdots = 0$$

Numerical Integration

Recall: Given a finite number of function value $f(x_i)$, $x_i \in [a, b]$.

Or the function $f(x)$ can be evaluated any $x \in [a, b]$, calculate

$$I(f) = \int_a^b f(x)dx.$$



Numerical Integration

Recall: Partition

$$a = x_1 < x_2 < \cdots < x_{n+1} = b.$$

and denote $h_i = x_{i+1} - x_i$. Then

$$I(f) = \sum_{i=1}^{n} I_i$$

$$I_i = \int_{x_i}^{x_{i+1}} f(x)dx.$$

Numerical Integration

Recall: Rules

- Rectangle Rule

  In each interval $[x_i, x_{i+1}]$, $f(x)$ is evaluated at the midpoint

  $$y_i = \frac{x_i + x_{i+1}}{2}.$$

  Then the rectangle ruls is:

  $$I_i \approx h_i f(y_i).$$

Numerical Integration

Recall: Rules

- Trapezoid Rule

  In each interval $[x_i, x_{i+1}]$, $f(x)$ is evaluated at the endpoints

  Then the rectangle ruls is:

  $$I_i \approx h_i \frac{f(x_i)+f(x_{i+1})}{2}.$$

Error In Rectangle Rule

Recall: Taylor expansion $f(x)$ about the midpoint $y_i = \frac{x_i + x_{i+1}}{2}$:

$$f(x) = f(y_i) + \sum_{p=1}^{\infty} \frac{(x-y_i)^p}{p!} f^{(p)}(y_i).$$

Integrate the both sides and notes that

$$\int_{x_i}^{x_{i+1}} (x - y_i)^p dx = \begin{cases} \frac{h_i^{p+1}}{(p+1)2^p} & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

Why ?

Error In Rectangle Rule

Set $\widehat{x} = x - y_i$**.**

Then

$$\int_{x_i}^{x_{i+1}} (x - y_i)^p dx = \int_{\frac{-h_i}{2}}^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

$$= \int_{\frac{-h_i}{2}}^{0} \widehat{x}^p d\widehat{x} + \int_{0}^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

Error In Rectangle Rule

- if p is odd

$$f(-\widehat{x}) = -f(\widehat{x})$$

Then

$$\int_{\frac{-h_i}{2}}^{0} \widehat{x}^p d\widehat{x} + \int_0^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

$$= -\int_0^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x} + \int_0^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

$$= 0$$

Error In Rectangle Rule

- if p is even

$$f(-\widehat{x}) = f(\widehat{x})$$

Then

$$\int_{\frac{-h_i}{2}}^{0} \widehat{x}^p d\widehat{x} + \int_{0}^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

$$= 2 \int_{0}^{\frac{h_i}{2}} \widehat{x}^p d\widehat{x}$$

$$= \frac{h_i^{p+1}}{(p+1)*2^p}$$

Error In Rectangle Rule

- Biggest error of terms
  $p = 2$
  $$\int_{x_i}^{x_{i+1}} \frac{(x-y_i)^2}{2} f''(y_i)dx = \frac{1}{24} h_i^3 f''(y_i)$$
- Error of Rectangle Rule
  $$I(f) - R(f) \approx \sum_1^n \frac{1}{24} h_i^3 f''(y_i) + O(\sum_1^n h_i^5 f^{(5)}(y_i))$$

Error In Trapezoid Rule

- Similarly, the error of Trapezoid Rule

$$I(f) - T(f) \approx -\sum_1^n \frac{1}{12} h_i^3 f''(y_i) - O(\sum_1^n h_i^5 f^{(5)}(y_i))$$

Error Estimation

- $I(f)$ unknown
- Compare the two error estimations

$$\begin{cases} I(f) - R(f) \approx \sum_1^n \frac{1}{24} h_i^3 f''(y_i) \\ \\ I(f) - T(f) \approx -\sum_1^n \frac{1}{12} h_i^3 f''(y_i) \end{cases}$$

We can estimate the errors

$$\begin{cases} I(f) - R(f) \approx \frac{1}{3}(T(f) - R(f)) \\ \\ I(f) - T(f) \approx \frac{2}{3}(R(f) - T(f)) \end{cases}$$

Thanks