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Structured condition numbers and small sample condition estimation of symmetric algebraic Riccati equations

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ABSTRACT

This paper is devoted to a structured perturbation analysis of the symmetric algebraic Riccati equations by exploiting the symmetry structure. Based on the analysis, the upper bounds for the structured normwise, mixed and componentwise condition numbers are derived. Due to the exploitation of the symmetry structure, our results are improvements of the previous work on the perturbation analysis and condition numbers of the symmetric algebraic Riccati equations. Our preliminary numerical experiments demonstrate that our condition numbers provide accurate estimates for the change in the solution caused by the perturbations on the data. Moreover, by applying the small sample condition estimation method, we propose a statistical algorithm for practically estimating the condition numbers of the symmetric algebraic Riccati equations.

1. Introduction

Algebraic Riccati equations arise in optimal control problems in continuous-time or discrete-time. The theory, applications, and numerical methods for solving the equations can be found in [1,26,27,30,33] and references therein. The continuous-time algebraic Riccati equation (CARE) is given in the form:

$$Q + A^H X + XA - XBR^{-1}B^H X = 0, \tag{1.1}$$

where X is the unknown matrix, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, A^H denotes the conjugate transpose of A, and Q, R are $n \times n$ Hermitian matrices with Q being positive semi-definite (p.s.d.) and R being positive definite. The discrete-time algebraic Riccati equation (DARE) is given in the form:

$$Y - A^H YA + A^H YB(R + B^H YB)^{-1}B^H YA - C^H C = 0, \tag{1.2}$$

where Y is the unknown matrix, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{r\times n}$, and $R \in \mathbb{C}^{m\times m}$ with R being Hermitian positive definite.

For the complex CARE (1.1), let $G = BR^{-1}B^H$, then it has the simplified form

$$Q + A^H X + XA - XGX = 0, \tag{1.3}$$

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where Q, G are Hermitian and p.s.d. For the complex DARE (1.2), let $Q = C^H C$ and $G = BR^{-1}B^H$, then it has the simplified form

$$Y - A^H Y(I + GY)^{-1}A - Q = 0, \tag{1.4}$$

where Q, G are Hermitian and p.s.d. In particular, when A, Q and G are real matrices, the real CARE becomes

$$Q + A^T X + XA - XGX = 0, \tag{1.5}$$

and the real DARE has the form

$$Y - A^T Y(I + GY)^{-1}A - Q = 0. \tag{1.6}$$

The existence and uniqueness of the solution is essential for perturbation analysis. Before making appropriate assumptions on the coefficient matrices necessary for the existence and uniqueness of Hermitian and p.s.d. stabilizing solution, we need some notions of stability, which play an important role in the study of the algebraic Riccati equations. An $n \times n$ matrix M is said to be c-stable if all of its eigenvalues lie in the open left-half complex plane, and M is said to be d-stable if its spectral radius $\rho(M) < 1$. Then to ensure the existence and uniqueness of the solution, we assume that (A, G) in the CARE (1.3) is a c-stabilizable pair, that is, there is a matrix $K \in \mathbb{C}^{n\times n}$ such that the matrix $A - GK$ is c-stable, and that (A, Q) is a c-detectable pair, that is, (A^T, Q^T) is c-stabilizable. It is known [5,28] that under these conditions there exists a unique Hermitian and p.s.d. solution X for the CARE (1.3) and the matrix $A - GX$ is c-stable. Similarly, for the DARE, we assume that (A, B) in the DARE (1.2) is a d-stabilizable pair, that is, if $\omega^T B = 0$ and $\omega^T A = \lambda\omega^T$ hold for some constant λ, then $|\lambda| < 1$ or $\omega = 0$, and that (A, C) is a d-detectable pair, that is, (A^T, C^T) is d-stabilizable. It is known [1,14,25] that under these conditions there exists a unique Hermitian and p.s.d. solution Y for the DARE (1.4), and the matrix $(I + GY)^{-1}A$ is d-stable, i.e., all the eigenvalues of $(I + GY)^{-1}A$ lie in the open unit disk.

Matrix perturbation analysis concerns the sensitivity of the solution to the perturbations in the data of a problem. A condition number is a measurement of the sensitivity. Liu studied mixed and componentwise condition numbers of non-symmetric algebraic Riccati equation in [29]. For the perturbation analysis of the CARE (1.3) or DARE (1.4), we refer papers [5,14,17,24,25] and their references therein. Sun [36] defined the structured normwise condition numbers for CARE and DARE and showed that the expressions of structured normwise condition numbers are the same as their unstructured counterparts for both real and complex cases. Later, Zhou et al. [39] performed componentwise perturbation analyses of CARE and DARE and obtained the exact expressions for mixed and componentwise condition numbers defined in [12] for the real case. However, in their paper, the perturbations on Q and G are general (unstructured). In this paper, we perform a structured perturbation analysis, define the structured normwise, mixed and componentwise condition numbers for complex CARE and DARE, and derive their expressions using the Kronecker product [13]. Specifically, we assume that the perturbation ΔG (ΔQ) has the same structure as G (Q). Furthermore, in the complex case, we separate the real part and the imaginary part. Thus, the real part of G (Q) or ΔG (ΔQ) is symmetric and the imaginary part of G (Q) or ΔG (ΔQ) is skew-symmetric. In our analysis, we exploit the structure and consider the perturbations on the real part and the imaginary part separately. In contrast, the analysis in [36] considers the perturbation on a complex matrix as whole. Apparently, separating real and imaginary parts gives more precise results.

Efficiently estimating the condition of a problem is one of the most fundamental topics in numerical analysis. Together with the knowledge of backward error, a good condition estimate can provide an estimate for the accuracy of the computed solution. Although the expressions of the condition numbers derived in [36,39] and this paper are explicit, they involve the solution matrix and require extensive computation, especially for large size problems. As pointed out in [36, p. 260], practical algorithms for estimating the condition numbers of the algebraic Riccati equations are worth studying. In this paper, we present a statistical method for practically estimating the structured normwise, mixed and componentwise condition numbers for CARE and DARE by applying the small sample condition estimation method (SCE) [18].

The SCE, proposed by Kenny and Laub [18], is an efficient method for estimating the condition numbers for linear systems [20,21], linear least squares problems [19], the Tikhonov regularization problem [8], the total least squares problem [9], eigenvalue problems [23], roots of polynomials [22], etc. Diao et al. [7,10,11] applied the SCE to the (generalized) Sylvester equations. Wang et al. [38] considered the mixed and componentwise condition numbers for the spectral projections, generalized spectral projections and sign functions for matrices and regular matrix pairs and derived explicit expressions of the condition numbers, which improved some known results of the normwise type and revealed the structured perturbations. Also, they applied the SCE to these problems to efficiently estimate the condition numbers. Wang et al. [37] studied the normwise, mixed and componentwise condition numbers for the following general nonlinear matrix equation $X + A^H F(X)A = Q$, where A is an n-by-n square matrix, Q an n-by-n positive definite matrix, X the unknown n-by-n positive semi-definite matrix, and F a differentiable mapping from the set of n-by-n positive semi-definite matrices to the set of n-by-n matrices. They derived corresponding explicit condition numbers and gave their statistical estimations with high reliability based on the SCE and a probabilistic spectral norm estimator. Differing from their algorithms, our methods produce estimated condition matrices instead of single condition numbers, that is, the entries of the condition matrices produced by our algorithms are the structured normwise or componentwise condition numbers of the corresponding entries of the solution matrices. Thus these condition matrices are more informative and precise about the conditioning of the solution.

Throughout this paper we adopt the following notations:

- $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denotes the set of complex (real) $m \times n$ matrices; $\mathbb{H}^{n\times n}$ the set of $n \times n$ Hermitian matrices; $\mathbb{S}^{n\times n}$ the set of $n \times n$ symmetric matrices; $\mathbb{SK}^{n\times n}$ the set of $n \times n$ skew-symmetric matrices.
- A^T denotes the transpose of A; A^H the complex conjugate and transpose of A; $A\dagger$ the Moore–Penrose inverse of A; I the identity matrix; 0 the zero matrix; $\mathrm{Re}(A)$ ($\mathrm{Im}(A)$) is the real (imaginary) part of a complex matrix A. The matrix $\mathrm{Diag}(A, B)$ denotes a block diagonal matrix with A and B being its diagonal.
- e_i denotes the ith column of I.
- The mapping $\mathrm{sym}(\cdot)$: $\mathbb{S}^{n\times n} \to \mathbb{R}^{n(n+1)/2}$ maps a symmetric matrix $A = [a_{ij}] \in \mathbb{S}^{n\times n}$ to a $(n(n+1)/2)$-vector:

$$[a_{11}, \dots, a_{1n}, a_{22}, \dots, a_{2n}, \dots, a_{n-1,n-1}, a_{n-1,n}, a_{nn}]^T.$$

- The mapping $\mathrm{skew}(\cdot)$: $\mathbb{SK}^{n\times n} \to \mathbb{R}^{n(n-1)/2}$ maps a skew-symmetric matrix $A = [a_{ij}] \in \mathbb{SK}^{n\times n}$ to the $(n(n-1)/2)$-vector:

$$[a_{12}, ..., a_{1n}, a_{23}, \dots, a_{2n}, \dots, a_{n-2,n-1}, a_{n-2,n} a_{n-1,n}]^T.$$

- $A \succ 0$ ($A \succeq 0$) means that A is positive definite (positive semi-definite).
- $\| \|_F$, $\| \|_2$ and $\| \|_\infty$ are the Frobenius norm, the spectral norm and infinity norm respectively. For $A \in \mathbb{C}^{m\times n}$, $\|A\|_{\max} = \max_{ij} |a_{ij}|$.
- $A \otimes B = [a_{ij}B]$ is the Kronecker product of $A = [a_{ij}]$ and matrix B and $\mathrm{vec}(A)$ is the vector defined by $\mathrm{vec}(A) = [a_1^T, \dots, a_n^T]^T \in \mathbb{C}^{mn}$; Π is an $n^2 \times n^2$ permutation matrix, such that, for an $n \times n$ real matrix A, $\mathrm{vec}(A^T) = \Pi \mathrm{vec}(A)$. For more properties of the Kronecker product and vec operation, see [13].
- $|A| \le |B|$ means $|a_{ij}| \le |b_{ij}|$ for $A, B \in \mathbb{C}^{m\times n}$; $A \oslash B$ is the componentwise division of matrices A and B of the same dimensions. In our context, it is used for componentwise relative error. So, when $b_{ij} = 0$, we assume its absolute error $a_{ij} = 0$ and set $(A \oslash B)_{ij} = 0$.

The rest of the paper is organized as follows. In Section 2, we present our structured perturbation analyses and expressions of the structured normwise condition numbers of the CARE (1.3) and the DARE (1.4). The expressions of the structured mixed and componentwise condition numbers are derived in Section 3. In Section 4, by applying the small sample condition estimation method, we propose our structured sensitivity estimation methods for the problems of solving the CARE and DARE. Our numerical experiment results are demonstrated in Section 5. Finally, Section 6 concludes this paper.

2. Structured normwise condition numbers

In this section, using the Kronecker product, we first present a structured perturbation analysis of the CARE (1.3) and derive expressions of the corresponding structured normwise condition number. In a similar way, a structured perturbation analysis of the DARE (1.4) can be performed and the corresponding structured normwise condition number can be obtained.

2.1. CARE

Let $\Delta A \in \mathbb{C}^{n\times n}$, $\Delta Q \in \mathbb{H}^{n\times n}$, and $\Delta G \in \mathbb{H}^{n\times n}$ be the perturbations to the data A, Q, and G respectively. Notice that the perturbations ΔQ and ΔG are also Hermitian. From Theorem 3.1 and its proof in [35], for $Q, G \succeq 0$ and sufficiently small $\|[\Delta A, \Delta Q, \Delta G]\|_F$, there is a unique Hermitian p.s.d. matrix $\tilde{X}$ such that $\tilde{A} - \tilde{G}\tilde{X}$ is c-stable, where $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$, and

$$\tilde{X}\tilde{G}\tilde{X} - \tilde{X}\tilde{A} - \tilde{A}^H\tilde{X} - \tilde{Q} = 0, \tag{2.1}$$

where $\tilde{Q} = Q + \Delta Q$. To perform a perturbation analysis, we define a linear operator $\mathbf{L}:\ \mathbb{H}^{n\times n} \to \mathbb{H}^{n\times n}$ by

$$\mathbf{L}W = (A - GX)^H W + W(A - GX), \quad W \in \mathbb{H}^{n\times n}. \tag{2.2}$$

Since the matrix $A - GX$ is c-stable, the operator $\mathbf{L}$ is invertible. Denote $\Delta X = \tilde{X} - X$, then

$$\Delta X = -\mathbf{L}^{-1}(\Delta Q + X\Delta A + \Delta A^H X - X\Delta G X) + O\left(\|[\Delta A, \Delta Q, \Delta G]\|_F^2\right), \tag{2.3}$$

as $\|[\Delta A, \Delta Q, \Delta G]\|_F \to 0$. In the first order approximation of (2.3), ΔX is the solution to the continuous Lyapunov equation:

$$(A - GX)^H \Delta X + \Delta X(A - GX) = -\Delta Q - X\Delta A - \Delta A^H X + X\Delta G X. \tag{2.4}$$

Numerical methods for solving the Lyapunov equations can be found in [4].

For the complex CARE (1.3), in addition to exploiting the symmetry structure of G and Q, to make our analysis precise, we separate the real part and the imaginary part. Since the real part of a Hermitian matrix is symmetric and the imaginary part is skew-symmetric, we introduce the function

$$\begin{aligned}
&\varphi:\ \mathbb{R}^{4n^2} \to \mathbb{C}^{n^2}\\
&[\mathrm{vec}(\mathrm{Re}(A))^T, \mathrm{vec}(\mathrm{Im}(A))^T, \mathrm{sym}(\mathrm{Re}(G))^T, \mathrm{skew}(\mathrm{Im}(G))^T,\\
&\ \mathrm{sym}(\mathrm{Re}(Q))^T, \mathrm{skew}(\mathrm{Im}(Q))^T]^T \mapsto \mathrm{vec}(X),
\end{aligned}$$

which maps the structured data vector to the solution vector. As we can see, our data vector exploits the structure and separates the real and imaginary parts. Applying the condition number theory of Rice [31] to the above mapping, we define the structured normwise condition numbers:

$$\kappa_i(\varphi) = \lim_{\epsilon\to 0} \sup_{\substack{\eta_i\le\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta Q\in\mathbb{H}^{n\times n},\ \Delta G\in\mathbb{H}^{n\times n} \\ Q+\Delta Q\succeq 0,\ G+\Delta G\succeq 0}} \frac{\|\Delta X\|_F}{\epsilon\|X\|_F}, \tag{2.5}$$

where

$$\begin{aligned} \eta_1 &= \left\| \left[\frac{\|\mathrm{Re}(\Delta A)\|_F}{\delta_1}, \frac{\|\mathrm{Im}(\Delta A)\|_F}{\delta_2}, \frac{\|\mathrm{sym}(\mathrm{Re}(\Delta G))\|_2}{\delta_3}, \frac{\|\mathrm{skew}(\mathrm{Im}(\Delta G))\|_2}{\delta_4}, \right.\right. \\ &\quad \times \left.\left. \frac{\|\mathrm{sym}(\mathrm{Re}(\Delta Q))\|_2}{\delta_5}, \frac{\|\mathrm{skew}(\mathrm{Im}(\Delta Q))\|_2}{\delta_6} \right] \right\|_2, \\ \eta_2 &= \max\left\{ \frac{\|\mathrm{Re}(\Delta A)\|_F}{\delta_1}, \frac{\|\mathrm{Im}(\Delta A)\|_F}{\delta_2}, \frac{\|\mathrm{sym}(\mathrm{Re}(\Delta G))\|_2}{\delta_3}, \frac{\|\mathrm{skew}(\mathrm{Im}(\Delta G))\|_2}{\delta_4}, \right. \\ &\quad \times \left. \frac{\|\mathrm{sym}(\mathrm{Re}(\Delta Q))\|_2}{\delta_5}, \frac{\|\mathrm{skew}(\mathrm{Im}(\Delta Q))\|_2}{\delta_6} \right\}, \end{aligned} \tag{2.6}$$

and the parameters $\delta_i > 0\,(i = 1, \dots, 6)$ are given. Generally, they are respectively chosen to be the functions of $\|\mathrm{Re}(A)\|_F$, $\|\mathrm{Im}(A)\|_F$, $\|\mathrm{sym}(\mathrm{Re}(G))\|_2$, $\|\mathrm{skew}(\mathrm{Im}(G))\|_2$, $\|\mathrm{sym}(\mathrm{Re}(Q))\|_2$, and $\|\mathrm{skew}(\mathrm{Im}(Q))\|_2$. Here, we set $\delta_1 = \|\mathrm{Re}(A)\|_F$, $\delta_2 = \|\mathrm{Im}(A)\|_F$, $\delta_3 = \|\mathrm{sym}(\mathrm{Re}(G))\|_2$, $\delta_4 = \|\mathrm{skew}(\mathrm{Im}(G))\|_2$, $\delta_5 = \|\mathrm{sym}(\mathrm{Re}(Q))\|_2$ and $\delta_6 = \|\mathrm{skew}(\mathrm{Im}(Q))\|_2$.

Now, we derive an explicit expression of $\kappa_1(\varphi)$ and an upper bound for $\kappa_2(\varphi)$ in (2.5). First, we present a matrix-tensor representation of the operator $\mathbf{L}$. Applying the identity

$$\mathrm{vec}(UVW) = (W^T\otimes U)\mathrm{vec}(V), \tag{2.7}$$

to the vectorized (2.2)

$$\mathrm{vec}(\mathbf{L}W) = \mathrm{vec}((A-GX)^H W + W(A-GX)),$$

we get the matrix

$$Z = I_n\otimes (A-GX)^H + (A-GX)^T\otimes I_n, \tag{2.8}$$

which transforms $\mathrm{vec}(W)$ into $\mathrm{vec}(\mathbf{L}W)$, as a matrix representation of the linear operator $\mathbf{L}$. Since $\mathbf{L}$ is invertible, Z is also invertible.

Since our structured data vector exploits the symmetry structure, to convert it back to original vector, we introduce the matrices $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows. For an $n\times n$ symmetric matrix J, $\mathcal{S}_1$ is the $n^2\times n(n+1)/2$ matrix such that

$$\mathrm{vec}(J) = \mathcal{S}_1\mathrm{sym}(J).$$

That is, $\mathcal{S}_1$ expands the $n(n+1)/2$-vector $\mathrm{sym}(J)$ to the n^2-vector $\mathrm{vec}(J)$ by copying its elements. The $n^2\times n(n-1)/2$ matrix $\mathcal{S}_2$ is defined by

$$\mathrm{vec}(K) = \mathcal{S}_2\mathrm{skew}(K),$$

where K is an $n\times n$ skew-symmetric matrix.

Then, applying (2.7) to (2.3), we get

$$\begin{aligned} \mathrm{vec}(\Delta X) &= -Z^{-1}[-(X^T\otimes X)\mathrm{vec}(\Delta G) + (I_n\otimes X)\mathrm{vec}(\Delta A) \\ &\quad + (X^T\otimes I_n)\mathrm{vec}(\Delta A^H) + \mathrm{vec}(\Delta Q)] + O(\|[\Delta A,\ \Delta Q,\ \Delta G]\|_F^2) \\ &= -Z^{-1}[-(X^T\otimes X)(\mathcal{S}_1\mathrm{sym}(\mathrm{Re}(\Delta G)) + \mathbf{i}\,\mathcal{S}_2\mathrm{skew}(\mathrm{Im}(\Delta G))) \\ &\quad + (I_n\otimes X)(\mathrm{vec}(\mathrm{Re}(\Delta A)) + \mathbf{i}\,\mathrm{vec}(\mathrm{Im}(\Delta A))) + (X^T\otimes I_n)\Pi(\mathrm{vec}(\mathrm{Re}(\Delta A)) \\ &\quad - \mathbf{i}\,\mathrm{vec}(\mathrm{Im}(\Delta A))) + \mathcal{S}_1\mathrm{sym}(\mathrm{Re}(\Delta Q)) + \mathbf{i}\,\mathcal{S}_2\mathrm{skew}(\mathrm{Im}(\Delta Q))] + O(\|[\Delta A,\ \Delta Q,\ \Delta G]\|_F^2) \\ &\to -Z^{-1}[(I_n\otimes X) + (X^T\otimes I_n)\Pi, \mathbf{i}\,((I_n\otimes X) - (X^T\otimes I_n)\Pi), \\ &\quad -(X^T\otimes X)\mathcal{S}_1,\ -\mathbf{i}\,(X^T\otimes X)\mathcal{S}_2,\ \mathcal{S}_1,\ \mathbf{i}\,\mathcal{S}_2]\cdot\Delta, \end{aligned} \tag{2.9}$$

as $\|[\Delta A,\ \Delta Q,\ \Delta G]\|_F\to 0$, where $\mathbf{i} = \sqrt{-1}$ and

$$\Delta = [\mathrm{vec}(\mathrm{Re}(\Delta A))^T,\ \mathrm{vec}(\mathrm{Im}(\Delta A))^T,\ \mathrm{sym}(\mathrm{Re}(\Delta G))^T,\ \mathrm{skew}(\mathrm{Im}(\Delta G))^T,\ \mathrm{sym}(\mathrm{Re}(\Delta Q))^T,\ \mathrm{skew}(\mathrm{Im}(\Delta Q))^T]^T \tag{2.10}$$

is the structured data perturbation vector.

Denoting

$$M_A = [(I_n\otimes X) + (X^T\otimes I_n)\Pi,\ \ \mathbf{i}((I_n\otimes X) - (X^T\otimes I_n)\Pi)]$$

corresponding to $[\mathrm{vec}(\mathrm{Re}(\Delta A))^T,\ \mathrm{vec}(\mathrm{Im}(\Delta A))^T]^T$,

$$M_G = [-(X^T\otimes X)\mathcal{S}_1,\ -\mathbf{i}(X^T\otimes X)\mathcal{S}_2]$$

corresponding to $[\mathsf{sym}(\mathsf{Re}(\Delta G))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta G))^T]^T$,

$$M_Q = [\mathcal{S}_1,\ \mathbf{i}\,\mathcal{S}_2]$$

corresponding to $[\mathsf{sym}(\mathsf{Re}(\Delta Q))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta Q))^T]^T$, and $\mathcal{M} = [M_A\ M_G\ M_Q]$, and using the definition of the directional derivative, we have the following lemma.

Lemma 1. *Using the above notations, the directional derivative $\mathcal{D}\varphi(X)$ of φ with respect to Δ (2.10) is given by*

$$\mathcal{D}\varphi(X)\Delta = -Z^{-1}\mathcal{M}\cdot\Delta. \tag{2.11}$$

Finally, the following theorem gives an explicit expression of $\kappa_1(\varphi)$ and an upper bound for $\kappa_2(\varphi)$.

Theorem 1. *Using the notations given above, the expression and upper bound for the normwise number of the complex CARE (1.3) are*

$$\kappa_1(\varphi) = \frac{\|Z^{-1}\mathcal{M}D\|_2}{\|X\|_F}, \tag{2.12}$$

$$\kappa_2(\varphi) \le \kappa_U(\varphi) := \min\left\{\sqrt{6}\kappa_1(\varphi), \alpha_c/\|X\|_F\right\}, \tag{2.13}$$

where

$$D = \mathsf{Diag}\big([\delta_1 I_{n^2},\ \delta_2 I_{n^2},\ \delta_3 I_{n(n+1)/2}, \delta_4 I_{n(n-1)/2},\ \delta_5 I_{n(n+1)/2},\ \delta_6 I_{n(n-1)/2}]\big) \tag{2.14}$$

and

$$\begin{aligned}\alpha_c &= \delta_1\|Z^{-1}[(I_n\otimes X) + (X^T\otimes I_n)\Pi]\|_2 + \delta_2\|Z^{-1}[(I_n\otimes X) - (X^T\otimes I_n)\Pi]\|_2\\ &\quad + \delta_3\|Z^{-1}(X^T\otimes X)\mathcal{S}_1\|_2 + \delta_4\|Z^{-1}(X^T\otimes X)\mathcal{S}_2\|_2 + \delta_5\|Z^{-1}\mathcal{S}_1\|_2 + \delta_6\|Z^{-1}\mathcal{S}_2\|_2.\end{aligned}$$

Proof. Introducing the positive parameters δ_i, $i = 1,\dots,6$, into (2.11), we get

$$\mathcal{D}\varphi(X)\Delta = -Z^{-1}\mathcal{M}DD^{-1}\Delta.$$

From the definition (2.5), we know that

$$\begin{aligned}\kappa_1(\varphi) &= \max_{\eta_1\le\epsilon}\frac{\|-Z^{-1}\mathcal{M}DD^{-1}\Delta\|_2}{\|X\|_F} = \max_{\|D^{-1}\Delta\|_2\le 1}\frac{\|-Z^{-1}\mathcal{M}DD^{-1}\Delta\|_2}{\|X\|_F}\\ &= \frac{\|Z^{-1}\mathcal{M}D\|_2}{\|X\|_F}.\end{aligned} \tag{2.15}$$

The last equality holds because Δ can vary freely.

Because $\|D^{-1}\Delta\|_2 \le \sqrt{6}\eta_2$, it is easy to see that

$$\|\Delta X\|_F \approx \|Z^{-1}\mathcal{M}DD^{-1}\Delta\|_2 \le \|Z^{-1}\mathcal{M}D\|_2\|D^{-1}\Delta\|_2 \le \sqrt{6}\|Z^{-1}\mathcal{M}D\|_2\eta_2,$$

which proves $\kappa_2(\varphi) \le \sqrt{6}\kappa_1(\varphi)$. On the other hand, since

$$\begin{aligned}\mathsf{vec}(\Delta X) &= -\delta_1 Z^{-1}[(I_n\otimes X) + (X^T\otimes I_n)\Pi]\frac{\mathsf{vec}(\mathsf{Re}(\Delta A))}{\delta_1}\\ &\quad - \mathbf{i}\delta_2 Z^{-1}[(I_n\otimes X) - (X^T\otimes I_n)\Pi]\frac{\mathsf{vec}(\mathsf{Im}(\Delta A))}{\delta_2}\\ &\quad + \delta_3 Z^{-1}(X^T\otimes X)\mathcal{S}_1\frac{\mathsf{sym}(\mathsf{Re}(\Delta G))}{\delta_3} + \mathbf{i}\delta_4 Z^{-1}(X^T\otimes X)\mathcal{S}_2\frac{\mathsf{skew}(\mathsf{Im}(\Delta G))}{\delta_4}\\ &\quad - \delta_5 Z^{-1}\mathcal{S}_1\frac{\mathsf{sym}(\mathsf{Re}(\Delta Q))}{\delta_5} - \mathbf{i}\delta_6 Z^{-1}\mathcal{S}_2\frac{\mathsf{skew}(\mathsf{Im}(\Delta Q))}{\delta_6},\end{aligned}$$

it is easy to see that $\|\Delta X\|_F \le \alpha_c\eta_2$. $\square$

In [36], Sun defined the structured normwise condition number for the complex CARE:

$$\kappa_{\mathrm{Sun}}^{\mathrm{CARE}} = \lim_{\epsilon\to 0}\sup_{\substack{\theta\le\epsilon\\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{H}^{n\times n}\\ G+\Delta G,\ Q+\Delta Q\ge 0}}\frac{\|\Delta X\|_F}{\epsilon\|X\|_F}, \tag{2.16}$$

where $\theta = \|[\Delta A/\mu_1,\ \Delta G/\mu_2,\ \Delta Q/\mu_3]\|_F$, and the parameters μ_i, $i = 1, 2, 3$, are positive.

Differently from the above Sun's definition, our definition separates the relative perturbations in the real and imaginary parts of a data matrix. Apparently, it is more precise than the Sun's definition. Moreover, it is more realistic, since, in computation, the real and imaginary parts of a complex matrix are stored and computed separately.

From the definitions and expressions of (2.5) and (2.16), if we choose $\delta_1=\delta_2=\mu_1,\ \delta_3=\delta_4=\mu_2$, and $\delta_5=\delta_6=\mu_3$, we can show that

$$\kappa_1(\varphi)\le \max\{\|\mathcal{S}_1\|_2,\ \|\mathcal{S}_2\|_2\}\cdot\kappa_{\text{Sun}}^{\text{CARE}}.$$

Note that each row of $\mathcal{S}_1$ or $\mathcal{S}_2$ is e_i^T for some i, that is, a row of the identity matrix.

Similarly to the complex case, for the real CARE (1.5), we introduce the mapping

$$\begin{aligned}\varphi^{\text{Re}}: \mathbb{R}^{n^2+n(n+1)} &\to \mathbb{R}^{n^2}\\ (\mathsf{vec}(A)^T, \mathsf{sym}(G))^T, \mathsf{sym}(Q)^T)^T &\mapsto \mathsf{vec}(X)\end{aligned}$$

and define the structured normwise condition number for the real CARE (1.5):

$$\kappa(\varphi^{\text{Re}})=\lim_{\epsilon\to 0}\sup_{\substack{\delta\le\epsilon\\ \Delta A\in\mathbb{R}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{S}^{n\times n}\\ G+\Delta G,\ Q+\Delta Q\ge 0}}\frac{\|\Delta X\|_F}{\epsilon\|X\|_F},$$

where

$$\delta=\max\left\{\frac{\|\Delta A\|_F}{\delta_1},\ \frac{\|\mathsf{sym}(\Delta G)\|_2}{\delta_2},\ \frac{\|\mathsf{sym}(\Delta Q)\|_2}{\delta_3}\right\},\tag{2.17}$$

with the positive parameters δ_i, $i=1,2,3$. We usually choose $\delta_1=\|A\|_F,\ \delta_2=\|\mathsf{sym}(G)\|_2$ and $\delta_3=\|\mathsf{sym}(Q)\|_2$.

Using the above notations, we can derive the following upper bound for the normwise condition number of the real CARE (1.5):

$$\kappa(\varphi^{\text{Re}})\le\kappa_U(\varphi^{\text{Re}}):=\min\left\{\sqrt{3}\frac{\|Z_1^{-1}\mathcal{M}_1 D_1\|_2}{\|X\|_F},\ \frac{\beta}{\|X\|_F}\right\},$$

where

$$\begin{aligned}Z_1&=I_n\otimes(A-GX)^T+(A-GX)^T\otimes I_n,\\ \mathcal{M}_1&=\left[I_n\otimes X+(X^T\otimes I_n)\Pi,\ -(X^T\otimes X)\mathcal{S}_1,\ \mathcal{S}_1\right],\\ D_1&=\mathsf{Diag}\big([\delta_1 I_{n^2},\ \delta_2 I_{n(n+1)/2},\ \delta_3 I_{n(n+1)/2}]\big),\end{aligned}$$

and

$$\beta=\delta_1\|Z_1^{-1}[(I_n\otimes X)+(X\otimes I_n)\Pi]\|_2+\delta_2\|Z_1^{-1}(X\otimes X)\mathcal{S}_1\|_2+\delta_3\|Z_1^{-1}\mathcal{S}_1\|_2.$$

In [39], Zhou et al. defined the following unstructured normwise condition number for the real CARE:

$$\kappa_1(\varphi^{\text{Re}})=\lim_{\epsilon\to 0}\sup_{\delta\le\epsilon}\frac{\|\Delta X\|_F}{\epsilon\|X\|_F},$$

where

$$\delta=\max\left\{\frac{\|\Delta A\|_F}{\delta_1},\ \frac{\|\Delta Q\|_F}{\delta_2},\ \frac{\|\Delta G\|_F}{\delta_3}\right\},$$

and derived the upper bound

$$\kappa_1(\varphi^{\text{Re}})\le\kappa_1^U(\varphi^{\text{Re}}):=\min\left\{\sqrt{3}\,\frac{\|Z_1^{-1}S_1\|_2}{\|X\|_F},\ \frac{\beta_c}{\|X\|_F}\right\},$$

where

$$S_1=[-(I_n\otimes X)-(X\otimes I_n)\Pi, X\otimes X,\ -I_{n^2}]\,\mathsf{Diag}([\delta_1 I_{n^2},\ \delta_2 I_{n^2},\ \delta_3 I_{n^2}])$$

and

$$\beta_c=\delta_1\|Z_1^{-1}[I_n\otimes X+(X\otimes I_n)\Pi]\|_2+\delta_2\|Z_1^{-1}(X\otimes X)\|_2+\delta_3\|Z_1^{-1}\|_2.$$

By setting $\delta_1=\|A\|_F,\ \delta_2=\|Q\|_F$ and $\delta_3=\|G\|_F$, we can prove that

$$\kappa_U(\varphi^{\text{Re}})\le\|\mathcal{S}_1\|_2\kappa_1^U(\varphi^{\text{Re}}).$$

However, our numerical experiments show that the difference between our $\kappa_U(\varphi^{\text{Re}})$ and $\kappa_1^U(\varphi^{\text{Re}})$ is marginal.

2.2. DARE

Following the structured perturbation analysis of CARE, for the complex DARE (1.4), we define a linear operator $\mathbf{L} : \mathbb{H}^{n\times n} \to \mathbb{H}^{n\times n}$ by

$$\mathbf{L}M = M - [(I_n + GY)^{-1}A]^H M (I_n + GY)^{-1}A,$$

for $M \in \mathbb{H}^{n\times n}$. Since the matrix $(I_n + GY)^{-1}A$ is d-stable, the operator $\mathbf{L}$ is invertible. Let

$$\tilde{Y} - \tilde{A}^H \tilde{Y}(I_n + \tilde{G}\tilde{Y})^{-1}\tilde{A} - \tilde{Q} = 0, \tag{2.18}$$

be the perturbed DARE, where $\tilde{A} = A + \Delta A,\ \tilde{G} = G + \Delta G$, and $\tilde{Q} = Q + \Delta Q$, then for sufficiently small $\|[\Delta A,\ \Delta Q,\ \Delta G]\|_F$, there is a unique Hermitian and p.s.d. solution $\tilde{Y}$ for the perturbed Eq. (2.18) and the change $\Delta Y = \tilde{Y} - Y$ in the solution is given by

$$\Delta Y \to \mathbf{L}^{-1}[\Delta Q + (A^H Y W)\Delta A + \Delta A^H (YWA) - (A^H Y W)\Delta G (YWA)], \tag{2.19}$$

as $\|[\Delta A,\ \Delta Q,\ \Delta G]\|_F \to 0$.

Denote $W = (I_n + GY)^{-1}$. In this case, the matrix-tensor representation of the linear operator $\mathbf{L}$ is

$$T = I_n - (A^T W^T) \otimes (A^H W^H), \tag{2.20}$$

which is invertible, since $\mathbf{L}$ is invertible.

For the structured perturbation analysis of the complex DARE (1.4), exploiting the symmetry structure and separating the real and imaginary parts, we define the mapping

$$\begin{aligned}\psi :\ & \mathbb{R}^{4n^2} \to \mathbb{C}^{n^2}\\ & [\mathrm{vec}(\mathrm{Re}(A))^T,\ \mathrm{vec}(\mathrm{Im}(A))^T,\ \mathrm{sym}(\mathrm{Re}(G))^T,\ \mathrm{skew}(\mathrm{Im}(G))^T,\ \mathrm{sym}(\mathrm{Re}(Q))^T,\ \mathrm{skew}(\mathrm{Im}(Q))^T]^T\\ & \mapsto \mathrm{vec}(Y).\end{aligned}$$

Similarly to (2.9), dropping the second and higher order terms, we have

$$\begin{aligned}T\mathrm{vec}(\Delta Y) \approx\ & [(I_n \otimes (A^H Y W)) + ((A^T W^T Y^T) \otimes I_n)\Pi,\ \mathbf{i}((I_n \otimes (A^H Y W)) - ((A^T W^T Y^T) \otimes I_n)\Pi),\\ & - (A^T W^T Y^T) \otimes (A^H Y W)\mathcal{S}_1,\ -\mathbf{i}((A^T W^T Y^T) \otimes (A^H Y W))\mathcal{S}_2,\ \mathcal{S}_1,\ \mathbf{i}\mathcal{S}_2] \cdot \Delta,\end{aligned}$$

where the data perturbation vector Δ is defined in (2.10). Denoting

$$\begin{aligned}N_A &= [(I_n \otimes (A^H Y W)) + ((A^T W^T Y^T) \otimes I_n)\Pi, \mathbf{i}((I_n \otimes (A^H Y W)) - ((A^T W^T Y^T) \otimes I_n)\Pi)],\\ N_G &= [-(A^T W^T Y^T) \otimes (A^H Y W)\mathcal{S}_1, -\mathbf{i}((A^T W^T Y^T) \otimes (A^H Y W))\mathcal{S}_2],\\ N_Q &= [\mathcal{S}_1,\ \mathbf{i}\mathcal{S}_2],\end{aligned}$$

and $\mathcal{N} = [N_A\ N_G\ N_Q]$, we have the following lemma.

Lemma 2. *With the above notations, the directional derivative $\mathcal{D}\psi(Y)$ of ψ with respect to Δ is*

$$\mathcal{D}\psi(Y)\Delta_1 = T^{-1}\mathcal{N} \cdot \Delta.$$

We then define the structured normwise condition numbers for solving the complex DARE (1.4):

$$\kappa_i(\psi) = \lim_{\epsilon \to 0} \sup_{\substack{\eta_i \le \epsilon \\ \Delta A \in \mathbb{C}^{n\times n},\ \Delta Q \in \mathbb{H}^{n\times n},\ \Delta G \in \mathbb{H}^{n\times n} \\ Q+\Delta Q \ge 0,\ G+\Delta G \ge 0}} \frac{\|\Delta Y\|_F}{\epsilon \|Y\|_F},$$

where $\eta_i,\ i = 1, 2$, are defined in (2.6).

Similarly to the proof of Theorem 1, we can obtain an explicit expression of $\kappa_1(\psi)$ and an upper bound for $\kappa_2(\psi)$ given in the following theorem.

Theorem 2. *Using the above notations, an explicit expression and an upper bound for the structured normwise numbers of the complex DARE (1.4) are*

$$\begin{aligned}\kappa_1(\psi) &= \frac{\|T^{-1}\mathcal{N}D\|_2}{\|Y\|_F},\\ \kappa_2(\psi) &\le \kappa_U(\psi) := \min\left\{\sqrt{6}\kappa_1(\psi), \alpha_d/\|Y\|_F\right\},\end{aligned}$$

where D is defined in (2.14) and

$$\begin{aligned}\alpha_d =\ & \delta_1 \|T^{-1}((I_n \otimes (A^H Y W)) + ((A^T W^T Y^T) \otimes I_n)\Pi)\|_2\\ & + \delta_2 \left\|T^{-1}((I_n \otimes (A^H Y W)) - ((A^T W^T Y^T) \otimes I_n)\Pi)\right\|_2\\ & + \delta_3 \|T^{-1}((A^T W^T Y^T) \otimes (A^H Y W)\mathcal{S}_1)\|_2 + \delta_4 \|T^{-1}((A^T W^T Y^T) \otimes (A^H Y W)\mathcal{S}_2)\|_2 + \delta_5 \|T^{-1}\mathcal{S}_1\|_2 + \delta_6 \|T^{-1}\mathcal{S}_2\|_2.\end{aligned}$$

Using the parameter θ in (2.16), Sun [36] studied the structured normwise condition number $\kappa_{\mathrm{Sun}}^{\mathrm{DARE}}$ for the complex DARE as follows

$$\kappa_{\mathrm{Sun}}^{\mathrm{DARE}} = \lim_{\epsilon\to 0} \sup_{\substack{\theta\le\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\ge 0}} \frac{\|\Delta Y\|_F}{\epsilon\|Y\|_F}.$$

Similarly to the complex CARE case, we can prove that

$$\kappa_1(\psi) \le \max\{\|\mathcal{S}_1\|_2,\ \|\mathcal{S}_2\|_2\}\cdot\kappa_{\mathrm{Sun}}^{\mathrm{DARE}},$$

when we choose $\delta_1=\delta_2=\mu_1$, $\delta_3=\delta_4=\mu_2$, and $\delta_5=\delta_6=\mu_3$.

For the real DARE (1.6), exploiting the symmetry structure, we define the mapping

$$\begin{aligned}\psi^{\mathrm{Re}} :\ & \mathbb{R}^{n^2+n(n+1)} \to \mathbb{C}^{n^2} \\ & [\mathrm{vec}(A)^T, \mathrm{sym}(G)^T, \mathrm{sym}(Q)^T]^T \mapsto \mathrm{vec}(Y)\end{aligned}$$

and the structured normwise condition number:

$$\kappa(\psi^{\mathrm{Re}}) = \lim_{\epsilon\to 0} \sup_{\substack{\delta\le\epsilon \\ \Delta A\in\mathbb{R}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{S}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\ge 0}} \frac{\|\Delta Y\|_F}{\epsilon\|Y\|_F},$$

where δ is defined in (2.17).

Using the above notations, we have the following upper bound for the structured normwise condition number of the real DARE:

$$\kappa(\psi^{\mathrm{Re}}) \le \kappa_U(\psi^{\mathrm{Re}}) := \min\left\{\frac{\sqrt{3}\,\|T_1^{-1}\mathcal{N}_1 D_1\|_2}{\|Y\|_F},\ \frac{\gamma}{\|Y\|_F}\right\}, \tag{2.21}$$

where

$$\begin{aligned}
T_1 &= I_n - (A^TW^T)\otimes(A^TW^T),\\
\mathcal{N}_1 &= [(I_n\otimes(A^TYW)) + ((A^TW^TY^T)\otimes I_n)\Pi,\ -(A^TW^TY^T)\otimes(A^TYW)\mathcal{S}_1, \mathcal{S}_1],\\
D_1 &= \mathrm{Diag}([\delta_1 I_{n^2},\ \delta_2 I_{n(n+1)/2},\ \delta_3 I_{n(n+1)/2}]),\\
\gamma &= \delta_1\left\|T_1^{-1}[(I_n\otimes(A^TYW)) + ((A^TW^TY^T)\otimes I_n)\Pi]\right\|_2\\
&\quad+\delta_2\left\|T_1^{-1}((A^TW^TY^T)\otimes(A^TYW))\mathcal{S}_1\right\|_2+\delta_3\left\|T_1^{-1}\mathcal{S}_1\right\|_2.
\end{aligned}$$

As expected, our upper bound (2.21) is an improvement of the condition number in [36, p. 260].

In [39], Zhou et al. defined the following unstructured normwise condition number for the real DARE:

$$\kappa_1(\psi^{\mathrm{Re}}) = \lim_{\epsilon\to 0}\sup_{\delta\le\epsilon}\frac{\|\Delta Y\|_F}{\epsilon\|Y\|_F},$$

where

$$\delta = \max\left\{\frac{\|\Delta A\|_F}{\delta_1},\ \frac{\|\Delta Q\|_F}{\delta_2},\ \frac{\|\Delta G\|_F}{\delta_3}\right\},$$

and derived the upper bound

$$\kappa_1(\psi^{\mathrm{Re}}) \le \kappa_1^U(\psi^{\mathrm{Re}}) := \min\left\{\sqrt{3}\,\frac{\|T_1^{-1}P_1\|_2}{\|Y\|_F},\ \frac{\gamma_d}{\|Y\|_F}\right\},$$

where

$$\begin{aligned}P_1 &= \left[(I_n\otimes(A^TYW)) + ((A^TW^TY)\otimes I_n)\Pi, -((A^TW^TY)\otimes(A^TYW)),\ I_{n^2}\right]\\ &\quad\cdot\mathrm{Diag}([\delta_1 I_{n^2},\ \delta_2 I_{n^2},\ \delta_3 I_{n^2}])\end{aligned}$$

and

$$\begin{aligned}\gamma_d &= \delta_1\|T_1^{-1}((I_n\otimes(A^TYW)) + ((A^TW^TY)\otimes I_n)\Pi)\|_2\\ &\quad+\delta_2\|T_1^{-1}((A^TW^TY)\otimes(A^TYW))\|_2+\delta_3\|T_1^{-1}\|_2.\end{aligned}$$

By setting $\delta_1=\|A\|_F$, $\delta_2=\|Q\|_F$ and $\delta_3=\|G\|_F$, we can prove that

$$\kappa_U(\psi^{\mathrm{Re}}) \le \|\mathcal{S}_1\|_2\kappa_1^U(\psi^{\mathrm{Re}}).$$

3. Structured mixed and componentwise condition numbers

Componentwise analysis [6,15,32,34] is more informative than its normwise counterpart when the data are badly scaled or sparse. Here, we consider the two kinds of condition numbers introduced by Gohberg and Koltracht [12]. The first kind, called the mixed condition number, measures the output errors in norm while the input perturbations componentwise. The second kind, called the componentwise condition number, measures both the output and the input perturbations componentwise.

Following [12], the structured mixed and componentwise condition numbers for the complex CARE (1.3) are defined by

$$m(\varphi) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta\in C_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\geq 0}} \frac{\|\Delta X\|_{\max}}{\epsilon\|X\|_{\max}},$$

$$c(\varphi) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta\in C_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\geq 0}} \frac{1}{\epsilon}\|\Delta X\oslash X\|_{\max},$$

where Δ is defined in (2.10) and

$$\begin{aligned} C_\epsilon = \Big\{&\Delta A,\ \Delta G,\ \Delta Q \ \big|\ |\mathrm{Re}(\Delta A))| \leq \epsilon|\mathrm{Re}(A)|,\ |\mathrm{Im}(\Delta A))| \leq \epsilon|\mathrm{Im}(A)|,\\ &\times|\mathrm{sym}(\mathrm{Re}(\Delta G))| \leq \epsilon|\mathrm{sym}(\mathrm{Re}(G))|,\ |\mathrm{skew}(\mathrm{Im}(\Delta G))| \leq \epsilon|\mathrm{skew}(\mathrm{Im}(G))|,\\ &\times|\mathrm{sym}(\mathrm{Re}(\Delta Q))| \leq \epsilon|\mathrm{sym}(\mathrm{Re}(Q))|,\ |\mathrm{skew}(\mathrm{Im}(\Delta Q))| \leq \epsilon|\mathrm{skew}(\mathrm{Im}(Q))|\Big\}. \end{aligned} \tag{3.1}$$

In Theorems 3 and 4, we present the structured mixed and componentwise condition numbers of the complex CARE (1.3) and the complex DARE (1.4).

Theorem 3. *For the structured mixed and componentwise condition numbers of the complex CARE (1.3), we have respectively*

$$\begin{aligned} m(\varphi) = \|X\|_{\max}^{-1}\Big\| &|Z^{-1}((I_n\otimes X)+(X^T\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Re}(A)|)\\ &+|Z^{-1}((I_n\otimes X)-(X^T\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Im}(A)|)+|Z^{-1}(X^T\otimes X)\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(G)|)\\ &+|Z^{-1}(X^T\otimes X)\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(G)|)+|Z^{-1}\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(Q)|)+|Z^{-1}\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(Q)|)\Big\|_\infty, \end{aligned}$$

$$\begin{aligned} c(\varphi) = \Big\|&\mathrm{Diag}(\mathrm{vec}(X))^\dagger\cdot\Big(|Z^{-1}((I_n\otimes X)+(X^T\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Re}(A)|)\\ &+|Z^{-1}((I_n\otimes X)-(X^T\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Im}(A)|)+|Z^{-1}(X^T\otimes X)\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(G)|)\\ &+|Z^{-1}(X^T\otimes X)\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(G)|)+|Z^{-1}\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(Q)|)+|Z^{-1}\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(Q)|)\Big)\Big\|_\infty, \end{aligned}$$

where $A\dagger$ is the Moore–Penrose inverse of A. Furthermore, we have their simpler upper bounds

$$\begin{aligned} m_U(\varphi) := \|X\|_{\max}^{-1}\|Z^{-1}\|_\infty\Big\|&\ |X|\,|\mathrm{Re}(A)|+|\mathrm{Re}(A)|^T|X|\\ &+|X|\,|\mathrm{Im}(A)|+|\mathrm{Im}(A)|^T|X|+|X||\mathrm{Re}(G)||X|+|X|\,|\mathrm{Im}(G)|\,|X|+|\mathrm{Re}(Q)|+|\mathrm{Im}(Q)|\ \Big\|_{\max} \end{aligned}$$

and

$$\begin{aligned} c_U(\varphi) := \|\mathrm{Diag}(\mathrm{vec}(X))^\dagger Z^{-1}\|_\infty\cdot\Big\|&\ |X|\,|\mathrm{Re}(A)|+|\mathrm{Re}(A)|^T|X|\\ &+|X|\,|\mathrm{Im}(A)|+|\mathrm{Im}(A)|^T|X|+|X|\,|\mathrm{Re}(G)||X|+|X|\,|\mathrm{Im}(G)|\,|X|+|\mathrm{Re}(Q)|+|\mathrm{Im}(Q)|\ \Big\|_{\max}. \end{aligned}$$

Proof. From Lemma 1, dropping the second and higher order terms, we have

$$\|\Delta X\|_{\max} = \|\mathrm{vec}(\Delta X)\|_\infty \approx \|Z^{-1}\mathcal{M}\Delta\|_\infty \leq \|Z^{-1}\mathcal{M}D_m\|_\infty\|D_m^\dagger\Delta\|_\infty,$$

where

$$\begin{aligned} D_m = \mathrm{Diag}([&\mathrm{vec}(\mathrm{Re}(A))^T,\ \mathrm{vec}(\mathrm{Im}(A))^T,\ \mathrm{sym}(\mathrm{Re}(G))^T,\ \mathrm{skew}(\mathrm{Im}(G))^T,\\ &\mathrm{sym}(\mathrm{Re}(Q))^T,\ \mathrm{skew}(\mathrm{Im}(Q))^T]^T). \end{aligned}$$

Since $\Delta\in C_\epsilon$, we have $\|D_m^\dagger\Delta\|_\infty \leq \epsilon$. Because Δ can be chosen arbitrarily, the upper bound is attainable. Recalling that $\mathbf{e}$ is the vector consisting all 1's, it can be verified that

$$\begin{aligned} \big\|Z^{-1}\mathcal{M}D_m\big\|_\infty &= \big\||Z^{-1}\mathcal{M}|\cdot|D_m|\mathbf{e}\big\|_\infty\\ &= \Big\||Z^{-1}\mathcal{M}|\cdot\Big[|\mathrm{vec}(\mathrm{Re}(A))|^T,\ |\mathrm{vec}(\mathrm{Im}(A))|^T,\ |\mathrm{sym}(\mathrm{Re}(G))|^T,\ |\mathrm{skew}(\mathrm{Im}(G))|^T, \end{aligned}$$

$$|\mathrm{sym}(\mathrm{Re}(Q))|^T, |\mathrm{skew}(\mathrm{Im}(Q))|^T\Big]^T\Big\|_\infty.$$

After some algebraic manipulation, we can get the explicit expression for $m(\varphi)$. From the following inequalities,

$$\begin{aligned}
&|Z^{-1}((I_n \otimes X) + (X^T \otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Re}(A)|) \\
&\leq |Z^{-1}|((I_n \otimes |X|) + (|X^T| \otimes I_n)\Pi)\mathrm{vec}(|\mathrm{Re}(A)|) = |Z^{-1}|\mathrm{vec}(|X||\mathrm{Re}(A)| + |\mathrm{Re}(A)|^T|X|), \\
&|Z^{-1}((I_n \otimes X) - (X^T \otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Im}(A)|) \leq |Z^{-1}|\mathrm{vec}(|X||\mathrm{Im}(A)| + |\mathrm{Im}(A)|^T|X|), \\
&|Z^{-1}(X^T \otimes X)\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(G)|) \\
&\leq |Z^{-1}|(|X^T| \otimes |X|)\mathrm{vec}(|\mathrm{Re}(G)|) = |Z^{-1}|\mathrm{vec}(|X||\mathrm{Re}(G)||X|), \\
&|Z^{-1}(X^T \otimes X)\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(G)|) \\
&\leq |Z^{-1}|(|X^T| \otimes |X|)\mathrm{vec}(|\mathrm{Im}(G)|) = |Z^{-1}|\mathrm{vec}(|X||\mathrm{Im}(G)||X|), \\
&|Z^{-1}\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(Q)|) \leq |Z^{-1}|\mathrm{vec}(|\mathrm{Re}(Q)|), \quad |Z^{-1}\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(Q)|) \leq |Z^{-1}|\mathrm{vec}(|\mathrm{Im}(Q)|),
\end{aligned}$$

and the monotonicity of the infinity norm, we can obtain the upper bound $m_U(\varphi)$. For the structured componentwise condition number $c(\varphi)$, noting that

$$\|\Delta X \oslash X\|_{\max} = \|\mathrm{Diag}(\mathrm{vec}(X))^\dagger \mathrm{vec}(\Delta X)\|_\infty,$$

similarly to the derivation of $m(\varphi)$, we can obtain the explicit expression for $c(\varphi)$. Also, the upper bound $c_U(\varphi)$ can be deduced similarly. □

The above theorem shows that an ill-conditioned Z is an indication of large $m(\varphi)$ or $c(\varphi)$.

For the real CARE (1.5), the structured mixed and componentwise condition numbers of φ^{Re} at X can be defined by

$$m(\varphi^{\mathrm{Re}}) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta_1 \in C^1_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{S}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\geq 0}} \frac{\|\Delta X\|_{\max}}{\epsilon\|X\|_{\max}},$$

$$c(\varphi^{\mathrm{Re}}) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta_1 \in C^1_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{S}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\geq 0}} \frac{1}{\epsilon}\|\Delta X \oslash X\|_{\max},$$

where $\Delta_1 = [\Delta A^T, \mathrm{sym}(\Delta G)^T, \mathrm{sym}(\Delta Q)^T]^T$ and

$$\begin{aligned}
C^1_\epsilon = \Big\{&\Delta A,\ \Delta G,\ \Delta Q,\ \mid |\Delta A| \leq \epsilon|A|,\ |\mathrm{sym}(\Delta G)| \leq \epsilon|\mathrm{sym}(\Delta G)|, \\
&|\mathrm{sym}(\Delta Q)| \leq \epsilon|\mathrm{sym}(\Delta Q)|\Big\}.
\end{aligned} \tag{3.2}$$

Now, we have expressions for $m(\varphi^{\mathrm{Re}})$ and $c(\varphi^{\mathrm{Re}})$.

Corollary 1. *For structured mixed and componentwise condition numbers of the real CARE (1.5), we have respectively*

$$\begin{aligned}
m(\varphi^{\mathrm{Re}}) &= \|X\|^{-1}_{\max}\Big\||Z_1^{-1}((I_n \otimes X) + (X^T \otimes I_n)\Pi)|\mathrm{vec}(|A|) \\
&\quad + |Z_1^{-1}(X^T \otimes X)\mathcal{S}_1|\mathrm{sym}(|G|) + |Z_1^{-1}\mathcal{S}_1|\mathrm{sym}(|Q|)\Big\|_\infty \\
c(\varphi^{\mathrm{Re}}) &= \Big\|\mathrm{Diag}(\mathrm{vec}(X))^\dagger\Big(|Z_1^{-1}((I_n \otimes X) + (X^T \otimes I_n)\Pi)| \\
&\quad \cdot \mathrm{vec}(|A|) + |Z_1^{-1}(X^T \otimes X)\mathcal{S}_1|\mathrm{sym}(|G|) + |Z_1^{-1}\mathcal{S}_1|\mathrm{sym}(|Q|)\Big)\Big\|_\infty,
\end{aligned}$$

where $Z_1 = I_n \otimes (A - GX)^T + (A - GX)^T \otimes I_n$. Furthermore, we have their simpler upper bounds

$$m_U(\varphi^{\mathrm{Re}}) := \|X\|^{-1}_{\max}\|Z_1^{-1}\|_\infty\Big\|\ |X|\,|A| + |A|^T|X| + |X|\,|G|\,|X| + |Q|\ \Big\|_{\max},$$

and

$$c_U(\varphi^{\mathrm{Re}}) := \|\mathrm{Diag}(\mathrm{vec}(X))^\dagger Z_1^{-1}\|_\infty\Big\|\ |X|\,|A| + |A|^T|X| + |X|\,|G|\,|X| + |Q|\ \Big\|_{\max}.$$

Zhou et al. [39] derived the mixed and componentwise condition numbers for the real CARE without exploiting the symmetry structure. It can be readily shown that our structured mixed and componentwise condition numbers are smaller than their counterparts in [39], although, they are empirically comparable.

Similarly, the structured mixed and componentwise condition numbers for the complex DARE (1.4) can be defined by

$$m(\psi) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta \in C_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\ \Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q\geq 0}} \frac{\|\Delta Y\|_{\max}}{\epsilon\|Y\|_{\max}},$$

$$c(\psi) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta\in C_\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\, Q+\Delta Q\geq 0}} \frac{1}{\epsilon}\|\Delta Y \oslash Y\|_{\max},$$

where Δ is defined in (2.10) and C_ϵ is defined in (3.1).

Following the proof of Theorem 3, we have the structured mixed and componentwise condition numbers for the complex DARE (1.4) in the following theorem. Its proof is omitted.

Theorem 4. *With the notations as before, we have*

$$\begin{aligned} m(\psi) = \|Y\|_{\max}^{-1}\Big\| & |T^{-1}(I_n\otimes(A^HYW)+((A^TW^TY^T)\otimes I_n)\Pi)|\cdot \mathrm{vec}(|\mathrm{Re}(A)|) \\ &+|T^{-1}(I_n\otimes(A^HYW)-((A^TW^TY^T)\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Im}(A)|) \\ &+|T^{-1}(A^TW^TY^T)\otimes(A^HYW)\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(G)|) \\ &+|T^{-1}(A^TW^TY^T)\otimes(A^HYW)\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(G)|) \\ &+|T^{-1}\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(Q)|)+|T^{-1}\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(Q)|)\Big\|_\infty \end{aligned}$$

and

$$\begin{aligned} c(\psi) = \Big\| & \mathrm{Diag}(\mathrm{vec}(Y))^\dagger\Big(|T^{-1}((I_n\otimes(A^HYW))+((A^TW^TY^T)\otimes I_n)\Pi)|\cdot\mathrm{vec}(|\mathrm{Re}(A)|) \\ &+|T^{-1}((I_n\otimes(A^HYW))-((A^TW^TY^T)\otimes I_n)\Pi)|\mathrm{vec}(|\mathrm{Im}(A)|) \\ &+|T^{-1}((A^TW^TY^T)\otimes(A^HYW)\mathcal{S}_1)|\mathrm{sym}(|\mathrm{Re}(G)|) \\ &+|T^{-1}((A^TW^TY^T)\otimes(A^HYW)\mathcal{S}_2)|\mathrm{skew}(|\mathrm{Im}(G)|) \\ &+|T^{-1}\mathcal{S}_1|\mathrm{sym}(|\mathrm{Re}(Q)|)+|T^{-1}\mathcal{S}_2|\mathrm{skew}(|\mathrm{Im}(Q)|)\Big)\Big\|_\infty. \end{aligned}$$

Furthermore, we have their simpler upper bounds:

$$\begin{aligned} m_U(\psi) = \|Y\|_{\max}^{-1}\|T^{-1}\|_\infty\Big\| & |A^H||Y||W||\mathrm{Re}(A)| \\ &+|\mathrm{Re}(A)|^T|Y||W||A|+|A^H||Y||W||\mathrm{Im}(A)| \\ &+|\mathrm{Im}(A)|^T|Y||W||A|+|A^H||Y||W||\mathrm{Re}(G)||Y||W||A| \\ &+|A^H||Y||W||\mathrm{Im}(G)||Y||W||A|+|\mathrm{Re}(Q)|+|\mathrm{Im}(Q)|\Big\|_{\max} \end{aligned}$$

and

$$\begin{aligned} c_U(\psi) = \|\mathrm{Diag}(\mathrm{vec}(Y))^\dagger T^{-1}\|_\infty\Big\| & |A^H||Y||W||\mathrm{Re}(A)| \\ &+|\mathrm{Re}(A)|^T|Y||W||A|+|A^H||Y||W||\mathrm{Im}(A)| \\ &+|\mathrm{Im}(A)|^T|Y||W||A|+|A^H||Y||W||\mathrm{Re}(G)||Y||W||A| \\ &+|A^H||Y||W||\mathrm{Im}(G)||Y||W||A|+|\mathrm{Re}(Q)|+|\mathrm{Im}(Q)|\Big\|_{\max}. \end{aligned}$$

The above expressions in Theorem 4 show that an ill conditioned T indicates large $m(\psi)$ or $c(\psi)$.

For the real DARE (1.6), we can define the structured mixed and componentwise condition numbers of ψ^{Re} at Y:

$$m(\psi^{\mathrm{Re}}) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta_1\in C_\epsilon^1 \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\Delta Q\in\mathbb{S}^{n\times n} \\ G+\Delta G,\, Q+\Delta Q\geq 0}} \frac{\|\Delta Y\|_{\max}}{\epsilon\|Y\|_{\max}},$$

$$c(\psi^{\mathrm{Re}}) = \lim_{\epsilon\to 0} \sup_{\substack{\Delta_1\in C_\epsilon^1 \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\Delta Q\in\mathbb{S}^{n\times n} \\ Q+\Delta Q\geq 0,\, G+\Delta G,\, Q+\Delta Q\geq 0}} \frac{1}{\epsilon}\|\Delta Y\oslash Y\|_{\max},$$

respectively, where $\Delta_1 = [\Delta A^T, \mathrm{sym}(\Delta G)^T, \mathrm{sym}(\Delta Q)^T]^T$ and C_ϵ^1 is defined in (3.2).

Similarly to Corollary 1, we then can obtain the structured mixed and componentwise condition numbers of the real DARE in the following corollary.

Corollary 2. *With the notations above, we have*

$$\begin{aligned} m(\psi^{\mathrm{Re}}) &= \|Y\|_{\max}^{-1}\Big\| |T^{-1}((I_n\otimes(A^TYW))+((A^TYW)\otimes I_n)\Pi)|\cdot\mathrm{vec}(|A|) \\ &\quad + |T^{-1}((A^TYW)\otimes(A^TYW)\mathcal{S}_1)|\mathrm{sym}(|G|)+|T^{-1}\mathcal{S}_1|\mathrm{sym}(|Q|)\Big\|_\infty, \\ c(\psi^{\mathrm{Re}}) &= \Big\|\mathrm{Diag}(\mathrm{vec}(Y))^\dagger\Big(|T^{-1}((I_n\otimes(A^TYW))+((A^TYW)\otimes I_n)\Pi)|\cdot\mathrm{vec}(|A|) \\ &\quad + |T^{-1}((A^TYW)\otimes(A^TYW)\mathcal{S}_1)|\mathrm{sym}(|G|)+|T^{-1}\mathcal{S}_1|\mathrm{sym}(|Q|)\Big)\Big\|_\infty, \end{aligned}$$

Furthermore, we have their simpler upper bounds:

$$\begin{aligned} m_U(\psi^{\mathrm{Re}}) &= \|Y\|_{\max}^{-1}\|T^{-1}\|_\infty\Big\| \, |A^T||Y||W||A|+|A|^T|Y||W||A| \\ &\quad +|A^T||Y||W||G||Y||W||A|+|Q|\,\Big\|_{\max}, \\ c_U(\psi^{\mathrm{Re}}) &= \|\mathrm{Diag}(\mathrm{vec}(Y))^\dagger T^{-1}\|_\infty\Big\| \, |A^T||Y||W||A|+|A|^T|Y||W||A| \\ &\quad +|A^T||Y||W||G||Y||W||A|+|Q|\,\Big\|_{\max}. \end{aligned}$$

4. Small sample condition estimation

Although the expressions of the condition numbers presented earlier are explicit, they involve the solution and their computation is intensive when the problem size is large. Thus, practical algorithms for approximating the condition numbers are worth studying [36, p. 260]. In this section, based on a small sample statistical condition estimation method, we present a practical method for estimating the condition numbers for the symmetric algebraic Riccati equations.

We first briefly describe our method. Given a differentiable function $f:\mathbb{R}^p\to\mathbb{R}$, we are interested in its sensitivity at some input vector x. From its Taylor expansion, we have

$$f(x+\delta d)-f(x)=\delta(\nabla f(x))^Td+O(\delta^2),$$

for a small scalar δ, where

$$\nabla f(x)=\left[\frac{\partial f(x)}{\partial x_1},\frac{\partial f(x)}{\partial x_2},\ldots,\frac{\partial f(x)}{\partial x_p}\right]^T$$

is the gradient of f at x. Then the local sensitivity, up to the first order in δ, can be measured by $\|\nabla f(x)\|_2$. The condition number of f at x is mainly determined by the norm of the gradient $\nabla f(x)$ ([18]). It is shown in [18] that if we select d uniformly and randomly from the unit p-sphere S_{p-1} (denoted $d\in U(S_{p-1})$), then the expected value $\mathbf{E}(|(\nabla f(x))^Td|/\omega_p)$ is $\|\nabla f(x)\|_2$, where ω_p is the Wallis factor, which depends only on p, given by

$$\omega_p=\begin{cases} 1, & \text{for } p\equiv 1,\\ \dfrac{2}{\pi}, & \text{for } p\equiv 2,\\ \dfrac{1\cdot3\cdot5\cdots(p-2)}{2\cdot4\cdot6\cdots(p-1)}, & \text{for } p \text{ odd and } p>2,\\ \dfrac{2}{\pi}\dfrac{2\cdot4\cdot6\cdots(p-2)}{1\cdot3\cdot5\cdots(p-1)}, & \text{for } p \text{ even and } p>2, \end{cases}$$

which can be accurately approximated by

$$\omega_p\approx\sqrt{\frac{2}{\pi(p-\frac{1}{2})}}. \tag{4.1}$$

Therefore,

$$\nu=\frac{|(\nabla f(x))^Td|}{\omega_p}$$

can be used to estimate $\|\nabla f(x)\|_2$, an approximation of the condition number, with high probability [18]. Specifically, for $\gamma>1$,

$$\mathrm{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma}\le\nu\le\gamma\|\nabla f(x)\|_2\right)\ge 1-\frac{2}{\pi\gamma}+O(\gamma^{-2}).$$

Multiple samples d_j can be used to increase the accuracy [18]. The k-sample condition estimation is given by

$$\nu(k) = \frac{\omega_k}{\omega_p}\sqrt{|\nabla f(x)^T d_1|^2 + |\nabla f(x)^T d_2|^2 + \cdots + |\nabla f(x)^T d_k|^2},$$

where $d_1, d_2, \ldots, d_k$ are orthonormalized after they are selected uniformly and randomly from $U(S_{p-1})$. In particular, the accuracy of $\nu(2)$ is given by

$$\text{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma} \le \nu(2) \le \gamma\|\nabla f(x)\|_2\right) \approx 1 - \frac{\pi}{4\gamma^2}.$$

Usually, a small set of samples is sufficient for good accuracy.

These results can be readily generalized to vector-valued or matrix-valued functions by viewing f as a map from $\mathbb{R}^s$ to $\mathbb{R}^t$ by applying the operations vec and unvec to transform data between matrices and vectors, where each of the t entries of f is a scalar-valued function.

4.1. Structured normwise case

In this subsection, by applying the small sample condition estimation method, we devise the algorithms for estimating the structured normwise condition numbers of CARE and DARE. Before that, we introduce the unvec operation: Given m and n, for $v = [v_1, v_2, \ldots, v_{mn}] \in \mathbb{R}^{1\times mn}$, $A = \text{unvec}(v)$ sets the (i, j)-entry of A to $v_{i+(j-1)n}$.

For CARE, from Lemma 1, the directional derivative $D_X \in \mathbb{H}^{n\times n}$ of φ at X with respect to the direction Δ, defined in (2.10), satisfies the continuous Lyapunov Eq. (2.4). Putting things together, we propose the following subspace structured condition number estimation algorithm for the complex CARE (1.3).

1. Generate vectors $f_i \in \mathbb{R}^{4n^2}$, $i = 1, \ldots, k$, with each entry in $\mathcal{N}(0, 1)$. Orthonormalize them using, for example, the QR factorization, to get $z_j \in \mathbb{R}^{4n^2}$, $j = 1, \ldots, k$. Each z_j can be converted into matrices $\widetilde{A}_j$, $\widetilde{G}_j$, and $\widetilde{Q}_j$ by applying the unvec operation, where $\widetilde{A}_j \in \mathbb{C}^{n\times n}$ and $\widetilde{G}_j$, $\widetilde{Q}_j \in \mathbb{H}^{n\times n}$;
2. For $i = 1, 2, \ldots, k$, solve for $D_i \in \mathbb{H}^{n\times n}$ in the following continuous Lyapunov equation
$$(A - GX)^H D_i + D_i(A - GX) = X\widetilde{G}_i X - \widetilde{Q}_i - X\widetilde{A}_i - \widetilde{A}_i^H X;$$
3. Approximate ω_k and ω_p $(p = 4n^2)$ by (4.1) and calculate the absolute condition number matrix
$$K_{\text{abs}}^{\text{CARE},(k)} := \|[A, G, Q]\|_F \frac{\omega_k}{\omega_p}\sqrt{|D_1|^2 + |D_2|^2 + \cdots + |D_k|^2},$$
where the square operation is applied to each entry of $|D_i|$, $i = 1, 2, \ldots, k$ and the square root is also applied componentwise;
4. Finally, the relative condition number matrix
$$K_{\text{rel}}^{\text{CARE},(k)} = K_{\text{abs}}^{\text{CARE},(k)} \oslash X$$
is obtained by componentwise division for nonzero entries of X, leaving the entries of $K_{\text{abs}}^{\text{CARE},(k)}$ corresponding to the zero entries of X unchanged.

The real CARE (1.5) is a special case.

Note that Step 2 in the above algorithm involves solving a sequence of Lyapunov equations. When a Lyapunov equation is ill-conditioned, the computed solution D_i can be inaccurate, consequently, the condition number for CARE computed in the following Step 3 can be inaccurate. However, the conditioning of the Lyapunov equation and that of CARE are related in that the ill-conditioning of the continuous Lyapunov equation implies the ill-conditioning of the original CARE, because solving the Lyapunov equation is essentially equivalent to finding Z^{-1} in the condition number for CARE presented in Theorem 3.

For the complex DARE (1.4), from Lemma 2, the directional derivative $D_Y \in \mathbb{H}^{n\times n}$ of ψ at Y with respect to the direction Δ is the solution of the discrete Lyapunov equation

$$D_Y - (WA)^H D_Y WA = \Delta Q + (A^H YW)\Delta A + \Delta A^H(YWA) - (A^H YW)\Delta G(YWA),$$

where $\Delta A \in \mathbb{C}^{n\times n}$ and ΔG, $\Delta Q \in \mathbb{H}^{n\times n}$.

Similarly to the complex CARE case, we propose the following algorithm for the complex DARE.

1. Generate vectors $f_i \in \mathbb{R}^{4n^2}$, $i = 1, \ldots, k$ with each entry in $\mathcal{N}(0, 1)$. Orthonormalize them using, for example, the QR factorization, to get $z_j \in \mathbb{R}^{4n^2}$, $j = 1, \ldots, k$. Each z_j can be converted into the corresponding matrices $\widetilde{A}_j$, $\widetilde{G}_j$, and $\widetilde{Q}_j$ by applying the unvec operation, where $\widetilde{A}_j \in \mathbb{C}^{n\times n}$ and $\widetilde{G}_j$, $\widetilde{Q}_j \in \mathbb{H}^{n\times n}$;
2. For $i = 1, 2, \ldots, k$, solve for $D_i \in \mathbb{H}^{n\times n}$ in the following discrete Lyapunov equation
$$D_i - (WA)^H D_i WA = \widetilde{Q}_i + (A^H YW)\widetilde{A}_i + \widetilde{A}_i^H(YWA)(A^H YW)\widetilde{G}_i(YWA);$$

3. Approximate ω_k and ω_p $(p = 4n^2)$ by (4.1) and calculate the absolute condition number matrix

$$K_{\text{abs}}^{\text{DARE},(k)} := \|[A, G, Q]\|_F \frac{\omega_k}{\omega_p}\sqrt{|D_1|^2 + |D_2|^2 + \cdots + |D_k|^2};$$

4. Finally, the relative condition number matrix

$$K_{\text{rel}}^{\text{DARE},(k)} = K_{\text{abs}}^{\text{DARE},(k)} \oslash Y.$$

4.2. Structured componentwise case

Componentwise condition number often leads to a more realistic indication of the accuracy of a computed solution than the normwise condition number. The sensitivity effects of componentwise perturbations can be measured by the SCE method [18]. For a perturbation $\Delta A = [\Delta a_{ij}]$ on a matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, it is a componentwise perturbation, if

$$|\Delta A| \le \varepsilon |A| \quad \text{or} \quad |\Delta a_{ij}| \le \varepsilon |a_{ij}|.$$

We can write $\Delta A = \delta \cdot \mathcal{A} \boxdot A$ with $|\delta| \le \varepsilon$ and the entries of $\mathcal{A}$ are in the interval $[-1, 1]$, where $\boxdot$ is a componentwise multiplication. We propose the following algorithm for a structured componentwise sensitivity estimate of the solution X of the complex CARE (1.3).

1. Generate vectors $f_i \in \mathbb{R}^{4n^2}$, $i = 1, \ldots, k$, with each entry in $\mathcal{N}(0, 1)$. Orthonormalize them using, for example, the QR factorization, to get $z_j \in \mathbb{R}^{4n^2}$, $j = 1, \ldots, k$. Each z_j can be converted into the corresponding matrices $\widetilde{A}_j$, $\widetilde{G}_j$, and $\widetilde{Q}_j$ by applying the unvec operation, where $\widetilde{A}_j \in \mathbb{C}^{n\times n}$ and $\widetilde{G}_j, \widetilde{Q}_j \in \mathbb{H}^{n\times n}$;
2. For $j = 1, 2, \ldots, k$, set $[\widetilde{A}_j, \widetilde{G}_j, \widetilde{Q}_j]$ equal to the componentwise product of $[A, G, Q]$ and $[\widetilde{A}_j, \widetilde{G}_j, \widetilde{Q}_j]$;
3. For $i = 1, 2, \ldots, k$, solve for $D_i \in \mathbb{H}^{n\times n}$ in the following continuous Lyapunov equation

$$(A - GX)^H D_i + D_i(A - GX) = X\widetilde{G}_i X - \widetilde{Q}_i - X\widetilde{A}_i - \widetilde{A}_i^H X;$$

4. Approximate ω_k and ω_p $(p = 4n^2)$ by (4.1) and calculate the absolute condition number matrix

$$C_{\text{abs}}^{\text{CARE},(k)} := \frac{\omega_k}{\omega_p}\sqrt{|D_1|^2 + |D_2|^2 + \cdots + |D_k|^2};$$

5. Finally, the relative condition number matrix

$$C_{\text{rel}}^{\text{CARE},(k)} = C_{\text{abs}}^{\text{CARE},(k)} \oslash X.$$

Analogously to the above complex case, we propose the following algorithm for the complex DARE (1.4).

1. Generate vectors $f_i \in \mathbb{R}^{4n^2}$, $i = 1 \ldots, k$, with each entry in $\mathcal{N}(0, 1)$. Orthonormalize them using, for example, the QR factorization, to get $z_j \in \mathbb{R}^{4n^2}$, $j = 1, \ldots, k$. Each z_j can be converted into the corresponding matrices $\widetilde{A}_j$, $\widetilde{G}_j$, and $\widetilde{Q}_j$ by applying the unvec operation, where $\widetilde{A}_j \in \mathbb{C}^{n\times n}$ and $\widetilde{G}_j, \widetilde{Q}_j \in \mathbb{H}^{n\times n}$;
2. For $j = 1, 2, \ldots, k$, set $[\widetilde{A}_j, \widetilde{G}_j, \widetilde{Q}_j]$ equal to the componentwise product of $[A, G, Q]$ and $[\widetilde{A}_j, \widetilde{G}_j, \widetilde{Q}_j]$;
3. For $i = 1, 2, \ldots, k$, solve for $D_i \in \mathbb{H}^{n\times n}$ in the following discrete Lyapunov equation

$$D_i - (WA)^H D_i WA = \widetilde{Q}_i + (A^H YW)\widetilde{A}_i + \widetilde{A}_i^H (YWA) - (A^H YW)\widetilde{G}_i (YWA);$$

4. Approximate ω_k and ω_p $(p = 4n^2)$ by (4.1) and calculate the absolute condition number matrix

$$C_{\text{abs}}^{\text{DARE},(k)} := \frac{\omega_k}{\omega_p}\sqrt{|D_1|^2 + |D_2|^2 + \cdots + |D_k|^2};$$

5. Finally, the relative condition number matrix

$$C_{\text{rel}}^{\text{DARE},(k)} = C_{\text{abs}}^{\text{DARE},(k)} \oslash Y.$$

5. Numerical examples

In this section, we adopt the examples in [2,3,39] to illustrate the effectiveness of our methods. All the experiments were performed using MATLAB 8.1, with the machine epsilon $\mu \approx 2.2 \times 10^{-16}$.

Given $A \in \mathbb{R}^{n\times n}$ and $G, Q \in \mathbb{S}^{n\times n}$, we generated the perturbations on A, G and Q as follows: $\Delta A = \epsilon(M_1 \boxdot A)$, $\Delta G = \epsilon(M_2 \boxdot G)$, and $\Delta Q = \epsilon(M_3 \boxdot Q)$, where $\epsilon = 10^{-j}$ for some nonnegative integer j, $\boxdot$ denotes the componentwise multiplication of two matrices, and $M_1 \in \mathbb{R}^{n\times n}$, and $M_2, M_3 \in \mathbb{S}^{n\times n}$ whose entries are random numbers uniformly distributed in the open interval $(-1, 1)$.

Example 1. Consider the CARE (1.3) from [[2], Example 9] with

$$A = \begin{bmatrix} 0 & \nu \\ 0 & 0 \end{bmatrix}, \quad Q = I_2, \quad G = BR^{-1}B^T,$$

Table 1
Comparison of the accurate relative changes in the solution with the estimates obtained by our condition numbers, where $\epsilon = 10^{-8}$.

ν	$\|\Delta X\|_F/\|X\|_F$	$\epsilon\,\kappa_U(\varphi^{\mathrm{Re}})$	$\epsilon\,\kappa_1^U(\varphi^{\mathrm{Re}})$	
1	3.0642×10^{-9}	3.7258×10^{-8}	4.0054×10^{-8}	
10^6	7.0865×10^{-9}	5.000×10^{-3}	5.000×10^{-3}	
10^{-6}	4.6983×10^{-9}	5.0000×10^{3}	5.0000×10^{3}	
ν	$\|\Delta X\|_{\max}/\|X\|_{\max}$	$\epsilon\, m(\varphi^{\mathrm{Re}})$	$\|\Delta X \oslash X\|_{\max}$	$\epsilon\, c(\varphi^{\mathrm{Re}})$
1	6.1630×10^{-9}	1.6667×10^{-8}	7.7288×10^{-9}	1.6667×10^{-8}
10^6	7.0865×10^{-9}	1.5000×10^{-8}	1.2161×10^{-8}	1.5000×10^{-8}
10^{-6}	4.6983×10^{-9}	2.0000×10^{-8}	7.5086×10^{-9}	2.0000×10^{-8}

where

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R = 1.$$

The pair (A, G) is c-stabilizable and the pair (A, Q) is c-detectable. The exact solution is

$$X = \begin{bmatrix} \frac{\sqrt{1+2\nu}}{\nu} & 1 \\ 1 & \sqrt{1+2\nu} \end{bmatrix}.$$

When ν is large or small, $\|X\|_F$ is approximately $\sqrt{\nu}\,(\nu \geq 1)$ or $1/\sqrt{\nu}\,(0 < \nu < 1)$ respectively and CARE becomes ill conditioned in terms of the normwise conditions $\kappa_U(\varphi^{\mathrm{Re}})$ and $\kappa_1^U(\varphi^{\mathrm{Re}})$. However, as shown in Table 1, from the componentwise perturbation analysis, $m(\varphi^{\mathrm{Re}})$ and $c(\varphi^{\mathrm{Re}})$ are always of $\mathcal{O}(1)$.

Let $\tilde{Q} = Q + \Delta Q, \tilde{A} = A + \Delta A, \tilde{G} = G + \Delta G$ be the coefficient matrices of the perturbed CARE (2.1). The perturbation size $\epsilon = 10^{-8}$. We used the MATLAB function `are` to compute the unique symmetric positive semidefinite solution $\tilde{X}$ to the perturbed equation. Let $\Delta X = \tilde{X} - X$.

For the bound $\kappa_U(\varphi^{\mathrm{Re}})$, we set $\delta_1 = \|A\|_F, \delta_2 = \|\mathrm{sym}(Q)\|_2, \delta_3 = \|\mathrm{sym}(G)\|_2$. For $\kappa_1^U(\varphi^{\mathrm{Re}})$ in [39] we choose $\delta_1 = \|A\|_F, \delta_2 = \|Q\|_F, \delta_3 = \|G\|_F$. Table 1 compares the accurate relative changes $\|\Delta X\|_F/\|X\|_F$, $\|\Delta X\|_{\max}/\|X\|_{\max}$ and $\|\Delta X \oslash X\|_{\max}$ obtained by MATLAB with the estimates obtained by our condition numbers. Our normwise condition numbers are consistent with those in [[2], p. 9] for $\nu = 1, 10^6, 10^{-6}$. Our mixed and componentwise condition numbers, however, give accurate estimates for the corresponding relative changes in the solution.

For the SCE algorithms, we set the sample number $k = 5$ and tested them for various values of ν. The results are shown as follows. For $\nu = 1$,

$$\Delta X \oslash X = 10^{-8} \begin{bmatrix} -0.4039 & -0.7729 \\ -0.7729 & -0.6163 \end{bmatrix},$$

$$\epsilon\, K_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{-8} \begin{bmatrix} 6.5364 & 7.4764 \\ 7.4764 & 3.8048 \end{bmatrix},$$

$$\epsilon\, C_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{-8} \begin{bmatrix} 0.7649 & 0.6111 \\ 0.6111 & 0.6727 \end{bmatrix}.$$

For $\nu = 10^6$,

$$\Delta X \oslash X = 10^{-7} \begin{bmatrix} -0.1216 & -0.0962 \\ -0.0962 & -0.0709 \end{bmatrix},$$

$$\epsilon\, K_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{-2} \begin{bmatrix} 0.9807 & 1.3288 \\ 1.3288 & 1.1426 \end{bmatrix},$$

$$\epsilon\, C_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{-8} \begin{bmatrix} 1.1150 & 0.8994 \\ 0.8994 & 1.2533 \end{bmatrix}.$$

For $\nu = 10^{-6}$,

$$\Delta X \oslash X = 10^{-8} \begin{bmatrix} -0.4698 & -0.4207 \\ -0.4207 & -0.7509 \end{bmatrix},$$

$$\epsilon\, K_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{4} \begin{bmatrix} 1.0725 & 1.0725 \\ 1.0725 & 0.0000 \end{bmatrix},$$

$$\epsilon\, C_{\mathrm{rel}}^{\mathrm{CARE},(5)} = 10^{-8} \begin{bmatrix} 1.1728 & 0.4670 \\ 0.4670 & 0.7666 \end{bmatrix}.$$

Table 2
Comparison of the accurate relative changes in the solution with the estimates obtained by our condition numbers, where $\epsilon = 10^{-12}$.

m	$\|\Delta Y\|_F/\|Y\|_F$	$\epsilon\,\kappa_U(\psi^{\mathrm{Re}})$	$\epsilon\,\kappa_1^U(\psi^{\mathrm{Re}})$	
1	1.2974×10^{-12}	6.6183×10^{-12}	7.1051×10^{-12}	
5	1.5931×10^{-8}	5.0002×10^{-8}	5.2934×10^{-8}	
7	1.0577×10^{-7}	5.0000×10^{-6}	5.2932×10^{-6}	
m	$\|\Delta Y\|_{\max}/\|Y\|_{\max}$	$\epsilon\, m(\psi^{\mathrm{Re}})$	$\|\Delta Y \oslash Y\|_{\max}$	$\epsilon\, c(\psi^{\mathrm{Re}})$
1	1.2945×10^{-12}	4.8227×10^{-12}	1.8177×10^{-12}	1.1056×10^{-11}
5	1.5931×10^{-8}	3.9507×10^{-8}	6.3715×10^{-8}	1.5801×10^{-7}
7	1.0577×10^{-7}	3.9506×10^{-6}	4.2307×10^{-7}	1.5802×10^{-5}

As we can see, for this particular example, the componentwise condition matrices $C_{\mathrm{rel}}^{\mathrm{CARE},(5)}$ for all values of ν can be used to accurately estimate the changes in the solution. In contrast, the normwise condition matrix $K_{\mathrm{rel}}^{\mathrm{CARE},(5)}$ can give good estimate only when $\nu = 1$ because the problem is well conditioned under the normwise perturbation analysis in this case.

Example 2. For DARE, we adopt the following example from [39]. Consider the DARE (1.4) with

$$Q = VQ_0V, \quad A = VA_0V, \quad G = VG_0V,$$

where

$$Q_0 = \mathrm{Diag}([10^m, 1, 10^{-m}]^T), \quad A_0 = \mathrm{Diag}([0, 10^{-m}, 1]^T),$$

$$G_0 = \mathrm{Diag}([10^{-m}, 10^{-m}, 10^{-m}]^T),$$

and

$$V = I - 2vv^T/3, \quad v = [1, 1, 1]^T.$$

Correspondingly, in the original DARE (1.2), $B = V$, $R = G_0^{-1}$, and $C = V\sqrt{Q_0}V$. The pair (A, B) is d-stabilizable and the pair (A, C) is d-detectable. The unique symmetric positive semidefinite solution Y to the DARE (1.4) is given by $Y = VY_0V$, where $Y_0 = \mathrm{Diag}([y_1, y_2, y_3]^T)$ with

$$y_i = (a_i^2 + q_ig_i - 1 + ((a_i^2 + q_ig_i - 1)^2 + 4q_ig_i)^{1/2})/(2g_i),$$

and q_i, a_i and g_i are respectively the diagonal elements of Q_0, A_0 and G_0. The perturbation matrices ΔA, ΔG and ΔQ were generated as described in the beginning of this section with $\epsilon = 10^{-12}$. Let $\tilde{Q} = Q + \Delta Q, \tilde{A} = A + \Delta A, \tilde{G} = G + \Delta G$ be the coefficient matrices of the perturbed DARE (1.4). We used MATLAB function `dare` to compute the unique symmetric positive semidefinite solution $\tilde{Y}$ of the perturbed Eq. (2.18). Let $\Delta Y = \tilde{Y} - Y$.

For the bound $\kappa_U(\psi^{\mathrm{Re}})$, we set $\delta_1 = \|A\|_F, \delta_2 = \|\mathrm{sym}(Q)\|_2, \delta_3 = \|\mathrm{sym}(G)\|_2$. For $\kappa_1^U(\psi^{\mathrm{Re}})$ in [39] we choose $\delta_1 = \|A\|_F, \delta_2 = \|Q\|_F, \delta_3 = \|G\|_F$. Table 2 shows that our condition numbers give reasonably good estimates for the changes in the solution.

For the SCE algorithms, we set the sample number $k = 5$ and tested them for various values of m with $\epsilon = 10^{-12}$. The results are shown as follows. For $m = 1$,

$$\Delta Y \oslash Y = 10^{-11}\begin{bmatrix} -0.1465 & 0.1818 & -0.1187 \\ 0.1818 & -0.1294 & 0.1432 \\ -0.1187 & 0.1432 & -0.0856 \end{bmatrix},$$

$$\epsilon\, K_{\mathrm{rel}}^{\mathrm{DARE},(5)} = 10^{-12}\begin{bmatrix} 25.5340 & -25.0489 & -15.4023 \\ -25.0489 & 10.8400 & 6.1395 \\ -15.4023 & 6.1395 & 3.1528 \end{bmatrix},$$

$$\epsilon\, C_{\mathrm{rel}}^{\mathrm{DARE},(5)} = 10^{-12}\begin{bmatrix} 3.0920 & -3.1064 & -2.1995 \\ -3.1064 & 1.4491 & 0.3301 \\ -2.1995 & 0.3301 & 0.9229 \end{bmatrix}.$$

For $m = 5$,

$$\Delta Y \oslash Y = 10^{-7}\begin{bmatrix} -0.6371 & 0.3186 & -0.1593 \\ 0.3186 & -0.1593 & 0.0797 \\ -0.1593 & 0.0797 & -0.0398 \end{bmatrix},$$

$$\epsilon\, K_{\mathrm{rel}}^{\mathrm{DARE},(5)} = 10^{-7}\begin{bmatrix} 5.1319 & -2.5662 & -1.2831 \\ -2.5662 & 1.2831 & 0.6415 \\ -1.2831 & 0.6415 & 0.3208 \end{bmatrix},$$

$$\epsilon\, C_{\rm rel}^{{\rm DARE},(5)} = 10^{-8}\begin{bmatrix} 6.1758 & -3.0882 & -1.5441 \\ -3.0882 & 1.5441 & 0.7721 \\ -1.5441 & 0.7721 & 0.3860 \end{bmatrix}.$$

For $m = 7$,

$$\Delta Y \oslash Y = 10^{-6}\begin{bmatrix} 0.4231 & -0.2115 & 0.1058 \\ -0.2115 & 0.1058 & -0.0529 \\ 0.1058 & -0.0529 & 0.0264 \end{bmatrix},$$

$$\epsilon\, K_{\rm rel}^{{\rm DARE},(5)} = 10^{-5}\begin{bmatrix} 4.0438 & -2.0219 & -1.0109 \\ -2.0219 & 1.0109 & 0.5055 \\ -1.0109 & 0.5055 & 0.2527 \end{bmatrix},$$

$$\epsilon\, C_{\rm rel}^{{\rm DARE},(5)} = 10^{-5}\begin{bmatrix} 1.0597 & -0.5299 & -0.2649 \\ -0.5299 & 0.2649 & 0.1325 \\ -0.2649 & 0.1325 & 0.0662 \end{bmatrix}.$$

As shown above, even for a small number of samples, the accuracy of the SCE method is within a factor between 10^{-1} and 10, which is considered acceptable [16, Chapter 15].

6. Concluding remarks

In this paper, by exploiting the symmetry structure and separating the real and imaginary parts, we present structured perturbation analyses of both the continuous-time and the discrete-time symmetric algebraic Riccati equations. From the analyses, we define the structured normwise, mixed and componentwise condition numbers and derive their upper bounds. Our bounds are improvements of the results in previous work [36,39]. Our preliminary experiments show that the three kinds of condition numbers provide accurate bounds for the change in the perturbed solution. Also, applying the small-sample condition estimation method, we propose statistical algorithms for practically estimating the structured condition numbers for continuous and discrete symmetric algebraic Riccati equations.

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