Exponential Decomposition and Hankel Matrix

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1 Introduction

We consider an important problem in mathematical modeling:

Exponential Decomposition. Given a finite sequence of signal values $\{s_1, s_2, \ldots, s_n\}$, determine a minimal positive integer r, complex coefficients $\{c_1, c_2, \ldots, c_r\}$, and distinct complex knots $\{z_1, z_2, \ldots, z_r\}$, so that the following exponential decomposition signal model is satisfied:

$$s_k = \sum_{i=1}^{r} c_i z_i^{k-1}, \quad \text{for } k = 1, \ldots, n. \tag{1.1}$$

We usually solve the above problem in three steps. First, determine the rank r of the signal sequence. Second, find the knots z_i. Third, calculate the coefficients $\{c_i\}$ by solving an $r \times r$ Vandermonde system:

$$W^{(r)}\mathbf{c} = \mathbf{s}^{(r)}, \tag{1.2}$$

where

$$W^{(r)} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_r \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{r-1} & z_2^{r-1} & \cdots & z_r^{r-1} \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} \text{ and } \mathbf{s}^{(r)} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix}.$$

A square Vandermonde system (1.2) can be solved efficiently and stably (see [1]).

There are two well known methods for solving our problem, namely, Prony [8] and Kung [6]. Vandevoorde [9] presented a single matrix pencil based framework that includes [8] and [6] as special cases. By adapting an implicit Lanczos process in our matrix pencil framework, we present an exponential decomposition algorithm which requires only $O(n^2)$ operations and $O(n)$ storage, compared to $O(n^3)$ operations and $O(n^2)$ storage required by previous methods.

The relation between this problem and a large subset of the class of Hankel matrices was fairly widely known, but a generalization was unavailable [3]. We propose a new variant of the Hankel-Vandermonde decomposition and prove that it exists for any given Hankel matrix. We also observe that this Hankel-Vandermonde decomposition of a Hankel matrix is rank-revealing. As with many other rank-revealing factorizations, the Hankel-Vandermonde decomposition can be used to find a low rank approximation to the given perturbed matrix. In addition to the computational savings, our algorithm has an advantage over many other popular methods in that it produces an approximation that exhibits the exact desired Hankel structure.

This paper is organized as follows. After introducing notations in §2, we discuss the techniques for determining rank and separating signals from noise in §3. In §4, we describe the matrix pencil framework and show how it unifies two previously known exponential decomposition methods. We devote §5 to our new fast algorithm based on the framework and §6 to our new Hankel-Vandermonde decomposition.

2 Notations

We adopt the Matlab notation; for example, the symbol $A_{i:j,k:l}$ denotes a submatrix formed by intersecting rows i through j and columns k through l of A, and $A_{:,k:l}$ is a submatrix consisting of columns k through l of A. We define $E^{(m)}$ as an $m \times m$ exchange matrix:

$$E^{(m)} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{for} \quad m = 1, 2, \ldots .$$

We also define a sequence of n-element vectors $\{\mathbf{w}^{(0)}(z), \mathbf{w}^{(1)}(z), \ldots, \mathbf{w}^{(n-1)}(z)\}$ as follows:

$$\mathbf{w}^{(0)}(z) = \begin{bmatrix} 1 \\ \vdots \\ z^{k-1} \\ z^k \\ z^{k+1} \\ \vdots \\ z^{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{w}^{(k)}(z) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ (k+1)z \\ \vdots \\ \frac{(n-1)!}{k!\,(n-1-k)!} z^{n-1-k} \end{bmatrix},$$

for $k = 1, 2, \ldots, n-1$. We see that $\mathbf{w}^{(k)}(z)$ is a *scaled* k-th derivative of $\mathbf{w}^{(0)}(z)$, normalized so that the first nonzero element equals unity. We introduce the concept of a *confluent* Vandermonde slice of order $n \times m$:

$$C^{(m)}(z) = \left[\mathbf{w}^{(0)}(z), \mathbf{w}^{(1)}(z), \ldots, \mathbf{w}^{(m-1)}(z)\right], \quad \text{for} \quad m = 1, 2, \ldots.$$

A confluent Vandermonde matrix [5] is a matrix of the form

$$\left[C^{(m_1)}(z_1), C^{(m_2)}(z_2), \ldots, C^{(m_j)}(z_j)\right].$$

The positive integer m_i is called the multiplicity of z_i in the confluent Vandermonde matrix. For our purposes, we define a new and more general structure that we call a *biconfluent* Vandermonde matrix:

$$\left[B^{(m_1)}(z_1), B^{(m_2)}(z_2), \ldots, B^{(m_j)}(z_j)\right],$$

where $B^{(m_i)}(z_i)$ equals either $C^{(m_i)}(z_i)$ or $E^{(n)}C^{(m_i)}(z_i)$.

3 Finding Rank and Denoising Data

Recall our sequence of signal values. We want to find r, the rank of the data. Since the values contain errors, finding the rank means separating the signals from the noise. We modify our notations and use $\{\hat{s}_j\}$ to represent the sequence of "polluted" signal values. Although we do not know the exact value of r, we would know a range in which r must lie. So we find two positive integers k and l such that

$$k + 1 \geq r, \quad l \geq r \quad \text{and} \quad k + l \leq n.$$

Since n is usually large and r quite small, choosing k and l to satisfy the above inequalities is not a problem. We construct a $(k+1) \times l$ Hankel matrix $\widehat{H}$ by

$$\widehat{H} = \begin{bmatrix} \hat{s}_1 & \hat{s}_2 & \cdots & \hat{s}_l \\ \hat{s}_2 & \hat{s}_3 & \cdots & \hat{s}_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{s}_k & \hat{s}_{k+1} & \cdots & \hat{s}_{k+l-1} \\ \hat{s}_{k+1} & \hat{s}_{k+2} & \cdots & \hat{s}_{k+l} \end{bmatrix}.$$

If there is no noise in the signal values, the rank of $\widehat{H}$ will be exactly r. Due to noise, the rank of $\widehat{H}$ will be greater than r. Compute a singular value decomposition (SVD) of $\widehat{H}$:

$$\widehat{H} = U\Sigma V^{\mathrm{H}},$$

where U and V are unitary matrices, and

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\min(k+1,l)}) \in R^{(k+1)\times l}.$$

We find r from looking for a "gap" in the singular values:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \gg \sigma_{r+1} \geq \sigma_{r+2} \geq \cdots \geq \sigma_{\min(k+1,l)} \geq 0. \tag{3.1}$$

There could be multiple gaps in the sequence of singular values, in which case we could either pick the first gap or use other information to choose r from the several possibilities. This is how we determine the numerical rank r of $\widehat{H}$, and thereby the rank of the signal data. A $(k+1)\times l$ matrix A of exact rank r is obtained from

$$A = U_{:,1:r}\Sigma_{1:r,1:r}(V_{:,1:r})^{\mathrm{H}}.$$

Unfortunately, the matrix A would have lost its Hankel structure. We want to find a Hankel matrix H that will be "close" to A.

Hankel Matrix Approximation. Given a $(k+1)\times l$ matrix A of rank-r, find a $(k+1)\times l$ Hankel matrix H of rank-r such that $\|A - H\|_F = \min$.

A simple way to get a Hankel structure from A is to average along the antidiagonals; that is,

$$s_1 = a_{11},\ s_2 = (a_{12} + a_{21})/2,\ s_3 = (a_{13} + a_{22} + a_{31})/3,\ \ldots$$

However, the rank of the resultant Hankel matrix would inevitably be greater than r. An SVD could be used to reduce the rank to r, but the decomposition would destroy the Hankel form. Cadzow [2] proposed that we iterate using a two-step cycle of an SVD followed by averaging along the antidiagonals. He proved that the iteration would converge under some mild assumptions.

Note that in the matrix approximation problem, the obvious selection of the Frobenius norm may not be appropriate, since this norm would give an unduly large weighting to the central elements of the Hankel structure. A weighted Frobenius matrix could be constructed, but such a norm is no longer unitarily invariant.

Nonetheless, let us assume that we have denoised the data and obtained a $(k+1)\times l$ Hankel matrix H of rank r:

$$H = \begin{bmatrix} s_1 & s_2 & \cdots & s_l \\ s_2 & s_3 & \cdots & s_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{k+l-1} \\ s_{k+1} & s_{k+2} & \cdots & s_{k+l} \end{bmatrix}. \tag{3.2}$$

4 Matrix Pencil Framework

For this section, we take

$$l = r$$

in (3.2) and we assume that $\text{rank}(H) = r$. We present a matrix pencil framework which includes the methods of Prony [8] and Kung [6] as special cases.

We begin by defining two k-by-r submatrices of H:

$$H_1 \equiv H_{1:k,:} \quad \text{and} \quad H_2 \equiv H_{2:k+1,:}. \tag{4.1}$$

So,

$$H_1 = \begin{bmatrix} s_1 & s_2 & \cdots & s_{r-1} & s_r \\ s_2 & s_3 & \cdots & s_r & s_{r+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_k & s_{k+1} & \cdots & s_{r+k-2} & s_{r+k-1} \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} s_2 & s_3 & \cdots & s_r & s_{r+1} \\ s_3 & s_4 & \cdots & s_{r+1} & s_{r+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{k+1} & s_{k+2} & \cdots & s_{r+k-1} & s_{r+k} \end{bmatrix}.$$

Note that column i of H_2 equals column $(i+1)$ of H_1, for $i = 1, 2, \ldots, r-1$. What about the last column of H_2? Let $\{z_i\}$ denote the roots of the polynomial

$$q_r(z) = z^r + \gamma_{r-1} z^{r-1} + \cdots + \gamma_1 z + \gamma_0. \tag{4.2}$$

For any $m = 1, 2, \ldots, k$, we can verify that

$$s_{r+m} + \sum_{i=0}^{r-1} \gamma_i s_{i+m} = c_1 z_1^{m-1} q_r(z_1) + c_2 z_2^{m-1} q_r(z_2) + \cdots + c_r z_r^{m-1} q_r(z_r) = 0.$$

It follows that

$$s_{r+m} = -\sum_{i=0}^{r-1} \gamma_i s_{i+m}.$$

Thus,

$$H_1 X = H_2, \tag{4.3}$$

where

$$X = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\gamma_0 \\ 1 & 0 & \cdots & 0 & -\gamma_1 \\ 0 & 1 & \cdots & 0 & -\gamma_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\gamma_{r-1} \end{bmatrix}.$$

Hence X is the companion matrix of the polynomial (4.2), and the knots z_i are the eigenvalues of the X.

Let $\mathbf{v}_i$ denote an eigenvector of X corresponding to z_i; that is

$$X\mathbf{v}_i = z_i\mathbf{v}_i.$$

It follows from (4.3) that

$$z_i H_1 \mathbf{v}_i = H_2 \mathbf{v}_i.$$

In general, the knots z_i are the eigenvalues of matrix Y satisfying the matrix equation:

$$FH_1GY = FH_2G, \tag{4.4}$$

for any nonsingular F and G. In other words, let F and G be any nonsingular matrices of orders k and r respectively, then the knots z_i are the eigenvalues of the pencil-equation

$$FH_2G\mathbf{v} - \lambda FH_1G\mathbf{v} = 0.$$

4.1 Prony's Method

For Prony's method, we consider the case when $n = 2r$ and set $k = r$. Let $F = G = I$ in the framework (4.4), then the knots are the eigenvalues of the matrix Y satisfying $H_1Y = H_2$. Now, (4.3) shows that the companion matrix X of the polynomial (4.2) satisfies this matrix equation. Thus the roots of the polynomial (4.2) are the knots. Moreover, the last column of (4.3) indicates that the coefficient vector $\mathbf{g} = (\gamma_0, \gamma_1, \dots, \gamma_{r-1})^{\mathrm{T}}$ of the polynomial (4.2) is the solution of the $r \times r$ Yule-Walker system:

$$H_1\mathbf{g} = -(s_{r+1}, s_{r+2}, \cdots, s_{2r})^{\mathrm{T}}. \tag{4.5}$$

This leads to the Prony's method:

Prony's Method

1. Solve a square equation (4.5), i.e., $n = 2r$, for the coefficients γ_i;
2. Determine the roots z_i of the polynomial $q_r(z)$ of (4.2);
3. Find the coefficients c_i by solving the square Vandermonde system: $W^{(r)}\mathbf{c} = \mathbf{s}^{(r)}$ of (1.2).

4.2 Kung's Method

Now, we derive Kung's Hankel SVD method [6] using our framework. Let

$$H = U\Sigma V^{\mathrm{H}}$$

be the SVD of H in (3.2). Denote

$$U_1 = U_{1:k,:} \quad \text{and} \quad U_2 = U_{2:k+1,:}.$$

Then we have

$$H_1 = U_1 \Sigma V^{\mathrm{H}} \quad \text{and} \quad H_2 = U_2 \Sigma V^{\mathrm{H}}.$$

Setting $F = I$ and $G = V\Sigma^{-1}$ in the framework (4.4), we know that the eigenvalues of Y in

$$U_1 Y = U_2 \tag{4.6}$$

are the knots z_i. The matrix Y can be solved by an approximation method, for example, least squares.

The following procedure is sometimes referred to as the *Hankel SVD* (HSVD) algorithm:

Kung's HSVD Method

1. Solve (4.6) for Y in a least squares sense;
2. Find the knots z_i as the eigenvalues of Y;
3. Find the coefficients c_i by solving (1.2).

Kung's method computes better results than Prony's method. See [9] for numerical examples. However, it is based on the SVD, which is expensive to compute.

5 Lanczos Process

In this section, we assume that $k = r$. First, we expand H into a $2r \times 2r$ Hankel matrix:

$$\widehat{H} = \begin{bmatrix} s_1 & \cdots & s_{2r} \\ \vdots & \iddots & \\ s_{2r} & & 0 \end{bmatrix}$$

and assume that it is strongly nonsingular, i.e., all its leading principal submatrices are nonsingular. Suppose that we have a triangular decomposition $\widehat{H} = R^{\mathrm{T}} D R$ where R is upper triangular and D diagonal. Since the Hankel matrix H in (3.2) equals $\widehat{H}_{1:k+1,1:r}$, we have

$$H = (R_{1:r,1:k+1})^{\mathrm{T}} D_{1:r,1:r} R_{1:r,1:r}. \tag{5.1}$$

Thus,

$$H_1 \equiv H_{1:k,:} = (R_{1:r,1:k})^{\mathrm{T}} D R_{1:r,1:r}$$

and

$$H_2 \equiv H_{2:k+1,:} = (R_{1:r,2:k+1})^{\mathrm{T}} D R_{1:r,1:r}.$$

Setting $F = I$ and $G = (R_{1:r,1:r})^{-1} (D_{1:r,1:r})^{-1}$ in our framework (4.4), we know that the eigenvalues of a matrix Y satisfying

$$(R_{1:r,1:k})^{\mathrm{T}} Y = (R_{1:r,2:k+1})^{\mathrm{T}} \tag{5.2}$$

are the desired knots. Now, using Lanczos process, we show that we can find a tridiagonal matrix satisfying (5.2).

Let $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $D(\mathbf{z}) = \mathrm{diag}(z_1, \ldots, z_r)$, then the transpose of the Vandermonde matrix

$$W^{\mathrm{T}} = [\mathbf{e}, D(\mathbf{z})\mathbf{e}, \ldots, D(\mathbf{z})^{k+1}\mathbf{e}]$$

is a Krylov matrix. Similar to the standard Lanczos method [5], we can find a tridiagonal matrix T such that

$$BT = D(\mathbf{z})B \quad \text{or} \quad T^{\mathrm{T}}B^{\mathrm{T}} = B^{\mathrm{T}}D(\mathbf{z}), \tag{5.3}$$

where B is orthogonal with respect to $D(\mathbf{c}) = \mathrm{diag}(c_1, \ldots, c_r)$, i.e., $B^{\mathrm{T}}D(\mathbf{c})B = D$, a diagonal matrix.

Lanczos method also gives a QR decomposition of the Krylov matrix

$$W^{\mathrm{T}} = BR_{1:r,1:k+1} \tag{5.4}$$

where R is an $r \times (k+1)$ upper trapezoidal matrix. It follows that

$$(W_{1:k,:})^{\mathrm{T}} = BR_{1:r,1:k} \quad \text{and} \quad (W_{2:k+1,:})^{\mathrm{T}} = BR_{1:r,2:k+1}.$$

It is obvious that $(W_{2:k+1,:})^{\mathrm{T}} = D(\mathbf{z})(W_{1:k,:})^{\mathrm{T}}$. Then we have

$$D(\mathbf{z})BR_{1:r,1:k} = BR_{1:r,2:k+1}.$$

From (5.3), substituting $D(\mathbf{z})B$ with BT in the above equation, we get

$$TR_{1:r,1:k} = R_{1:r,2:k+1} \quad \text{or} \quad (R_{1:r,1:k})^{\mathrm{T}}T^{\mathrm{T}} = (R_{1:r,2:k+1})^{\mathrm{T}}. \tag{5.5}$$

On the other hand, it can be easily verified from straightforward multiplication that

$$H = W_{1:k+1,:}D(\mathbf{c})(W_{1:r,:})^{\mathrm{T}}. \tag{5.6}$$

Then, from (5.4), we have the decomposition $H = (R_{1:r,1:k+1})^{\mathrm{T}}DR_{1:r,1:r}$. This says that the upper trapezoidal matrix $R_{1:r,1:k+1}$ computed by the Lanczos method gives the decomposition (5.1) of H and we can find a tridiagonal T^{T} satisfying (5.2). Thus the eigenvalues of T^{T} are the knots. In fact, the equations in (5.3) also show that the eigenvalues of T^{T} are the knots z_i. Moreover, (5.5) gives a three-term recursion of the rows of R, which leads to an efficient method for generating the tridiagonal T and the rows of R. To start the process, we note that the first row of R, or the first column of R^{T}, is the scaled first column of $\widehat{H}$ since $\widehat{H} = R^{\mathrm{T}}DR$ and R is upper triangular and D is diagonal. The process is outlined as follows. For details, see [9].

Lanczos Process

1. Initialize $R_{1,1:2r} = (s_1, \ldots, s_{2r})/s_1$;
2. Set $\alpha_1 = R_{1,2}$;
3. Use the recurrence

$$R_{i-2,i:2r-i+1} = R_{i-1,i+1:2r-i+2} - \alpha_{i-1} R_{i-1,i:2r-i+1} - \beta_{i-1}^2 R_{i,i:2r-i+1}$$

   to compute a new row $R_{i,i:2r-i+1}$;

4. Compute new $\alpha_i = R_{i,i+1} - R_{i-1,i}$; $\beta_{i-1}^2 = R_{i,i+2} - \alpha_i R_{i,i+1} - R_{i-1,i+1}$.

The tridiagonal T^{T} is of the form

$$T^{\mathrm{T}} = \begin{bmatrix} \alpha_1 & \beta_1^2 & & & 0 \\ 1 & \alpha_2 & \beta_2^2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha_{r-1} & \beta_{r-1}^2 \\ 0 & & & 1 & \alpha_r \end{bmatrix}.$$

Then we transform T into a complex symmetric tridiagonal matrix by scaling its rows and columns. Since the transformed T is complex symmetric, we apply the complex-orthogonal transformations in the QR method [4, 7] to compute its eigenvalues, i.e., the knots.

6 A New Hankel-Vandermonde Decomposition

The matrix product formulation (5.6) does not apply to an arbitrary Hankel matrix. Even when it exists, the exponential decomposition is not always unique. In this section, we will generalize this Hankel factorization so that the new decomposition will exist for any Hankel matrix and exhibit the rank of the given Hankel matrix. We start by establishing some useful properties of Hankel matrices which are closely related to Prony's method.

Observe that the signal model (1.1) describes the solution of a *linear difference equation* (LDE) and that Prony's method computes the coefficients γ_i in its characteristic polynomial

$$y_{k+r} + \gamma_{r-1} y_{k+r-1} + \cdots + \gamma_0 y_k = 0,$$

whose solution starts with the sequence $(s_1, s_2, \ldots, s_{2r-1})$. Indeed, the coefficients γ_i form the solution of the Hankel system:

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_r \\ s_2 & s_3 & \cdots & s_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{r-1} & s_r & \cdots & s_{2r-2} \\ s_r & s_{r+1} & \cdots & s_{2r-1} \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{r-1} \end{bmatrix} = - \begin{bmatrix} s_{r+1} \\ s_{r+2} \\ \vdots \\ s_{2r-1} \\ \eta \end{bmatrix}, \tag{6.1}$$

where η can be chosen arbitrarily. Thus γ_i are determined by the parameter η.

In preparation of a general result, we introduce a new concept of a *Prony η-continuation*.

Let H be an $r\times r$ nonsingular Hankel matrix. The $k\times m$ Prony η-continuation of H is the Hankel matrix

$$\mathrm{Pro}(H,\eta)=\begin{bmatrix} s_1(\eta) & s_2(\eta) & \cdots & s_m(\eta) \\ s_2(\eta) & s_3(\eta) & \cdots & s_{m+1}(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ s_k(\eta) & s_{k+1}(\eta) & \cdots & s_{k+m-1}(\eta) \end{bmatrix}$$

whose entries $s_i(\eta)$ are determined by the solution $(\gamma_0,\dots,\gamma_{r-1})^{\mathrm{T}}$ of (6.1).

Note that $s_i(\eta)=s_i$ for $i<2r$ and $s_{2r}(\eta)=\eta$ and $\mathrm{rank}(\mathrm{Pro}(H,\eta))=r$ if $k\ge r$ and $m\ge r$. Thus we embed a square and nonsingular H into a general Hankel matrix. Using the Prony continuation, we are able to prove that a Hankel matrix can be decomposed into a sum of two Prony continuations of two nonsingular Hankel matrices. We refer the proof to [9].

Let H be a $k\times k$ Hankel matrix of rank r, then there exist values η_1 and η_2, and nonsingular Hankel matrices H_1 and H_2, with respective sizes $r_1\times r_1$ and $r_2\times r_2$, such that $r=r_1+r_2$ and

$$H=\mathrm{Pro}(H_1,\eta_1)+E_k\mathrm{Pro}(E_kH_2E_k,\eta_2)E_k.$$

Then, using confluent Vandermonde matrix, we obtain the following theorem of Hankel-Vandermonde decomposition [9].

Let H be an arbitrary $k\times m$ real or complex Hankel matrix. There exist positive integers m_i $(i=1,\dots,j)$ and distinct complex numbers z_i $(i=1,\dots,j)$ such that

$$H=W_k\mathrm{diag}(C_1,C_2,\dots,C_j)W_m^{\mathrm{T}}, \tag{6.2}$$

where W_l denotes an $l\times r$ biconfluent Vandermonde matrix:

$$W_l=\left[B_{m_1}(z_1),\dots,B_{m_j}(z_j)\right],$$

with $r=m_1+m_2+\cdots+m_j$, and the C_i are nonsingular "left" (upper) triangular $m_i\times m_i$ Hankel matrices:

$$C_i=\begin{bmatrix} c_1^{(i)} & c_2^{(i)} & \cdots & c_{m_i}^{(i)} \\ c_2^{(i)} & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \\ c_{m_i}^{(i)} & 0 & & 0 \end{bmatrix}.$$

Obviously, (6.2) is a generalization of (5.6) in that H can be any Hankel matrix. Also, we note that

- $\text{rank}(H) = \min(k, m, r)$;
- The values z_i can be chosen to lie on the unit disk, i.e., $|z_i| \leq 1$.

Conclusion We have proposed a matrix-pencil framework, which unifies previously known exponential decomposition methods; developed a new fast Lanczos based exponential decomposition algorithm; presented a general Hankel-Vandermonde decomposition. A low rank Hankel approximation to a given Hankel matrix can be derived from our general Hankel-Vandermonde decomposition (see [9]).

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