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RESEARCH ARTICLE

Acute perturbation of Drazin inverse and oblique projectors

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Abstract For an $n\times n$ complex matrix A with $\mathrm{ind}(A)=r$, let A^D and $A^\pi = I-AA^D$ be respectively the Drazin inverse and the eigenprojection corresponding to the eigenvalue 0 of A. For an $n\times n$ complex singular matrix B with $\mathrm{ind}(B)=s$, it is said to be a *stable* perturbation of A, if $I-(B^\pi-A^\pi)^2$ is nonsingular, equivalently, if the matrix B satisfies the condition $\mathscr{R}(B^s)\cap\mathscr{N}(A^r)=\{\mathbf{0}\}$ and $\mathscr{N}(B^s)\cap\mathscr{R}(A^r)=\{\mathbf{0}\}$, introduced by Castro-González, Robles, and Vélez-Cerrada. In this paper, we call B an *acute* perturbation of A with respect to the Drazin inverse if the spectral radius $\rho(B^\pi-A^\pi)<1$. We present a perturbation analysis and give sufficient and necessary conditions for a perturbation of a square matrix being acute with respect to the matrix Drazin inverse. Also, we generalize our perturbation analysis to oblique projectors. In our analysis, the spectral radius, instead of the usual spectral norm, is used. Our results include the previous results on the Drazin inverse and the group inverse as special cases and are consistent with the previous work on the spectral projections and the Moore-Penrose inverse.

Keywords Drazin inverse, acute perturbation, stable perturbation, spectral radius, spectral norm, oblique projection
MSC 15A09, 65F20

1 Introduction and preliminaries

The Drazin inverse has been extensively investigated and widely applied. For instance, it is applied to the solution of singular linear systems [51], the theory of finite Markov chains [5,20,28,29], control theory [5], and numerical analysis [15,19,27,40,44,49]. Perturbation analysis is an important part of the study of

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the Drazin inverse. A number of papers on explicit formulas for the perturbation of the Drazin inverse, the spectral norm of the upper bounds derived from these formulas, and the error estimation have been published [6,8,10,16,21,23–26,31,37,41–43,46–48,50,52,55].

In this paper, we use the following notations. The set of $m \times n$ complex matrices is denoted by $\mathbb{C}^{m\times n}$; the idenity and null matrices are denoted by I and $\mathbf{O}$, respectively; the range and null spaces of a matrix A are denoted by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively.

Recall that the *Drazin inverse* of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^D \in \mathbb{C}^{n\times n}$ satisfying

$$A^D A = AA^D, \quad A^D AA^D = A^D, \quad A^{l+1}A^D = A^l, \quad \forall\, l \geqslant r, \tag{1}$$

where r is the smallest nonnegative integer such that $\operatorname{rank}(A^{r+1}) = \operatorname{rank}(A^r)$, called the *Drazin index* of A and denoted by $\operatorname{ind}(A)$. Clearly, $\operatorname{ind}(A) = 0$, if and only if A is nonsingular. In the case when $\operatorname{ind}(A) = 1$, the Drazin inverse is called the *group inverse*, denoted by $A^{\#}$.

Now, we briefly review stable matrices and oblique projectors. A matrix A is said to be *stable*, if its eigenvalues lie in the open left-half complex plane, that is, $\operatorname{Re}(\lambda(A)) < 0$, and it is said to be *semi-stable*, if all of its eigenvalues, except a few semi-simple [18] zero eigenvalues, lie on the open left-half complex plane. Semi-stability implies that A is singular and that the index of A is one, consequently, A has a group inverse and

$$\lim_{t\to\infty} \exp(At) = I - AA^{\#}.$$

The stable matrices are used in the stability analysis of ordinary differential equations. The results presented in this paper can be applied to the perturbation analysis of $I - AA^{\#}$.

A projection P is an *idempotent* operator, that is,

$$P^2 = P.$$

The operator P is a projection along its null space $\mathscr{N}(P)$ onto its range $X = \mathscr{R}(P)$. If these two subspaces are orthogonal, the projection is said to be *orthogonal*, which is the case if and only if P is Hermitian. Otherwise, it is called an *oblique projection*. A nice and useful identity between the norm of a projection P and that of its complementary projection $I-P$ in an inner-product space [36] is

$$\|P\| = \|I - P\| \geqslant 1, \quad \rho(P) = \rho(I - P) = 1,$$

where P is neither null nor the identity and the operator norm is the standard norm induced by the vector norm defined by the inner product. From the first equation in (1), $A^D A$ is a projection. Its complementary projection $I - A^D A$, denoted by A^π, is the spectral projection of A corresponding to the eigenvalue 0 with

$$\mathscr{R}(A^\pi) = \mathscr{N}(A^r), \quad \mathscr{N}(A^\pi) = \mathscr{R}(A^r).$$

Since $\mathscr{R}(A^\pi)$ and $\mathscr{N}(A^\pi)$ need not be orthogonal, A^π can be an oblique projection.

This paper focuses on the stable and acute perturbations with respect to the Drazin inverse.

The concept of stable perturbation with respect to the Drazin inverse was introduced by Castro-González, Robles and Vélez-Cerrada [9, Theorem 2.1]. A perturbation B of A is said to be *stable* if $I-(B^\pi - A^\pi)^2$ is nonsingular. A perturbation B of A is said to be *acute*, if $\|B-A\|$ is sufficiently small and the spectral radius $\rho(B^\pi - A^\pi) < 1$. Let A be a square matrix with $\mathrm{ind}(A) = r > 0$, and let B be a perturbation of A with $\mathrm{ind}(B) = s$. A formula for B^π is given in [9, Theorem 2.3] under the assumption

$$(\mathscr{C}_s) \qquad \mathscr{R}(B^s)\cap\mathscr{N}(A^r) = \{\mathbf{0}\}, \quad \mathscr{N}(B^s)\cap\mathscr{R}(A^r) = \{\mathbf{0}\}, \tag{2}$$

in which case, A^r and B^s are called disjoint matrices [12].

Wei [53] conjectured that a perturbation B is an *acute* perturbation with respect to the Drazin inverse provided that the spectral norm $\|B^\pi - A^\pi\| < 1$. If B is a stable perturbation of A, does $\|B^\pi - A^\pi\| < 1$ always hold? Here is a counterexample.

Example 1 [32, Example 2.1] Let

$$A = \begin{pmatrix} A_{11} & \mathbf{O} \\ \mathbf{O} & A_{22} \end{pmatrix}, \quad B = A + E = \begin{pmatrix} A_{11} & A_{12} \\ \mathbf{O} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_{12} = \frac{2}{5}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

It can be verified that

$$\mathrm{ind}(A) = \mathrm{ind}(B) = 1$$

and

$$(A_{11})^\# = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad (A_{22})^\# = A_{22}, \quad B^\# = \begin{pmatrix} (A_{11})^\# & L \\ \mathbf{O} & (A_{22})^\# \end{pmatrix},$$

where

$$L = \frac{1}{5}\begin{pmatrix} 0 & 0 \\ 8+2\sqrt{2} & 2\sqrt{2} \end{pmatrix}.$$

Then

$$\|A^\#\|\,\|B-A\| = \sqrt{2}\,\frac{2\sqrt{2}}{5} = \frac{4}{5} < 1$$

and

$$B^\pi - A^\pi = \frac{1}{5}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$I - (B^\pi - A^\pi)^2 = I,$$

so B is a stable perturbation of A. However,

$$\|B^\pi - A^\pi\| = \frac{4\sqrt{2}}{5} > 1.$$

Notice that in this example, $\rho(B^\pi - A^\pi) = 0$. This suggests that the spectral radius is a better choice than the spectral norm.

In this paper, we present necessary and sufficient condtions for a perturbation being acute with respect to the Drazin inverse. Our results are generalization of those with respect to the group inverse [45]. Moreover, following [9,10], we derive new formulae for the spectral radii $\rho((I - B^\pi)A^\pi)$ and $\rho((I - A^\pi)B^\pi)$ under the condition $(\mathscr{C}_s)$ in (2) and upper bounds for $\rho(B^\pi - A^\pi)$.

The paper is organized as follows. In Section 2, we present equivalent conditions for the stable perturbation of the Drazin inverse, which is the same as condition $(\mathscr{C}_s)$ for matrices [9, Theorem 2.1]. In Section 3, we derive sufficient and necessary conditions for the acute perturbation of the Drazin inverse in Theorem 1. In Section 4, we investigate the perturbation of oblique projectors with the spectral radius. We present some applications in Section 5 and conclude with some remarks in the last section.

2 Equivalent conditions on stable perturbation

In the following discussion, we assume that $A \in \mathbb{C}^{n\times n}$ is singular with

$$\operatorname{ind}(A) = r \geqslant 1, \quad \operatorname{rank}(A^r) = d.$$

Thus, A is similar to a matrix in the Jordan form, which can be rearranged so that the Jordan block corresponding to the zero eigenvalue is at the lower-right corner. That is, we can write A in the *core-nilpotent block form* [3,5,38]

$$A = V\begin{pmatrix} A_1 & \mathbf{O} \\ \mathbf{O} & A_2 \end{pmatrix}V^{-1}, \tag{3}$$

for some nonsingular matrix V such that $A_1 \in \mathbb{C}^{d\times d}$ is nonsingular, $A_2^r = \mathbf{O}$ but $A_2^{r-1} \neq \mathbf{O}$. Corresponding to the matrix V in (3), we denote

$$\theta(B) = V^{-1}BV, \quad \forall\, B \in \mathbb{C}^{n\times n}. \tag{4}$$

By [5, Theorem 7.2.1], corresponding to the decomposition (3), $\theta(A^D)$ and $\theta(A^\pi)$ are given by

$$\theta(A^D) = \begin{pmatrix} A_1^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \theta(A^\pi) = \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & I \end{pmatrix}. \tag{5}$$

Denoting $C_B = B(BB^D)$ with $(C_B)^\# = B^D$, for any $B \in \mathbb{C}^{n\times n}$, the core-nilpotent decomposition (5) can be written as

$$A = C_A + N_A, \quad C_A N_A = N_A C_A = \mathbf{O},\ \mathrm{ind}(C_A) \leqslant 1,\ N_A \text{ nilpotent}, \tag{6}$$

where $C_A = A(AA^D)$ with $(C_A)^\# = A^D$, and $N_A = AA^\pi$ satisfying $(N_A)^r = \mathbf{O}$.

For a perturbation B of A, the following lemma gives equivalent conditions for B being a stable perturbation.

Lemma 1 [9, Theorem 2.1] *The following statements on $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ are equivalent*:

(a) *B is a stable perturbation of A ($I - (B^\pi - A^\pi)^2$ is nonsingular)*;

(b) *B satisfies condition ($\mathscr{C}_s$) in* (2);

(c) $\mathrm{rank}(B^s) = \mathrm{rank}(A^r) = \mathrm{rank}(A^r B^s) = \mathrm{rank}(B^s A^r) = \mathrm{rank}(A^r B^s A^r)$.

The following lemma for matrices is stated in [5] and extended to bounded linear operators on Banach spaces in [10].

Lemma 2 *Let $B_1 \in \mathbb{C}^{m\times m}$ be nonsingular, let $T \in \mathbb{C}^{m\times n}$ and $S \in \mathbb{C}^{n\times m}$ be arbitrary, and define*

$$B = \begin{pmatrix} B_1 & B_1 T \\ SB_1 & SB_1 T \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}.$$

Then B is group invertible if and only if $I + TS$ is nonsingular. In this case, $B^\#$ and B^π can be given, respectively, by

$$B^\# = \begin{pmatrix} [(I+TS)B_1(I+TS)]^{-1} & [(I+TS)B_1(I+TS)]^{-1}T \\ S[(I+TS)B_1(I+TS)]^{-1} & S[(I+TS)B_1(I+TS)]^{-1}T \end{pmatrix}, \tag{7}$$

$$B^\pi = \begin{pmatrix} I-(I+TS)^{-1} & -(I+TS)^{-1}T \\ -S(I+TS)^{-1} & I-S(I+TS)^{-1}T \end{pmatrix}. \tag{8}$$

The key point of the above two formulae is that $B^\#$ and B^π can be rewritten in a more practical way as follows.

Lemma 3 [54, Lemma 2.5] *Let B be as in Lemma 2 such that B_1 and $I+TS$ both are nonsingular. Then*

$$B^\# = \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix} \begin{pmatrix} \left[B_1(I+TS)\right]^{-1} & \left[B_1(I+TS)\right]^{-2} B_1 T \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix}, \tag{9}$$

$$B^\pi = \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix} \begin{pmatrix} \mathbf{O} & -(I+TS)^{-1}T \\ \mathbf{O} & I \end{pmatrix} \begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix}. \tag{10}$$

Now, we have more equivalent conditions for B being a stable perturbation of A.

Lemma 4 [54, Lemma 2.6] *The following statements on $B \in \mathbb{C}^{n\times n}$ are equivalent*:

(a) *B is a stable perturbation of A;*

(b) *C_B is a stable perturbation of A;*

(c) *$I + A^D(C_B - A)$ is nonsingular and $C_B[I + A^D(C_B - A)]^{-1}A^\pi = \mathbf{O}$;*

(d) *there exist $B_1 \in \mathbb{C}^{d\times d}$, $T \in \mathbb{C}^{d\times(n-d)}$, and $S \in \mathbb{C}^{(n-d)\times d}$ with B_1 and $I + TS$ are nonsingular, such that*

$$\theta(C_B) = \begin{pmatrix} B_1 & B_1T \\ SB_1 & SB_1T \end{pmatrix}; \tag{11}$$

(e) *for any $k \in \mathbb{N}$, $(\theta(C_B))^k$ can be written as*

$$(\theta(C_B))^k = \begin{pmatrix} [B_1(I + TS)]^{k-1}B_1 & [B_1(I + TS)]^{k-1}B_1T \\ S[B_1(I + TS)]^{k-1}B_1 & S[B_1(I + TS)]^{k-1}B_1T \end{pmatrix}$$

for some $B_1 \in \mathbb{C}^{d\times d}$, $T \in \mathbb{C}^{d\times(n-d)}$, and $S \in \mathbb{C}^{(n-d)\times d}$, such that B_1 and $I+TS$ are nonsingular.

3 Acute perturbation of Drazin inverse

In this section, we investigate the *acute* perturbation with respect to the Drazin inverse. Using the notations in Section 2 and denoting

$$E_{k,l} = B^k - A^l, \quad k, l \in \mathbb{N},$$

in the case when $I + (A^D)^l E_{k,l}$ is nonsingular, we define

$$Y_{k,l} = [I + (A^D)^l\, E_{k,l}]^{-1}(A^D)^l\, E_{k,l}A^\pi, \tag{12}$$

$$Z_{k,l} = A^\pi\, E_{k,l}(A^D)^l[I +\, E_{k,l}(A^D)^l]^{-1}. \tag{13}$$

From

$$\theta(B^k) = \theta((C_B)^k) = (\theta(C_B))^k,$$

which can be verified by using the definition (4), we have

$$\theta(Y_{k,l}) = \theta([I + (A^D)^l E_{k,l}]^{-1}(A^D)^l)[(\theta(C_B))^k - (\theta(A))^l]\theta(A^\pi).$$

Let

$$\begin{aligned} G &= \theta([I + (A^D)^l E_{k,l}]^{-1}(A^D)^l) \\ &= [I + (\theta(A^D))^l[(\theta(C_B))^k - (\theta(A))^l]]^{-1}(\theta(A^D))^l. \end{aligned}$$

Then, when B is a stable perturbation of A, applying Lemma 4, we get

$$G = \begin{pmatrix} B_1^{-1}[B_1(I + TS)]^{1-k} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix},$$

implying that

$$\theta(Y_{k,l}) = G[(\theta(C_B))^k - (\theta(A))^l]\theta(A^\pi) = \begin{pmatrix} \mathbf{O} & T \\ \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

Similarly, we have

$$\theta(Z_{k,l}) = \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ S & \mathbf{O} \end{pmatrix}.$$

Consequently,

$$\theta(I + Y_{k,l}Z_{k,l}) = \begin{pmatrix} I + TS & \mathbf{O} \\ \mathbf{O} & I \end{pmatrix} \tag{14}$$

is nonsingular.

To summarize, we have the following lemma.

Lemma 5 [54, Lemma 3.1] *Let $B \in \mathbb{C}^{n\times n}$ be a stable perturbation of A with* $\mathrm{ind}(B) = s$. *Then $Y_{k,l}$ and $Z_{k,l}$ defined in* (12) *and* (13), *respectively, are independent of the indices k, l with $k \geqslant s$. Moreover, $I + Y_{k,l}Z_{k,l}$ is nonsingular for $k \geqslant s$.*

Using the notations above, we have explicit expressions of B^D and B^π in the following lemma.

Lemma 6 [54, Theorem 3.2] *Let $B \in \mathbb{C}^{n\times n}$ be a stable perturbation of A with* $\mathrm{ind}(B) = s$. *Then, for any $k, l \in \mathbb{N}$ with $k \geqslant s$,*

$$B^D = W_{k,l}^{-1}[I + (A^D)^{l+1}\, E_{k+1,l+1}]^{-1}A^D[I + (A^D)^l\, E_{k,l}]W_{k,l}, \tag{15}$$

$$B^\pi = W_{k,l}^{-1}[I + (A^D)^l\, E_{k,l}]^{-1}A^\pi[I + (A^D)^l\, E_{k,l}]W_{k,l}, \tag{16}$$

where $Y_{k,l}$ and $Z_{k,l}$ are defined in (12) *and* (13), *respectively, and*

$$W_{k,l} = (I + Y_{k,l}Z_{k,l})(I - Z_{k,l}), \quad W_{k,l}^{-1} = (I + Z_{k,l})(I + Y_{k,l}Z_{k,l})^{-1}. \tag{17}$$

Now, we are ready for an upper bound for the spectral radius $\rho(B^\pi - A^\pi)$.

Theorem 1 *Let $B = A + E \in \mathbb{C}^{n\times n}$ with*

$$\mathrm{ind}(A) = r, \quad \mathrm{ind}(B) = s, \quad \mathrm{rank}(A^r) = \mathrm{rank}(B^s).$$

If the perturbation $E_{k,l} = B^k - A^l$ satisfies

$$\max\{\|(A^D)^l E_{k,l}\|, \|E_{k,l}(A^D)^l\|\} < \frac{1}{1 + \sqrt{2\|A^\pi\|}}, \tag{18}$$

then we have

(a) $\rho(Y_{k,l}Z_{k,l}) < 1/2$;

(b) $\rho(BB^D(I - AA^D)) = \rho(AA^D(I - BB^D)) \leqslant \rho(Y_{k,l}Z_{k,l})/(1 - \rho(Y_{k,l}Z_{k,l}))$;

(c) $[\rho(B^\pi - A^\pi)]^2 = \rho(BB^D(I - AA^D)) = \rho(AA^D(I - BB^D)) < 1$;

(d) $I-BB^D(I-AA^D)$, $I-AA^D(I-BB^D)$, $I-(B^\pi-A^\pi)$ *and* $I-(B^\pi-A^\pi)^2$ *are invertible.*

Proof We first show that

$$\rho(BB^D(I-AA^D))=\rho((I+TS)^{-1}TS). \tag{19}$$

Indeed,

$$\theta(BB^D(I-AA^D))=\theta(C_B(C_B)^{\#})\theta(I-AA^D).$$

Let

$$\theta(C_B)=\begin{pmatrix} B_1 & B_1T \\ SB_1 & SB_1T \end{pmatrix}.$$

Then, from (9), we get

$$\begin{aligned}
\theta(C_B(C_B)^{\#}) &= \theta(C_B)(\theta(C_B))^{\#} \\
&= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} B_1(I+TS) & B_1T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} [B_1(I+TS)]^{-1} & [B_1(I+TS)]^{-2}B_1T \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix} \\
&= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} I & (I+TS)^{-1}T \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\theta(BB^D(I-AA^D)) \\
&\quad= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} I & (I+TS)^{-1}T \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix}\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & I \end{pmatrix} \\
&\quad= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} (I+TS)^{-1} & (I+TS)^{-1}T \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & I \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix} \\
&\quad= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} (I+TS)^{-1}TS & (I+TS)^{-1}T \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix},
\end{aligned}$$

implying (19). Similarly,

$$\begin{aligned}
\theta(AA^D(I-BB^D)) &= \theta(AA^D)\theta(I-C_B(C_B)^{\#}) \\
&= \begin{pmatrix} I & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+TS)^{-1}T \\ \mathbf{O} & I \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix} \\
&= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+TS)^{-1}T \\ \mathbf{O} & \mathbf{0} \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix} \\
&= \begin{pmatrix} I & \mathbf{O} \\ S & I \end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+TS)^{-1}T \\ \mathbf{O} & S(I+TS)^{-1}T \end{pmatrix}\begin{pmatrix} I & \mathbf{O} \\ -S & I \end{pmatrix},
\end{aligned}$$

which implies that

$$\rho(AA^D(I-BB^D))=\rho(S(I+TS)^{-1}T).$$

Recalling (14), we have

$$\begin{aligned}\rho(BB^D(I-AA^D)) &= \rho((I+TS)^{-1}TS)\\ &= \rho(S(I+TS)^{-1}T)\\ &= \rho(AA^D(I-BB^D))\\ &= \rho((I+Y_{k,l}Z_{k,l})^{-1}Y_{k,l}Z_{k,l})\\ &\leqslant \rho((I+Y_{k,l}Z_{k,l})^{-1})\rho(Y_{k,l}Z_{k,l})\\ &\leqslant \frac{\rho(Y_{k,l}Z_{k,l})}{1-\rho(Y_{k,l}Z_{k,l})}.\end{aligned}$$

For (c), under condition (18), we have

$$\begin{aligned}\rho(Y_{k,l}Z_{k,l}) &\leqslant \|Y_{k,l}Z_{k,l}\|\\ &\leqslant \|[I+(A^D)^lE_{k,l}]^{-1}(A^D)^lE_{k,l}A^\pi A^\pi E_{k,l}(A^D)^l[I+E_{k,l}(A^D)^l]^{-1}\|\\ &\leqslant \frac{\|(A^D)^lE_{k,l}\|\,\|E_{k,l}(A^D)^l\|\,\|A^\pi\|}{(1-\|(A^D)^lE_{k,l}\|)(1-\|E_{k,l}(A^D)^l\|)}\\ &< \frac{1}{2},\end{aligned}$$

noticing that $(A^\pi)^2 = A^\pi$. Consequently,

$$\rho(BB^D(I-AA^D)) = \rho(AA^D(I-BB^D)) \leqslant \frac{\rho(Y_{k,l}Z_{k,l})}{1-\rho(Y_{k,l}Z_{k,l})} < 1.$$

Next, we consider $\rho(B^\pi - A^\pi)$. Applying (10), we have

$$\begin{aligned}\theta(B^\pi - A^\pi) &= \theta(B^\pi) - \theta(A^\pi)\\ &= \begin{pmatrix} I & \mathbf{O}\\ S & I\end{pmatrix}\begin{pmatrix}\mathbf{O} & -(I+TS)^{-1}T\\ \mathbf{O} & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -S & I\end{pmatrix} - \begin{pmatrix}\mathbf{O} & \mathbf{O}\\ \mathbf{O} & I\end{pmatrix}\\ &= \begin{pmatrix} I & \mathbf{O}\\ S & I\end{pmatrix}\left[\begin{pmatrix}\mathbf{O} & -(I+TS)^{-1}T\\ \mathbf{O} & I\end{pmatrix} - \begin{pmatrix}\mathbf{O} & \mathbf{O}\\ S & I\end{pmatrix}\right]\begin{pmatrix} I & \mathbf{O}\\ -S & I\end{pmatrix}\\ &= \begin{pmatrix} I & \mathbf{O}\\ S & I\end{pmatrix}\begin{pmatrix}\mathbf{O} & -(I+TS)^{-1}T\\ -S & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -S & I\end{pmatrix},\end{aligned}$$

and then

$$[\theta(B^\pi - A^\pi)]^2 = \begin{pmatrix} I & \mathbf{O}\\ S & I\end{pmatrix}\begin{pmatrix}(I+TS)^{-1}TS & \mathbf{O}\\ \mathbf{O} & S(I+TS)^{-1}T\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -S & I\end{pmatrix}.$$

Thus, we have

$$[\rho(B^\pi - A^\pi)]^2 = \rho((B^\pi - A^\pi)^2) = \rho(BB^D(I-AA^D)) = \rho(AA^D(I-BB^D)) < 1.$$

Finally, (d) follows immediately from (c). □

From the definition of acute perturbation, we have the following corollary.

Corollary 1 *If*

$$\operatorname{rank}(A^r) = \operatorname{rank}(B^s), \quad \max\{\|(A^D)^l E_{k,l}\|, \|E_{k,l}(A^D)^l\|\} < \frac{1}{1+\sqrt{\|2A^\pi\|}},$$

then B is an acute perturbation of A.

Finally, we have the following necessary and sufficient conditions for the *acute* perturbation with respect to the Drazin inverse, which are the same as those for the *stable* perturbation with respect to the Drazin inverse [9, Theorem 2.1].

Theorem 2 *A perturbation B of A is acute with respect to the Drazin inverse, if and only if* $(\mathscr{C}_s)$ *in* (2) *holds.*

Proof If $\mathscr{R}(B^s) \cap \mathscr{N}(A^r) \neq \{\mathbf{0}\}$, then there exists a nonzero vector $x \in \mathscr{R}(B^s) \cap \mathscr{N}(A^r)$ such that

$$BB^D x = x, \quad AA^D x = \mathbf{0}.$$

Thus,

$$(BB^D - AA^D)x = x,$$

i.e.,

$$\rho(BB^D - AA^D) = \rho(B^\pi - A^\pi) \geqslant 1,$$

B is not an acute perturbation of A.

It follows from Lemma 1 that $(\mathscr{C}_s)$ in (2) is equivalent to

$$\operatorname{rank}(B^s) = \operatorname{rank}(A^r) = \operatorname{rank}(A^r B^s A^r). \tag{20}$$

With the help of Theorem 1, we can prove that $\rho(BB^D - AA^D) < 1$. □

Corollary 2 *A perturbation B of A is acute with respect to the Drazin inverse, if and only if* (20) *holds.*

Remark 1 If B is not an acute perturbation of A and $\operatorname{rank}(B^s) \geqslant \operatorname{rank}(A^k)$, then $\|B^\pi - A^\pi\| \geqslant 1$ (see [52]).

4 Oblique projector

Applying the theory of condition developed by Rice [30], Sun [35] defined a condition number of the spectral projection, which plays an important role in the perturbation theory of eigenvalue problems. Although there is a substantial number of literature on the numerical properties of orthogonal projectors, the literature on oblique projectors is largely confined to purely mathematical investigations [4] and applications and specific algorithms [13]. Stewart [34] analyzed the numerical properties of oblique projectors, including their perturbation theory, their various representations, their behavior in the presence of rounding error, the computation of complementary projections, and updating algorithms.

Inspired by the use of spectral radius, in this section, we investigate the perturbation of oblique projectors by using the spectral radius of the difference of two oblique projectors P and Q, instead of the spectral norm $\|P-Q\|$ of their difference. We generalized the previous discussion of A^π and B^π to general oblique projectors.

First, we consider the case when $\operatorname{rank}(P) > \operatorname{rank}(Q)$. Note that if the rank is not preserved, an oblique projector is discontinuous.

Theorem 3 *Let P and Q be oblique projectors. If* $\operatorname{rank}(P) > \operatorname{rank}(Q)$, *then*

$$1 \leqslant \rho(P-Q) \leqslant \|P-Q\|.$$

Proof If $\operatorname{rank}(P) > \operatorname{rank}(Q)$, then there exists a nonzero vector $x \in \mathscr{R}(P) \cap \mathscr{N}(Q)$, that is,

$$Px = x, \quad Qx = \mathbf{0}.$$

Thus, $(P-Q)x = x$, implying that $\rho(P-Q) \geqslant 1$. □

In the following discussion, we assume

$$\operatorname{rank}(P) = \operatorname{rank}(Q) = \operatorname{rank}(PQP), \quad P \approx Q.$$

Theorem 4 *Let P and Q be oblique projectors. If* $\operatorname{rank}(P) = \operatorname{rank}(Q) = \operatorname{rank}(PQP)$ *and P is sufficiently close to Q, then the spectral radii of $Q(I-P)$ and $P(I-Q)$ satisfy*

(a) $\rho(Q(I-P)) = \rho(P(I-Q))$;

(b) $[\rho(P-Q)]^2 = \rho(Q(I-P)) = \rho(P(I-Q)) < 1$.

Proof From [35], an oblique projector P can be decomposed into

$$P = S\begin{pmatrix} I & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} S^{-1} \tag{21}$$

for some nonsingular matrix S.

Let

$$Q = S\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} S^{-1} = S\begin{pmatrix} I \\ E \end{pmatrix} Q_{11} (\, I \;\; F\,) S^{-1},$$

where, since $P \approx Q$, Q_{11} is nonsingular,

$$\operatorname{rank}(Q_{11}) = \operatorname{rank}(PQP) = \operatorname{rank}(Q) = \operatorname{rank}(P),$$

$$E = Q_{21}Q_{11}^{-1}, \quad F = Q_{11}^{-1}Q_{12}, \quad \rho(FE) = \rho(Q_{12}Q_{21}) < \frac{1}{2},$$

which implies that $I + FE$ is invertible.

Since $Q^2 = Q$, we have

$$S^{-1}QS = (S^{-1}QS)^2 = \begin{pmatrix} I \\ E \end{pmatrix} Q_{11}(I+FE)Q_{11}(\, I \;\; F\,).$$

As $(I\ E^{\mathrm{T}})^{\mathrm{T}}$ is of full column rank and $(I\ F)$ is of full row rank, we have

$$Q_{11}(I+FE)Q_{11}=Q_{11},$$

implying that

$$Q_{11}=(I+FE)^{-1},\quad Q_{21}=E(I+FE)^{-1},\quad Q_{12}=(I+FE)^{-1}F.$$

Thus, we can write $\theta(I-Q)$ and $\theta(Q)$ as

$$\begin{aligned}
I-\theta(Q)&=I-S^{-1}QS\\
&=\begin{pmatrix} I-(I+FE)^{-1} & -(I+FE)^{-1}F\\ -E(I+FE)^{-1} & I-E(I+FE)^{-1}F\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+FE)^{-1}F\\ \mathbf{O} & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix},\\
\theta(Q)&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & (I+FE)^{-1}F\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\theta(P(I-Q))&=\theta(P)\theta(I-Q)\\
&=\begin{pmatrix} I & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+FE)^{-1}F\\ \mathbf{O} & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+FE)^{-1}F\\ \mathbf{O} & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+FE)^{-1}F\\ \mathbf{O} & E(I+FE)^{-1}F\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix},\\
\theta[Q(I-P)]&=\theta(Q)\theta(I-P)\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & (I+FE)^{-1}F\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\begin{pmatrix} \mathbf{O} & \mathbf{O}\\ \mathbf{O} & I\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & (I+FE)^{-1}F\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} \mathbf{O} & \mathbf{O}\\ \mathbf{O} & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} (I+FE)^{-1}FE & (I+FE)^{-1}F\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\theta(P-Q)&=\theta(P)-\theta(Q)\\
&=\begin{pmatrix} I & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}-\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} I & (I+FE)^{-1}F\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}\\
&=\begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix} \mathbf{O} & -(I+FE)^{-1}F\\ -E & \mathbf{O}\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix},
\end{aligned}$$

and then

$$\begin{aligned}\theta((P-Q)^2) &= [\theta(P-Q)]^2\\ &= \begin{pmatrix} I & \mathbf{O}\\ E & I\end{pmatrix}\begin{pmatrix}(I+FE)^{-1}FE & \mathbf{O}\\ \mathbf{O} & E(I+FE)^{-1}F\end{pmatrix}\begin{pmatrix} I & \mathbf{O}\\ -E & I\end{pmatrix}.\end{aligned}$$

Putting things together, we have

$$\begin{aligned}\rho(Q(I-P)) &= \rho((I+FE)^{-1}FE)\\ &= \rho(E(I+FE)^{-1}F)\\ &= \rho(P(I-Q))\\ &= [\rho(P-Q)]^2\\ &\leqslant \frac{\rho(FE)}{1-\rho(FE)}\\ &< 1,\end{aligned}$$

since $\rho(FE) < 1/2$ when P is sufficiently close to Q. $\square$

Since

$$\operatorname{ind}(P) = \operatorname{ind}(Q) = 1, \quad P^{\#} = P, \quad Q^{\#} = Q,$$

from [45], we have the following necessary and sufficient condition for $\rho(P-Q) < 1$.

Corollary 3 *Let P and Q be two oblique projectors, and assume that P approximates Q. Then $\rho(P-Q) < 1$ if and only if*

$$\mathscr{R}(P)\cap\mathscr{N}(Q) = \{\mathbf{0}\}, \quad \mathscr{R}(Q)\cap\mathscr{N}(P) = \{\mathbf{0}\}. \tag{22}$$

Example 2 Let us consider the following modified example of the two oblique projectors P and Q from [22]:

$$P = \begin{pmatrix} 0 & 0.01 & 0\\ 0 & 1.0 & 0\\ 0 & 0 & 1.0\end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0\\ 1.44 & 1.0 & 0\\ 0 & 0 & 1.0\end{pmatrix}.$$

It can be verified that

$$\operatorname{rank}(P) = \operatorname{rank}(Q) = \operatorname{rank}(PQP) = 2, \quad P-Q = \begin{pmatrix} 0 & 0.01 & 0\\ -1.44 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$Q(I-P) = \begin{pmatrix} 0 & 0 & 0\\ 1.44 & -0.0144 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad P(I-Q) = \begin{pmatrix} -0.0144 & 0 & 0\\ -1.44 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\rho(P-Q) = 0.12 < 1, \quad \rho(Q(I-P)) = \rho(P(I-Q)) = 0.0144 = [\rho(P-Q)]^2.$$

In contrast,

$$\|P - Q\| = 1.44 > 1.$$

This shows that the spectral radius is a better measurement than the spectral norm.

5 Applications

Wei [42] investigated the one-sided perturbation of the Drazin inverse, which actually is a special kind of acute perturbation.

Theorem 5 [42, Theorem 1] *Let $B = A + E$ with $\operatorname{ind}(A) = r$, such that*

$$E = AA^D E, \quad \|A^D E\| < 1.$$

Then

$$B^D = (I + A^D E)^{-1} A^D + \sum_{i=0}^{r-1} [(I + A^D E)^{-1} A^D]^{i+2} E (I - AA^D) A^i, \tag{23}$$

$$BB^D = AA^D + AA^D A (I + A^D E) \sum_{i=0}^{r-1} [(I + A^D E)^{-1} A^D]^{i+2} E (I - AA^D) A^i. \tag{24}$$

Moreover, B is an acute perturbation of A.

Proof Suppose

$$A = V \begin{pmatrix} A_1 & \mathbf{O} \\ \mathbf{O} & A_2 \end{pmatrix} V^{-1}, \quad A^D = V \begin{pmatrix} A_1^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} V^{-1},$$

for some nonsingular V and A_1. Since $E = AA^D E$, we have the exact form of the perturbation matrix

$$E = V \begin{pmatrix} E_{11} & E_{12} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} V^{-1}$$

for some E_{11} and E_{12}. Thus,

$$B = A + E = V \begin{pmatrix} A_1 + E_{11} & E_{12} \\ \mathbf{O} & A_2 \end{pmatrix} V^{-1}$$

$$B^D = V \begin{pmatrix} (A_1 + E_{11})^{-1} & X \\ \mathbf{O} & \mathbf{O} \end{pmatrix} V^{-1},$$

$$B^D B = V \begin{pmatrix} I & (A_1 + E_{11}) X \\ \mathbf{O} & \mathbf{O} \end{pmatrix} V^{-1},$$

where

$$X = \sum_{i=0}^{r-1} [(A_1 + E_{11})^{-1}]^{i+2} E_{12} A_2^i.$$

From the above expressions of A, A^D, E, and X, it can be verified that

$$V\begin{pmatrix} (A_1+E_{11})^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}V^{-1} = (I+A^DE)^{-1}A^D,$$

$$V\begin{pmatrix} \mathbf{O} & X \\ \mathbf{O} & \mathbf{O} \end{pmatrix}V^{-1} = \sum_{i=0}^{r-1}[(I+A^DE)^{-1}A^D]^{i+2}E(I-AA^D)A^i.$$

Equation (23) then follows. Similarly, equation (24) can be obtained.

Also, we have

$$\rho(BB^D - AA^D) = 0, \quad \mathscr{R}(B^s) = \mathscr{R}(A^r),$$

implying $(\mathscr{C}_s)$ in (2). Thus, B is an acute perturbation of A. $\square$

Theorem 5 includes the following result in [7,50] as a special case when $E = AA^DE = EAA^D$.

Corollary 4 ([7], [50, Theorem 3.2]) *Let $B = A + E$ with*

$$\operatorname{ind}(A) = r, \quad E = AA^DE = EAA^D, \quad \|A^DE\| < 1.$$

Then

$$B^D = (I+A^DE)^{-1}A^D, \quad BB^D = AA^D.$$

Furthermore, B is an acute perturbation of A.

Example 3 Sun [35, Remark 2.2] presented an example for the spectral projection. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha > 0.$$

The spectral projection P of A corresponding to the eigenvalue $\lambda = 0$ is

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose that the matrix A is perturbed to

$$B = \begin{pmatrix} 0 & \varepsilon \\ 0 & \alpha \end{pmatrix}.$$

Then the spectral projection is perturbed to

$$Q = \begin{pmatrix} 1 & -\varepsilon/\alpha \\ 0 & 0 \end{pmatrix}$$

and

$$\operatorname{rank}(P) = \operatorname{rank}(Q) = \operatorname{rank}(PQP).$$

Also, we have

$$\rho(Q-P)=0, \quad \frac{\|Q-P\|_2}{\|B-A\|_2}=\frac{|\varepsilon|/\alpha}{|\varepsilon|}=\frac{1}{\alpha},$$

and Q is an acute perturbation of P.

Wedin [39] and Stewart [33] presented the concept of the acute perturbation with respect to the Moore-Penrose inverse. They stated that the range spaces $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are *acute*, when

$$\rho(BB^\dagger - AA^\dagger)=\|BB^\dagger - AA^\dagger\|<1,$$

if and only if

$$\mathscr{R}(A)\cap\mathscr{N}(B^*)=\{\mathbf{0}\}, \quad \mathscr{R}(B)\cap\mathscr{N}(A^*)=\{\mathbf{0}\}.$$

Similarly, the range spaces $\mathscr{R}(A^*)$ and $\mathscr{R}(B^*)$ are said to be *acute*, if

$$\rho(B^\dagger B - A^\dagger A)=\|B^\dagger B - A^\dagger A\|<1.$$

The matrices A and B are called *acute*, if $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are *acute* and $\mathscr{R}(A^*)$ and $\mathscr{R}(B^*)$ are *acute*. In this case, they called B an *acute* perturbation of A [33,39].

Example 4 For an important class of matrices [1], the *core inverse* $A^{\oplus}$, in relation with the Moore-Penrose inverse and the group inverse, satisfies

$$AA^{\oplus}=AA^\dagger, \quad A^{\oplus}A=A^{\#}A.$$

If B is a perturbation of A and has the *core inverse*, then

$$\rho(BB^{\oplus}-AA^{\oplus})=\rho(BB^\dagger - AA^\dagger), \quad \rho(B^{\oplus}B-A^{\oplus}A)=\rho(B^{\#}B-A^{\#}A).$$

A perturbation B of A is acute, if and only if

$$\mathscr{R}(A)\cap\mathscr{N}(B^*)=\{\mathbf{0}\}, \quad \mathscr{R}(B)\cap\mathscr{N}(A^*)=\{\mathbf{0}\},$$
$$\mathscr{R}(B)\cap\mathscr{N}(A)=\{\mathbf{0}\}, \quad \mathscr{N}(B)\cap\mathscr{R}(A)=\{\mathbf{0}\}.$$

The above examples show that our results on the acute perturbation with respect to the Drazin inverse are consistent with previous results on other generalized inverses.

6 Concluding remarks

In this paper, we prove that a perturbation B of $A\in\mathbb{C}^{n\times n}$ is *acute* with respect to the Drazin inverse ($\rho(BB^D - AA^D)<1$), if and only if ($\mathscr{C}_s$) in (2) or (20) holds. We then generalize the spectral projections AA^D and BB^D to general oblique projectors P and Q and show that if P approximates Q, then

$\rho(P-Q) < 1$ if and only if (22) holds. Our results are a generalization of the previous work on the Drazin inverse [42,50] and the group inverse [45], and consistent with the previous work on the Moore-Penrose inverse [33,39] and the core inverse [1].

The spectral radius is used in our analysis. Is the spectral radius $\rho(P-Q)$ closely related to the measurement of separation or the minimal angle between subspaces [2,11,14,17]? It will be our future work.

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