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The representation and approximation of the Drazin inverse of a linear operator in Hilbert space

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Abstract

We present a unified representation theorem for the Drazin inverse of linear operators in Hilbert space and a general error bound. Five specific expressions, computational procedures, and their error bounds for the Drazin inverse are uniformly derived from the unified representation theorem.

Keywords: Drazin inverse; Linear operator; Hilbert space; Moore–Penrose inverse; Iterative method; Error bound

1. Introduction

In 1958, Drazin [7] introduced a generalized inverse of an element in an associative ring or a semigroups. It is defined as follows.

Let S be an algebraic semigroup (or associative ring). Then an element $a \in S$ is said to have a Drazin inverse, or to be Drazin invertible [7] if there exists $x \in S$ such that

$$a^{k+1}x = a^k, \quad xax = x, \quad ax = xa. \tag{1.1}$$

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If a has a Drazin inverse, then the smallest non-negative integer k in (1.1) is called the index, denoted by $\operatorname{Ind}(a)$, of a. For any a, there is at most one x such that (1.1) holds. The unique x is denoted by a^{D} and called the Drazin inverse of a. The Drazin inverse does not have the reflexivity property, but it commutes with the element as shown in (1.1).

In particular, when $\operatorname{Ind}(a) = 1$, the x satisfying (1.1) is called the group inverse of a and denoted by $x = a^{\#}$.

The Drazin inverse has various applications in the areas such as singular differential and difference equations and Markov chain [2–4,6,10,12,15,16,18–21].

In this paper, we present a unified representation of the Drazin inverse in Hilbert space and a general error bound. For a sequence of continuous real valued functions converging to $1/x$, we can represent the Drazin inverse T^{D} as the limit of a sequence of corresponding functions of the operator T. Then, we apply the unified representation and the error bound to five special cases and derive five computational methods for the Drazin inverse and their error bounds. These results are analogous to those of the Moore–Penrose inverse [8].

We conclude this section by introducing some notations and basic properties of the Drazin inverse.

Let X and Y be infinite-dimensional complex Hilbert spaces. We denote the set of bounded linear operators from X into Y by $B(X, Y)$. In this paper, we consider linear operators in $B(X) = B(X, X)$. For T in $B(X)$, $N(T)$ and $R(T)$ denote the null space and the range space of T, respectively.

The spectrum of T is denoted by $\sigma(T)$ and the spectral radius by $\rho(T)$. The notation $\|\cdot\|$ stands for the spectral norm.

Recall that $\operatorname{asc}(T)$ $(\operatorname{des}(T))$, the ascent (descent) of $T \in B(X)$, is the smallest non-negative integer n such that

$$N(T^n) = N(T^{n+1})(R(T^n) = R(T^{n+1})).$$

If no such n exists, then $\operatorname{asc}(T) = \infty$ $(\operatorname{des}(T) = \infty)$. A square matrix always has its Drazin inverse. An operator $T \in B(X)$ has its Drazin inverse T^{D} if and only if it has finite ascent and descent [9,11,14]. In such a case,

$$\operatorname{Ind}(T) = \operatorname{asc}(T) = \operatorname{des}(T) = k.$$

2. A unified representation of Drazin inverse

In this section we present a unified representation theorem for the Drazin inverse of a linear operator in Hilbert space. We also give a general error bound for the approximation of the Drazin inverse. The key to the representation theorem is the following lemma.

Lemma 2.1. *Let $T \in B(X)$ with $\mathrm{Ind}(T) = k$ and $R(T^k)$ closed. Then*

$$T^{\mathrm{D}} = \tilde{T}^{-1} T^k T^{*^{2k+1}} T^k,$$

where $\tilde{T} = (T^k T^{*^{2k+1}} T^{k+1})|_{R(T^k)}$ is the restriction of $T^k T^{*^{2k+1}} T^{k+1}$ on $R(T^k)$.

Proof. It follows from [3, p. 247] that

$$T^{\mathrm{D}} = T^k (T^{2k+1})^{\dagger} T^k,$$

where $(T^{2k+1})^{\dagger}$ is the Moore–Penrose inverse of T^{2k+1}. Also, it is easy to prove that

$$R(T^k T^{*^{2k+1}} T^k) = R(T^{\mathrm{D}}) \quad \text{and} \quad N(T^k T^{*^{2k+1}} T^k) = N(T^{\mathrm{D}}).$$

The conclusion then follows from Theorem 2 in [18] and Lemma 3.1 in [20]. □

The above lemma is a generalization of a result in [10] in that the conditions $N(T^k) \subseteq N(T^{k^*})$ and $R(T^k) \subseteq R(T^{k^*})$ required in [10] are removed. Also, this lemma generalizes Corollary 2.1 in [6] from matrices to linear operators.

Now we are ready to present the representation theorem.

Theorem 2.2. *Let $T \in B(X)$ with $\mathrm{Ind}(T) = k$ and $R(T^k)$ closed, and define $\tilde{T} = (T^k T^{*^{2k+1}} T^{k+1})|_{R(T^k)}$. If Ω is an open set such that $\sigma(\tilde{T}) \subset \Omega \subset (0, \infty)$ and $\{S_n(x)\}$ is a sequence of continuous real valued functions on Ω with $\lim_{n\to\infty} S_n(x) = 1/x$ uniformly on $\sigma(\tilde{T})$, then*

$$T^{\mathrm{D}} = \lim_{n\to\infty} S_n(\tilde{T}) T^k T^{*^{2k+1}} T^k. \tag{2.1}$$

Furthermore, for any $\varepsilon > 0$, there is a operator norm $\|\cdot\|_*$ on X such that

$$\frac{\|S_n(\tilde{T}) T^k T^{*^{2k+1}} T^k - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant \max_{x\in\sigma(\tilde{T})} |S_n(x)x - 1| + \mathrm{O}(\varepsilon). \tag{2.2}$$

Proof. It follows from [1] that

$$\sigma(T^k T^{*^{2k+1}} T^{k+1}) = \sigma[(T^{2k+1})^*(T^{2k+1})]$$

is non-negative. Thus, the spectrum of $\tilde{T}$ is positive since $\tilde{T}$ is non-singular.

Using Theorem 10.27 in [17], we have

$$\lim_{n\to\infty} S_n(\tilde{T}) = \tilde{T}^{-1}$$

uniformly in $B(R(T^k))$.

It then follows from Lemma 2.1 that

$$\lim_{n\to\infty} S_n(\tilde{T})T^k T^{*^{2k+1}} T^k = \tilde{T}^{-1} T^k T^{*^{2k+1}} T^k = T^{\mathrm{D}}.$$

To obtain the error bound (2.2), we note that $T^k T^{*^{2k+1}} T^k = \tilde{T} T^{\mathrm{D}}$. Therefore,

$$S_n(\tilde{T})T^k T^{*^{2k+1}} T^k - T^{\mathrm{D}} = [S_n(\tilde{T})\tilde{T} - I]T^{\mathrm{D}}.$$

Also, for any $\varepsilon > 0$, there is an operator norm $\|\cdot\|_*$ such that $\|T\|_* \leqslant \rho(T) + \varepsilon$. See [5, p. 77]. Thus

$$\begin{aligned}\|S_n(\tilde{T})T^k T^{*^{2k+1}} T^k - T^{\mathrm{D}}\|_* &\leqslant \|S_n(\tilde{T})\tilde{T} - I\|_* \|T^{\mathrm{D}}\|_* \\ &\leqslant \left[\max_{x\in\sigma(\tilde{T})} |S_n(x)x - 1| + \mathrm{O}(\varepsilon)\right] \|T^{\mathrm{D}}\|_*,\end{aligned}$$

which completes the proof. □

In the above theorem, we relax the restriction on the spectrum of T and extend the main results in [12,21].

In the following section, we apply the above unified representation (2.1) of the Drazin inverse and the general error bound (2.2) to five specific cases. In deriving the specific error bounds, we need lower and upper bounds for $\sigma(\tilde{T})$ given by the following theorem.

Theorem 2.3. *Let* $T \in B(X)$ *with* $\mathrm{Ind}(T) = k$ *and* $R(T^k)$ *closed, and define* $\tilde{T} = (T^k T^{*^{2k+1}} T^{k+1})|_{R(T^k)}$, *then for each* $\lambda \in \sigma(\tilde{T})$,

$$\|(T^{2k+1})^\dagger\|^{-2} \leqslant \lambda \leqslant \|T\|^{4k+2}.$$

Proof. For any $\lambda \in \sigma(\tilde{T})$,

$$0 < \lambda \in \sigma(\tilde{T}) \subseteq \sigma(T^k T^{*^{2k+1}} T^{k+1}) = \sigma[(T^{2k+1})^*(T^{2k+1})].$$

It is obvious that

$$\mathrm{Ind}((T^{2k+1})^* T^{2k+1}) = 1$$

and

$$1/\lambda \in \sigma([(T^{2k+1})^* T^{2k+1}]^\#) = \sigma([(T^{2k+1})^* T^{2k+1}]^\dagger) = \sigma[(T^{2k+1})^\dagger (T^{2k+1})^{\dagger^*}].$$

It then follows that

$$1/\lambda \leqslant \|(T^{2k+1})^\dagger (T^{2k+1})^{\dagger^*}\| = \|T^{2k+1}\|^2,$$

i.e.,

$$\lambda \geqslant \|(T^{2k+1})^\dagger\|^{-2}.$$

On the other hand, since

$$\|T^kT^{*^{2k+1}}T^{k+1}\| \geqslant \|(T^kT^{*^{2k+1}}T^{k+1})|_{R(T^k)}\|,$$

we get $\|\tilde{T}\| \leqslant \|T\|^{4k+2}$. Thus $\lambda \leqslant \|T\|^{4k+2}$ for all $\lambda \in \sigma(\tilde{T})$. □

3. Approximations of the Drazin inverse in Hilbert space

In this section, we apply Theorem 2.2 to five specific cases to derive five specific representations and five computational procedures for the Drazin inverse in Hilbert space and their corresponding error bounds.

3.1. Euler–Knopp method

Consider the following sequence:

$$S_n(x) = \alpha\sum_{j=0}^{n}(1-\alpha x)^j,$$

which can be viewed as the Euler–Knopp transform of the series $\sum_{n=0}^{\infty}(1-x)^n$. Clearly $\lim_{n\to\infty} S_n(x) = 1/x$ uniformly on any compact subset of the set

$$E_\alpha = \{x : |1-\alpha x| < 1\} = \{x : 0 < x < 2/\alpha\}.$$

By Theorem 2.3, we get

$$\sigma(\tilde{T}) \subseteq \left[\|(T^{2k+1})^\dagger\|^{-2}, \|T\|^{4k+2}\right] \subseteq (0, \|T\|^{4k+2}].$$

If we choose the parameter α, $0 < \alpha < 2\|T\|^{-(4k+2)}$, such that $(0, \|T\|^{4k+2}] \subseteq E_\alpha$, then we have the following representation of the Drazin inverse:

$$T^{\mathrm{D}} = \alpha\sum_{n=0}^{\infty}(I-\alpha T^kT^{*^{2k+1}}T^{k+1})^nT^kT^{*^{2k+1}}T^k.$$

Setting

$$T_n = \alpha\sum_{j=0}^{n}(I-\alpha T^kT^{*^{2k+1}}T^{k+1})^jT^kT^{*^{2k+1}}T^k,$$

we have the following iterative procedure for the Drazin inverse:

$$T_0 = \alpha T^kT^{*^{2k+1}}T^k, \quad T_{n+1} = (I-\alpha T^kT^{*^{2k+1}}T^{k+1})T_n + \alpha T^kT^{*^{2k+1}}T^k. \tag{3.1}$$

For the error bound, we note that the sequence $\{S_n(x)\}$ satisfies

$$S_{n+1}(x)x - 1 = (1-\alpha x)(S_n(x)x - 1).$$

Thus

$$|S_n(x)x - 1| = |1 - \alpha x|^n |S_0(x)x - 1| = |1 - \alpha x|^{n+1}.$$

If $x \in \sigma(\tilde{T})$ and $0 < \alpha < 2/\|T\|^{4k+2}$, then we see that $|1 - \alpha x| \leqslant \beta < 1$, where

$$\beta = \max\left\{|1 - \alpha\|T\|^{4k+2}|, |1 - \alpha\|(T^{2k+1})^{\dagger}\|^{-2}|\right\}. \tag{3.2}$$

Therefore,

$$|S_n(x)x - 1| \leqslant \beta^{n+1} \to 0 \quad (n \to \infty). \tag{3.3}$$

It follows from (3.3) and Theorem 2.2 that the error bound is

$$\frac{\|T_n - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant \beta^{n+1} + \mathrm{O}(\varepsilon). \tag{3.4}$$

In summary, we have the following corollary:

Corollary 3.1. *Let $T \in B(X)$ with $\mathrm{Ind}(T) = k$ and $R(T^k)$ closed. Then the sequence $\{T_n\}$ defined by* (3.1) *converges to T^{D}, if $0 < \alpha < 2/\|T\|^{4k+2}$. The error bound is given by* (3.4).

3.2. Newton–Raphson method

To develop another iterative method, we regard $1/x$ as the root of the function

$$s(y) = y^{-1} - x.$$

The Newton–Raphson method can be used to approximate this root. This is done by generating a sequence $\{y_n\}$ defined by

$$y_{n+1} = y_n - s(y_n)/s'(y_n) = y_n(2 - xy_n)$$

for a suitable initial y_0. Suppose that for $\alpha > 0$, we define a sequence $\{S_n(x)\}$ of functions by

$$S_0(x) = \alpha, \quad S_{n+1}(x) = S_n(x)[2 - xS_n(x)]. \tag{3.5}$$

Clearly, the sequence (3.5) satisfies

$$xS_{n+1}(x) - 1 = -[xS_n(x) - 1]^2.$$

Iterating on the above equality, we have

$$|xS_n(x) - 1| = |\alpha x - 1|^{2^n} \leqslant \beta^{2^n} \to 0, \quad \text{as } n \to \infty,$$

for $0 < \alpha < 2/\|T\|^{4k+2}$, where β is given by (3.2).

One attractive feature of the Newton–Raphson method is its generally quadratic rate of convergence. Using the above argument in conjunction with Theorem 2.2, we see that the sequence $\{S_n(\tilde{T})\}$ defined by

$$S_0(\tilde{T}) = \alpha I, \quad S_{n+1}(\tilde{T}) = S_n(\tilde{T})[2I - \tilde{T}S_n(\tilde{T})]$$

has the property that $\lim_{n\to\infty} S_n(\tilde{T})T^kT^{*^{2k+1}}T^k = T^{\mathrm{D}}$. Setting $T_n = S_n(\tilde{T})T^k \times T^{*^{2k+1}}T^k$, we have the following iterative procedure for the Drazin inverse:

$$T_0 = \alpha T^kT^{*^{2k+1}}T^k, \quad T_{n+1} = T_n(2I - TT_n). \tag{3.6}$$

The following corollary summarizes the Newton–Raphson method for the Drazin inverse.

Corollary 3.2. *Let* $T \in B(X)$ *with* $\mathrm{Ind}(T) = k$ *and* $R(T^k)$ *closed. Then the sequence* $\{T_n\}$ *determined by* (3.6) *converges to* T^{D}, *for* $0 < \alpha < 2/\|T\|^{4k+2}$. *Furthermore the error bound is*

$$\frac{\|T_n - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant \beta^{2^n} + \mathrm{O}(\varepsilon).$$

3.3. Limit expression

The limit expression of the Drazin inverse was first given by Meyer [13], that is, for $k = \mathrm{Ind}(T)$ it holds that $T^{\mathrm{D}} = \lim_{t\to 0^+}(tI + T^{k+1})^{-1}T^k$. It can be rewritten as [6]

$$T^{\mathrm{D}} = \lim_{t\to 0^+}(tI + T^kT^{*^{2k+1}}T^{k+1})^{-1}T^kT^{*^{2k+1}}T^k.$$

Setting $S_t(x) = (t + x)^{-1}$ $(t > 0)$, for $x \in \sigma(\tilde{T})$, we can derive the following error bound for this method:

$$|xS_t(x) - 1| = \frac{t}{x + t} \leqslant \frac{t}{\|(T^{2k+1})^\dagger\|^{-2} + t} = \frac{\|(T^{2k+1})^\dagger\|^2 t}{1 + \|(T^{2k+1})^\dagger\|^2 t}.$$

Therefore, from Theorem 2.2 we have the following corollary.

Corollary 3.3. *Let* $T \in B(X)$ *with* $\mathrm{Ind}(T) = k$ *and* $R(T^k)$ *closed. Then for* $t > 0$, *the following holds*:

$$\frac{\|(tI + T^kT^{*^{2k+1}}T^{k+1})^{-1}T^kT^{*^{2k+1}}T^k - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant \frac{\|(T^{2k+1})^\dagger\|^2 t}{1 + \|(T^{2k+1})^\dagger\|^2 t} + \mathrm{O}(\varepsilon).$$

The methods we have considered so far are based on approximating the function $f(x) = 1/x$. Next, we will apply Theorem 2.2 to polynomial interpolations of the function $f(x) = 1/x$ to derive iterative methods for computing T^{D} and their corresponding asymptotic error bounds.

3.4. Newton–Gregory interpolation

Let $p_n(x)$ denote the unique polynomial of degree n which interpolates the function $f(x) = 1/x$ at the points $x = 1, 2, \ldots, n+1$, then the Newton–Gregory interpolation formula gives the representation

$$p_n(x) = \sum_{j=0}^{n} \binom{x-1}{j} \Delta^j f(1), \tag{3.7}$$

where Δ is the forward difference operator defined by

$$\Delta f(x) = f(x+1) - f(x), \quad \Delta^j f(x) = \Delta(\Delta^{j-1} f)(x),$$

and

$$\binom{x}{0} = 1, \quad \binom{x-1}{j} = \frac{(x-1)(x-2)\cdots(x-j)}{j!}.$$

It is not hard to see that when $f(x) = 1/x$, $\Delta^j f(1) = (-1)^j/(j+1)$. A routine calculation shows that the interpolating polynomial (3.7) becomes

$$p_n(x) = \sum_{j=0}^{n} \frac{1}{j+1} \prod_{l=0}^{j-1} \left(1 - \frac{x}{l+1}\right), \tag{3.8}$$

where, by convention, the product term equals 1 when $j = 0$. From the above definition, it is easy to verify that

$$1 - xp_n(x) = \prod_{l=0}^{n} \left(1 - \frac{x}{l+1}\right).$$

Lemma 3.4. *The polynomials $p_n(x)$ in* (3.8) *satisfy*

$$\lim_{n\to\infty} p_n(x) = x^{-1}$$

uniformly on any compact subset of $(0, \infty)$.

Proof. The proof is analogous to that of [3, pp. 220–232].

It follows from Theorem 2.2 and Lemma 3.4 that

$$T^{\mathrm{D}} = \lim_{n\to\infty} p_n(\tilde{T}) T^k T^{*^{2k+1}} T^k.$$

In order to phrase this result in a form which is convenient for computation we derive

$$p_{n+1}(x) = p_n(x) + \frac{1}{n+2} \prod_{l=0}^{n} \left(1 - \frac{x}{l+1}\right) = p_n(x) + \frac{1}{n+2}[1 - xp_n(x)]$$

note that $p_0(x) = 1$. Therefore, setting $T_n = p_n(\tilde{T})T^kT^{*^{2k+1}}T^k$, we have the following iterative method for computing the Drazin inverse T^{D}:

$$T_0 = T^kT^{*^{2k+1}}T^k,$$
$$T_{n+1} = p_{n+1}(\tilde{T})T^kT^{*^{2k+1}}T^k = T_n + \frac{T_0}{n+2}(I - TT_n). \tag{3.9}$$

To derive an asymptotic error bound for this method, note that for

$$x \in \sigma(\tilde{T}) \subseteq [\|(T^{2k+1})^{\dagger}\|^{-2}, \|T\|^{4k+2}]$$

and for $l \geqslant L = \|T\|^{4k+2}$, we have

$$1 - \frac{x}{l+1} \leqslant \exp(-\frac{x}{l+1}) \quad \text{for all } x \in \sigma(\tilde{T}).$$

Therefore,

$$\prod_{l=L}^{n}\left(1 - \frac{x}{l+1}\right) \leqslant \exp\left(-x\sum_{l=L}^{n}\frac{1}{l+1}\right), \quad n \geqslant L.$$

Also,

$$\sum_{l=L}^{n}\frac{1}{l+1} \geqslant \int_{L+1}^{n+2}\frac{\mathrm{d}t}{t} = \ln(n+2) - \ln(L+1)$$

and hence

$$\exp\left(-x\sum_{l=L}^{n}\frac{1}{l+1}\right) \leqslant (L+1)^x(n+2)^{-x} = (1 + \|T\|^{4k+2})^x(n+2)^{-x}.$$

If we set the constant

$$c = \max_{x\in\sigma(\tilde{T})}\left\{(1 + \|T\|^{4k+2})^x\prod_{l=0}^{L-1}\left|1 - \frac{x}{l+1}\right|\right\},$$

then

$$|1 - xp_n(x)| = \prod_{l=0}^{L-1}\left|1 - \frac{x}{l+1}\right|\prod_{l=L}^{n}\left|1 - \frac{x}{l+1}\right| \leqslant c(n+2)^{-x}.$$

Finally, it follows from Theorem 2.2 that

$$\frac{\|T_n - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant c(n+2)^{-\|(T^{2k+1})^{\dagger}\|^{-2}} + \mathrm{O}(\varepsilon), \tag{3.10}$$

for sufficiently large n. $\square$

We conclude Newton–Gregory interpolation method in the following corollary.

Corollary 3.5. *Let* $T \in B(X)$ *with* $\mathrm{Ind}(T) = k$ *and* $R(T^k)$ *closed, and define* $\tilde{T} = (T^k T^{*^{2k+1}} T^{k+1})|_{R(T^k)}$. *Then the sequence defined by* (3.9) *converges to* T^{D} *and the error bound is given by* (3.10).

3.5. Hermite interpolation

We now take the natural next step of approximating the Drazin inverse T^{D} by Hermite interpolation of the function $f(x) = 1/x$ and deriving its asymptotic error bound.

We seek the unique polynomial $q_n(x)$ of degree $2n+1$ which satisfies

$$q_n(i) = 1/i, \quad q_n'(i) = -1/i^2 \quad (i = 1, 2, \ldots, n+1)$$

then the Hermite interpolation formula yields the representation

$$q_n(x) = \sum_{i=0}^{n} [2(i+1) - x] \prod_{l=1}^{i} \left(\frac{1-x}{1+l} \right)^2. \tag{3.11}$$

Here, by convention, the product term equals 1 when $i = 0$.

From the definition of $q_n(x)$ in (3.11), an inductive argument gives

$$1 - xq_n(x) = \prod_{l=0}^{n} \left(1 - \frac{x}{l+1} \right)^2.$$

Lemma 3.6. *The polynomials* $q_n(x)$ *in* (3.11) *satisfy*

$$\lim_{n\to\infty} q_n(x) = 1/x$$

uniformly on any compact subset in $(0, \infty)$.

Proof. The proof is analogous to that of [3, pp. 220–232].

It follows from Theorem 2.2 and Lemma 3.6 that $T^{\mathrm{D}} = \lim_{n\to\infty} q_n(\tilde{T}) T^k \times T^{*^{2k+1}} T^k$. Letting

$$q_0(x) = 2 - x,$$

$$
\begin{aligned}
q_{n+1}(x) &= q_n(x) + [2(n+2) - x]\prod_{l=1}^{n+1}\left(\frac{l-x}{l+1}\right)^2 \\
&= q_n(x) + [2(n+2) - x]\prod_{l=0}^{n}\left(\frac{1+l-x}{1+l}\right)^2 \Big/ (n+2)^2 \\
&= q_n(x) + \frac{1}{n+2}\left(2 - \frac{x}{n+2}\right)[1 - xq_n(x)]
\end{aligned}
$$

and

$$T_n = q_n(\tilde{T})T^k T^{*^{2k+1}} T^k,$$

we can deduce the following iterative method for computing the Drazin inverse T^{D}:

$$
\begin{aligned}
T_0 &= (2I - MT)M, \\
T_{n+1} &= T_n + \frac{1}{n+2}\left(2I - \frac{1}{n+2}MT\right)M(I - TT_n),
\end{aligned}
\tag{3.12}
$$

where $M = T^k T^{*^{2k+1}} T^k$ or $\tilde{T} = MT$.

Similar to Newton–Gregory interpolation method, we can establish the error bound as follows. For $l \geqslant L = \|T\|^{4k+2}$ and $x \in \sigma(\tilde{T}) \subseteq [\|(T^{2k+1})^{\dagger}\|^{-2}, \|T\|^{4k+2}]$, we have

$$\prod_{l=L}^{n}\left(1 - \frac{x}{l+1}\right)^2 \leqslant (1 + \|T\|^{4k+2})^{2x}(n+2)^{-2x}.$$

Let the constant

$$d = \max_{x \in \sigma(\tilde{T})}(1 + \|T\|^{4k+2})^{2x}\prod_{l=0}^{L-1}\left(1 - \frac{x}{l+1}\right)^2.$$

Then

$$|1 - xq_n(x)| = \prod_{l=0}^{L-1}\left(1 - \frac{x}{l+1}\right)^2 \prod_{l=L}^{n}\left(1 - \frac{x}{l+1}\right)^2 \leqslant d(n+2)^{-2x}.$$

By Theorem 2.2, we arrive at the error bound:

$$\frac{\|T_n - T^{\mathrm{D}}\|_*}{\|T^{\mathrm{D}}\|_*} \leqslant d(n+2)^{-2\|(T^{2k+1})^{\dagger}\|^{-2}} + \mathrm{O}(\varepsilon), \tag{3.13}$$

for sufficiently large n. □

The following corollary summarizes the Hermite interpolation method.

Corollary 3.7. *Let* $T \in B(X)$ *with* $\mathrm{Ind}(T) = k$ *and* $R(T^k)$ *closed, and define* $\tilde{T} = (T^k T^{*^{2k+1}} T^{k+1})|_{R(T^k)}$. *Then the sequence* $\{T_n\}$ *defined by* (3.12) *converges to* T^{D}. *Each approximation* T_n *has the error bound* (3.13).

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