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Solving constrained matrix equations and Cramer rule ☆

Guorong Wang [a,*], Sanzheng Qiao [b]

[a] Department of Mathematics, Shanghai Normal University, Shanghai 200234, People's Republic of China
[b] Department of Computing and Software, McMaster University, Hamilton, Ont., Canada L8S 4L7

Abstract

This paper presents the solution of a general constrained matrix equation using generalized inverses and gives an explicit expression for the elements of the solution matrix using Cramer rule.

Keywords: Constrained matrix equations; Generalized inverses; Cramer rule

1. Introduction

In [8], there is a Cramer rule for the unique solution of the constrained linear system of equations

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in M,$$

where $A \in C^{m\times n}$, $\mathbf{b} \in R(A)$, which the symbol $R(A)$ denotes the range space of A, and M is a complementary subspace of $N(A)$, which the symbol $N(A)$ denotes the null space of A. A Cramer rule for the unique solution of

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* Corresponding author.
E-mail addresses: grwang@online.sh.cn (G. Wang), qiao@mcmaster.ca (S. Qiao).

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$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R(A^k),$$

where $A \in C^{n\times n}$, $k = \mathrm{Ind}(A)$ and $\mathbf{b} \in R(A^k)$, is given in [6]. A Cramer rule for the unique solution of

$$WAW\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R((AW)^{k_1}),$$

where $A \in C^{m\times n}$, $W \in C^{n\times m}$, $k_1 = \mathrm{Ind}(AW)$, $k_2 = \mathrm{Ind}(WA)$ and $\mathbf{b} \in R((WA)^{k_2})$, is given in [7].

The Cramer rule for the unique solution or one of solution of the constrained linear system of equations

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in T,$$

where $A \in C^{m\times n}$, $\mathrm{rank}(A) = r$, and T is an arbitrary but fixed subspace of C^n, is discussed in [4]. The conditions for the existence and uniqueness of the solution is given there. This paper considers the following more general problem.

Given an m-by-n matrix A ($\mathrm{rank}(A) = r$), a p-by-q matrix B ($\mathrm{rank}(B) = \tilde{r}$), and an m-by-q matrix D, solve for X in the matrix equation:

$$AXB = D, \tag{1}$$

with the constraints

$$R(X) \subseteq T \quad \text{and} \quad N(X) \supseteq \widetilde{S}, \tag{2}$$

for the predetermined subspaces $T \subseteq C^n$ ($\dim(T) = t \leqslant r$) and $\widetilde{S} \subseteq C^p$ ($\dim(\widetilde{S}) = p - \tilde{t} \geqslant p - \tilde{r}$). Chen [5] gives a solution to this constrained matrix equation. In the method there, the matrix equation

$$AXB = D, \quad R(X) \subseteq T, \;\; N(X) \supseteq S,$$

is first transformed into

$$A^*AXBB^* = A^*DB^*, \quad R(X) \subseteq T, \;\; N(X) \supseteq S.$$

Then the Bott–Duffin inverse is used in the solution. In this paper, we use the $\{2\}$ inverse with prescribed range and null space, which includes the Bott–Duffin inverse as a special case. Thus, our results are more general and, moreover, our derivations are simpler than those in [5].

The following notations are used in this paper.

Given a matrix A and a subspace S, AS denotes the subspace obtained by applying the transformation A to S.

Given a matrix A and a column vector $\mathbf{b}$, $A(i \to \mathbf{b})$ is the matrix obtained by replacing the ith column of A with $\mathbf{b}$; $A(\mathbf{b}^{\mathrm{T}} \leftarrow j)$ is the matrix obtained by replacing the jth row of A with $\mathbf{b}^{\mathrm{T}}$.

Following the notations in [1,2], $A^{\dagger}$ denotes the Moore–Penrose inverse of A, A_{d} the Drazin inverse, $A^{(i)}$ ($i = 1, 2$) the $\{i\}$ inverse, and $A^{(2)}_{T,S}$ the $\{2\}$ inverse with range T and null space S. It is shown in [4] that $AA^{(2)}_{T,S}$ is the projector $P_{AT,S}$ and $A^{(2)}_{T,S}A$ is the projector $P_{T,(A^*S^\perp)^\perp}$. In particular,

$$P_{AT,S}X = X \quad \text{if } R(X) \subseteq AT, \tag{3}$$

and

$$XP_{AT,S} = X \quad \text{if } N(X) \supseteq S. \tag{4}$$

This paper is organized as follows. In Section 2, we present a proof of the existence and uniqueness of the solution for the matrix equation (1) with the constraints (2). Then, in Section 3, applying Cramer rule, we show an explicit expression for the elements x_{ij} of the solution X. Finally, in Section 4, we consider the special case when A and B in (1) are square and express the solution in terms of the Drazin inverse.

2. Existence and uniqueness

In this section, we give the conditions for the existence of the solution for the matrix equation (1) with the constraints (2) and express the solution in terms of $\{2\}$ inverses with specified ranges and null spaces. We also show that the solution is unique if it exists.

If we define the range and null space of a pair of matrices A and B as sets of matrices:

$$R(A,B) = \{Y = AXB, \ \text{for some } X\},$$

and

$$N(A,B) = \{X \ \text{such that } AXB = 0\},$$

then obviously the unconstrained matrix equation (1) has a solution if $D \in R(A,B)$.

Now, we consider the solution of (1) with the constraints (2). Let $S \subseteq C^m$ and $\widetilde{T} \subseteq C^q$ be two subspaces such that

$$\dim(S^\perp) = \dim(T) = t \leqslant r \quad \text{and} \quad AT \oplus S = C^m,$$

which is equivalent to $T \oplus (A^{\mathrm{H}}S^\perp)^\perp = C^n$, and

$$\dim(\widetilde{T}) = \dim(\widetilde{S}^\perp) = \tilde{t} \leqslant \tilde{r} \quad \text{and} \quad B\widetilde{T} \oplus \widetilde{S} = C^p,$$

which is equivalent to $\widetilde{T} \oplus (B^*\widetilde{S}^\perp)^\perp = C^q$.

We have the following theorem of the existence and uniqueness of the solution for the matrix equation (1) with the constraints (2).

Theorem 1. *Given the matrices A, B, D, and the subspaces T, S, $\widetilde{T}$, and $\widetilde{S}$ as above. If*

$$D \in R(AG, \widetilde{G}B), \tag{5}$$

for some matrices $G \in C^{n\times m}$ *and* $\widetilde{G} \in C^{q\times p}$ *satisfying*

$$R(G) = T, \quad N(G) = S, \quad R(\widetilde{G}) = \widetilde{T}, \quad \textit{and} \quad N(\widetilde{G}) = \widetilde{S}, \tag{6}$$

then the matrix equation (1) *with the constraints* (2) *has the unique solution*

$$X = A^{(2)}_{T,S} D B^{(2)}_{\widetilde{T},\widetilde{S}}. \tag{7}$$

Proof. From the definition of the range of a pair of matrices, (5) implies that $D = AGY\widetilde{G}B$ for some Y. Consequently, from (6),

$$R(D) \subseteq R(AG) = AT, \tag{8}$$

and

$$N(D) \supseteq (B^*\widetilde{S}^{\perp})^{\perp}, \tag{9}$$

since $R(D^{\mathrm{H}}) \subseteq B^*R(\widetilde{G}^{\mathrm{H}}) = B^*\widetilde{S}^{\perp}$. From (8), (3), (9), and (4), we have

$$AA^{(2)}_{T,S} D B^{(2)}_{\widetilde{T},\widetilde{S}} B = P_{AT,S} D P_{\widetilde{T},(B^*\widetilde{S}^{\perp})^{\perp}} = D,$$

that is, X in (7) is a solution of the matrix equation (1). The solution $X = A^{(2)}_{T,S} D B^{(2)}_{\widetilde{T},\widetilde{S}}$ also satisfies the constraints (2), because $R(X) \subseteq R(A^{(2)}_{T,S}) = T$ and $N(X) \supseteq N(B^{(2)}_{\widetilde{T},\widetilde{S}}) = \widetilde{S}$.

Finally, we prove the uniqueness. If X_0 is a solution of (1) satisfying (2), then

$$X = A^{(2)}_{T,S} D B^{(2)}_{\widetilde{T},\widetilde{S}} = A^{(2)}_{T,S} A X_0 B B^{(2)}_{\widetilde{T},\widetilde{S}} = P_{T,(A^*S^{\perp})^{\perp}} X_0 P_{B\widetilde{T},\widetilde{S}} = X_0,$$

since $R(X_0) \subseteq T$ and $N(X_0) \supseteq \widetilde{S}$. $\square$

3. Cramer rule

As we know, Cramer rule can be used to express the solution of a nonsingular linear system. In this section, we give an explicit expression for the elements x_{ij} of the solution matrix X.

When A and B in the matrix equation (1) are nonsingular, the solution $X = A^{-1}DB^{-1}$ can be computed by applying Cramer rule. Let Y be the solution of the matrix equation $AY = D$ and partition $Y = [\mathbf{y}_1, \ldots, \mathbf{y}_n]$ and $D = [\mathbf{d}_1, \ldots, \mathbf{d}_n]$, then, applying Cramer rule, the elements y_{ik} $(i = 1, \ldots, n)$ of the kth column $\mathbf{y}_k$ are given by

$$y_{ik} = \frac{\det(A(i \to \mathbf{d}_k))}{\det(A)}.$$

Similarly, consider the solution Z of the matrix equation $ZB = I$, then the elements z_{kj} $(j = 1, \ldots, n)$ of the kth row of Z can be computed by

$$z_{kj} = \frac{\det(B(\mathbf{e}_k^{\mathrm{T}} \leftarrow j))}{\det(B)},$$

where $\mathbf{e}_k^{\mathrm{T}}$ is the kth row of the identity matrix I. Thus the elements x_{ij} of the matrix $X = A^{-1}DB^{-1} = YZ$ can be obtained by

$$x_{ij} = \sum_{k=1}^{n} y_{ik} z_{kj} = \frac{\sum_{k=1}^{n} \det(A(i \to \mathbf{d}_k)) \det(B(\mathbf{e}_k^{\mathrm{T}} \leftarrow j))}{\det(A)\det(B)}.$$

In our case, A and B are general matrices. They may be rectangular or singular. In order to apply Cramer rule, we extend A and B into nonsingular matrices and imbed Eq. (1) in a larger but equivalent matrix equation.

Let

$$L \in C^{m\times(m-t)}, \quad M^* \in C^{n\times(n-t)}, \quad \widetilde{L} \in C^{p\times(p-\tilde{t})}, \quad \text{and} \quad \widetilde{M} \in C^{q\times(q-\tilde{t})} \tag{10}$$

be matrices of full column rank such that

$$R(L) = S, \quad N(M) = T, \quad R(\widetilde{L}) = \widetilde{S}, \quad \text{and} \quad N(\widetilde{M}) = \widetilde{T}, \tag{11}$$

then the bordered matrices

$$\begin{bmatrix} A & L \\ M & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B & \widetilde{L} \\ \widetilde{M} & 0 \end{bmatrix} \tag{12}$$

are nonsingular [3] and

$$\begin{bmatrix} A & L \\ M & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{(2)}_{T,S} & (I - A^{(2)}_{T,S}A)M^\dagger \\ L^\dagger(I - AA^{(2)}_{T,S}) & L^\dagger(AA^{(2)}_{T,S}A - A)M^\dagger \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} B & \widetilde{L} \\ \widetilde{M} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} B^{(2)}_{\widetilde{T},\widetilde{S}} & (I - B^{(2)}_{\widetilde{T},\widetilde{S}}B)\widetilde{M}^\dagger \\ \widetilde{L}^\dagger(I - BB^{(2)}_{\widetilde{T},\widetilde{S}}) & \widetilde{L}^\dagger(BB^{(2)}_{\widetilde{T},\widetilde{S}}B - B)\widetilde{M}^\dagger \end{bmatrix}. \tag{14}$$

Since X is the solution of the constrained matrix equation, from (10), we have

$$R(X) \subseteq T = N(M) \quad \text{and} \quad N(X) \supseteq S = R(\widetilde{L}).$$

It then follows that

$$MX = 0 \quad \text{and} \quad X\widetilde{L} = 0,$$

and

$$\begin{bmatrix} A & L \\ M & 0 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & \widetilde{L} \\ \widetilde{M} & 0 \end{bmatrix} = \begin{bmatrix} AXB & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus we can imbed the matrix equation (1) in the equation

$$\begin{bmatrix} A & L \\ M & 0 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & \widetilde{L} \\ \widetilde{M} & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \tag{15}$$

where the coefficient matrices are nonsingular, and $X = A^{(2)}_{T,S}DB^{(2)}_{\widetilde{T},\widetilde{S}}$ is the unique solution of the above extended equation. Applying Cramer rule to (15), we have the following theorem.

Theorem 2. *Given the matrices A, B, and D and the subspaces T, S, $\widetilde{T}$, and $\widetilde{S}$ as above. If the matrices L, M, $\widetilde{L}$, and $\widetilde{M}$ in* (10) *satisfy* (11), *then the elements x_{ij} of the solution X in* (7) *are given by*

$$x_{ij} = \frac{\sum_{k=1}^{q} \det\left(\begin{bmatrix} A(i \to \mathbf{d}_k) & L \\ M(i \to 0) & 0 \end{bmatrix}\right) \det\left(\begin{bmatrix} B(\mathbf{e}_k^{\mathrm{T}} \leftarrow j) & \widetilde{L}(0 \leftarrow j) \\ \widetilde{M} & 0 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} A & L \\ M & 0 \end{bmatrix}\right) \det\left(\begin{bmatrix} B & \widetilde{L} \\ \widetilde{M} & 0 \end{bmatrix}\right)}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, p$, where $\mathbf{d}_k$ is the kth *column of D and $\mathbf{e}_k$ is the* kth *column of the q-by-q identity matrix.*

4. Application

Noting that the Bott–Duffin inverse

$$(A^*A)^{(-1)}_{(T)} = (A^*A)^{(2)}_{T,T^\perp}, \qquad (BB^*)^{(-1)}_{(S^\perp)} = (BB^*)^{(2)}_{S^\perp,S}, \tag{16}$$

where

$$T = R(P_T), \qquad T^\perp = N(P_T), \qquad S^\perp = R(P_{S^\perp}), \qquad S = N(P_{S^\perp}), \tag{17}$$

we have the following result.

Corollary 1 [5]. *Given $A \in C^{m\times n}$, $B \in C^{p\times q}$, $D \in C^{m\times q}$, the matrix equation*

$$A^*AXBB^* = A^*DB^*, \tag{18}$$

with the constraints

$$R(X) \subseteq T, \qquad N(X) \supseteq S, \tag{19}$$

has the unique solution

$$X = (A^*A)^{(-1)}_{(T)} A^*DB^*(BB^*)^{(-1)}_{(S^\perp)}, \tag{20}$$

if $A^*DB^* \in R(A^*AP_T, P_{S^\perp BB^*})$ *for the orthogonal projectors* P_T *and* $P_{S^\perp}$ *such that*

$$R(P_T) = T, \qquad N(P_T) = T^\perp, \tag{21}$$

$$R(P_{S^\perp}) = S^\perp, \qquad N(P_{S^\perp}) = S. \tag{22}$$

Proof. From the assumption that

$$A^*DB^* = A^*AP_T YP_{S^\perp}BB^*,$$

for some Y. Consequently, from (21) and (22),

$$R(A^*DB^*) \subseteq A^*AT, \tag{23}$$

and

$$N(A^*DB^*) \supseteq (BB^*S^\perp)^\perp, \tag{24}$$

since $R(BD^*A) \subseteq BB^*R(P^*_{S^\perp}) = BB^*S^\perp$. From

$$(A^*A)(A^*A)^{(2)}_{T,T^\perp} = P_{A^*AT,T^\perp}, \tag{25}$$

and

$$(BB^*)^{(2)}_{S^\perp,S}(BB^*) = P_{S^\perp,(BB^*S^\perp)^\perp}, \tag{26}$$

we have

$$(A^*A)(A^*A)^{(2)}_{T,T^\perp}A^*DB^*(BB^*)^{(2)}_{S^\perp,S}(BB^*) = A^*DB^*,$$

that is

$$(A^*A)(A^*A)^{(-1)}_{(T)}A^*DB^*(BB^*)^{(-1)}_{(S^\perp)}(BB^*) = A^*DB^*.$$

Thus X in (20) is a solution of the matrix equation (18). The solution X also satisfy the constraints (19) because $R(X) \subseteq R((A^*A)^{(-1)}_{(T)}) = T$ and $N(X) \supseteq N((BB^*)^{(-1)}_{(S^\perp)}) = S$.

Finally, we prove the uniqueness. If X_0 is a solution of (18) satisfying (19), then

$$\begin{aligned} X &= (A^*A)^{(-1)}_{(T)}A^*DB^*(BB^*)^{(-1)}_{(S^\perp)} = (A^*A)^{(-1)}_{(T)}A^*AX_0BB^*(BB^*)^{(-1)}_{(S^\perp)} \\ &= (A^*A)^{(2)}_{T,T^\perp}(A^*A)X_0(BB^*)(BB^*)^{(2)}_{S^\perp,S} = P_{T,A^*AT}X_0P_{BB^*S^\perp,S} = X_0, \end{aligned}$$

since $R(X_0) \subseteq T$ and $N(X_0) \supseteq S$. $\square$

In particular, when $m = n$ and $p = q$, i.e., A and B are square, if $\mathrm{Ind}(A) = k$ ($\mathrm{rank}(A^k) = r < m$) and $\mathrm{Ind}(B) = \tilde{k}$ ($\mathrm{rank}(B^{\tilde{k}}) = \tilde{r} < p$), then we have the following results, noting that $R(A^k) \oplus N(A^k) = C^m$ and the Drazin inverse $A_d = A^{(2)}_{R(A^k),N(A^k)}$.

Corollary 2. *Given the square matrices A and B as above, the matrix equation*

$$AXB = D,$$

with the constraints

$$R(X) \subseteq R(A^k) \quad \text{and} \quad N(X) \supseteq N(B^{\tilde{k}}),$$

has the unique solution

$$X = A_d D B_d,$$

if $D \in R(AG, \widetilde{G}B)$ *for some matrices* $G \in C^{m\times m}$ *and* $\widetilde{G} \in C^{p\times p}$ *such that*

$$R(G) = R(A^k), \qquad N(G) = N(A^k),$$

and

$$R(\widetilde{G}) = R(B^{\tilde{k}}), \qquad N(\widetilde{G}) = N(B^{\tilde{k}}).$$

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