An Empirical Study of Algebraic-Reconstruction Techniques

Ehsan Mohamad Kazemi  
Computing and Software  
McMaster University  
Hamilton, Ontario, Canada  
email: mohame3@mcmaster.ca

Nedialko S. Nedialkov  
Computing and Software  
McMaster University  
Hamilton, Ontario, Canada  
email: nedialk@mcmaster.ca

Abstract—We study empirically four algebraic-reconstruction techniques (ART methods): Kaczmarz’s method, Symmetric Kaczmarz, Randomized Kaczmarz, and Simultaneous ART (SART). The performance of these methods depends on the choice of a relaxation parameter. The subject of our study is the role of this parameter with respect to their performance.

Section II gives a brief overview of these methods. Section III discusses the relaxation parameter. Section IV outlines our heuristic. Section V presents numerical results. Conclusions are in Section VI.

II. METHODS OVERVIEW

a) Kaczmarz’s method: This method starts with an initial guess for \( x \), a vector of prospective pixel values, and then computes the \( i \)th component of \( x \) on the \( k \)th iteration by

\[
x^{k,i} = x^{k,i-1} - \frac{r_i^T x^{k,i-1} - b_i}{r_i^T r_i} r_i,
\]

where \( r_i \) is the \( i \)th row of \( A \), and \( \lambda_{i,k} \) is a relaxation parameter for which \( 0 < \lambda_{i,k} < 2 \). Here, on each iteration \( k \) and substep \( i \), the last iterate is projected orthogonally onto a hyperplane associated with row \( r_i \). On each iteration, the sweep through the rows is from row 1 to row \( m \).

b) Symmetric Kaczmarz: In the symmetric Kaczmarz’s method [6], on each iteration given by (1), the equations are processed in order \( 1, \ldots, m \) and then from \( m-1 \) down to 1, then from 2 to \( m \) and so own.

c) Randomized Kaczmarz: In this method [7], the index \( i \), indicating the projection number, is chosen randomly from \( \{1, \ldots, m\} \), where the \( i \)th equation is selected with probability proportional to \( ||r_i||^2 \).

d) SART: This method was developed as a major refinement of the ART method [8]. The idea behind SART is that it considers a subset of the ray sums in the projection matrix, which is related to a specific angle. This method can generate a good reconstruction in only one iteration, and has a computational advantage over the traditional implementation of ART [8]. In fact, SART, unlike Kaczmarz’s method that updates the solution \( x \) in each projection, only updates the solution per iteration after computing all the projections of the current solution.

This method can be written in the form [8],

\[
x_j^k = x_j^{k-1} - \frac{\lambda}{A_{:,j}} \sum_{i=1}^{M} A_{i,j} (A_{i,:} \cdot x^{k-1} - b_i),
\]
where $A_{i,j}$, $A_{i,:}$, and $A_{:,j}$ denote the coefficients of the matrix $A$, its row sums, and its column sums, respectively.

III. On the Relaxation Parameter

As introduced in the variations of Kaczmarz’s method, we need a relaxation parameter for each iteration of an ART method. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images. If this parameter is carefully adjusted to the reconstruction procedure, it can produce efficiently high-quality images.

On the one hand, larger values of $\lambda$ in the interval $(0, 2)$ lead to faster convergence of ART, but also to noisy reconstruction images; on the other hand, smaller values of $\lambda$ lead to smoother images, but slower convergence [10], [11]. Obviously, the value of $\lambda$ need to be chosen to ensure fast convergence and good image quality.

However, it is not an easy task to find the relaxation parameter that best fits the reconstruction process, and there is no unique choice for the best relaxation parameter [11]. Generally speaking, the choice of relaxation parameter depends on [12]:

- Medical purpose of reconstruction
- Method of X-ray data acquisition
- Existence of noise on the measurements
- Number of iterations that we intend to do

Although several approaches have been proposed for finding an optimal value of $\lambda$, no specific method is generally accepted (see e.g. [13], [14], [15] and the references there in). In [15], two strategies are introduced to choose this parameter, but they are derived for Simultaneous Iterative Reconstruction Techniques (SIRT) [11] methods and are not suitable for ART methods.

An excellent work on choosing a relaxation parameter is Hansen’s strategy [13]. It proposes a training method for finding a value of the relaxation parameter ($\lambda$) for two of the ART methods, Kaczmarz and Symmetric Kaczmarz that gives rise to fast convergence and small relative error in the solution.

This strategy includes two parts. The first part determines a resolution limit (how accurate a solution can get), and the second part determines the optimal value of $\lambda$, with which the the resolution limit is reached using fewest number of iterations. In the second part, a modified version of the golden section search is used to find the value of $\lambda$.

IV. A Simple Heuristic for Finding $\lambda$

In this study, we use the following simple heuristic. For each of the four methods, we run the first iteration with different values for $\lambda$ between 0 and 2 and measure the error corresponding to each $\lambda$. Then, we use spline interpolation to interpolate the error versus $\lambda$, and find the global minimum of this interpolant. This minimum is the relaxation parameter for the first iteration. We repeat the same procedure on each iteration. We refer to the relaxation parameter determined on iteration $i$ as $\lambda_i$.

For a given image, one can generate such a sequence of $\lambda_i$’s, which can then be used to reconstruct images of the same type (e.g. knee images). In our experiments, we take the average of these $\lambda_i$ over all the iterations, and use this average as a constant relaxation parameter over all iterations.

Although very simple, and without theory to back it up, this approach led to some interesting results, which we report and discuss in the next section.

V. Numerical Results

A. Phantoms

For the results presented here, we use the Shepp-Logan phantom the para-sagittal MRI scan of a head and the sagittal MRI scan image of a knee, see Figure 1.

The structure of the Shepp-Logan phantom, Fig. 1(a) is defined by mathematical formulas [16], and it is one of the standard phantoms for testing. The other two phantoms are from real MRI images.

B. Measuring Image Quality

For a given phantom, denote the vector of true pixel values by $x$ and the vector of computed pixel values of the reconstructed image by $\tilde{x}$. To measure its quality, we use three types of measurements:

1) relative error $\|x - \tilde{x}\|/\|x\|$,  
2) relative residual $\|b - A\tilde{x}\|/\|b\|$, and the  
3) Colsher’s discrepancy metric [17]  
   $$\delta = \sqrt{\frac{\sum (\tilde{x}_i - x_i)^2}{\sum (x_i - \bar{x})^2}},$$
   where $\bar{x}$ is the mean value of the $x_i$’s. (The denominator is the standard deviation from the true image, and the numerator is the root mean square error of the reconstructed image.) Generally, the closer $\delta$ is to zero, the smaller is the discrepancy between the reconstructed image and the original phantom [17].

1) Image sizes: We use images of size $128 \times 128$, $256 \times 256$, and $512 \times 512$ for each phantom. We simulated X-ray data with 180 angles, $\theta = 1, 2, \ldots, 179$, and 100 projections per angle. The systems corresponding to $128 \times 128$ are overdetermined, and the systems corresponding to $256 \times 256$ and $512 \times 512$ are underdetermined.

C. Results

1) Relaxation parameter: In Tables I and II we report the $\lambda$’s determined by our heuristic on the Head phantom over 20 iterations and image sizes 256 and 512, respectively. For the four methods and the two image sizes, the averages of the $\lambda$’s are very similar.

In Table III we show the average $\lambda$’s found by our strategy for all of the above phantoms and methods. It is interesting to note that these values are close to each other, irrespective of the size and type of image. For a comparison, we give in Table IV the $\lambda$’s selected by Hansen’s approach, where we cannot make a similar conclusion.

http://en.wikipedia.org/wiki/Neuroimaging
2) Image quality: The corresponding image qualities, produced by both methods, are nearly the same, as Table I suggests. Here, we report the relative errors, relative residuals, and the discrepancies for the \( \lambda \)'s selected by our and Hansen's methods. Except for SART, where the measurements corresponding to our \( \lambda \) selection are worse compared to Hansen's \( \lambda \) selection, the rest are nearly the same.

To our surprise, for completely different phantoms, our strategy suggests nearly the same relaxation parameter for each of the methods.

Furthermore, we find similar values for \( \lambda \) on the phantoms in Figures 2 and 3; see Table II.

Fig. 1. Test Phantoms.
Experimental results of applying our strategy for finding a "good" value for this parameter does not depend on the type of the image and the image size. In particular, it seems that good value for $\lambda$ would be

method $\lambda$
Kaczmarz 0.27
Symmetric 0.2
Randomized 1.0
SART 1.93

Further, more rigorous investigations would be desirable to confirm our observations.

ACKNOWLEDGMENT

This work is supported in part by the Natural Sciences and Engineering Research Council of Canada. The authors thank Christopher Anand (McMaster University) for pointing out the phantoms in Figures 2 and 5.

REFERENCES


Fig. 2. Heart and head phantoms


Fig. 3. Cardiac and breast phantoms