Remarks on Mereology of Direct Products and Relations

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Abstract

The concept of being a *part of* for direct products and relations is analysed. Some operations based on this concept are introduced and analysed.

1 Introduction

One of the roles of mathematics is to provide formal definitions for the concepts that are intuitively well understood for many special instances, but their general and precise definition is by no means obvious or ever informally possible (for example *continuity*). One of the concepts that is frequently used in computer science, on various levels of abstractions, is the concept of being a "part of". It should not surprise anyone since practically every specification techniques uses the notion of "modularity". The concepts "whole" and "part of" abound in human experience but their fully adequate conceptualization has yet eluded our most able thinkers. The relation *part of* is the basic notion of *Leśniewski's Mereology* [15, 16], which is a version of set theory proposed as an antinomy-free counterpart of naive Cantor set theory. Leśniewski's systems are different from the standard set theory based on Zermelo-Fraenkel axioms[13].

The relation *part of* was also a partial motivation for introduction the cylindric algebras ("*a circle is a part of a cylinder*", see [6]), but this concept never become a formal part of cylindric algebras[1] Unfortunately the formal translation of Leśniewski's ideas into the standard set theory framework is not obvious, although

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[1]Tarski, a student of Leśniewski, was involved in the development of Leśniewski's mereology in an early stage. He presented an axiomatization of n-dimensional geometry on the basis of Leśniewski's "part of" and the notion of a sphere [17]. However the relation "part of" does not appear explicitly in cylindric algebras.

possible [16], but the results are not easy to read and apply. We are much less ambitious. We believe that eventually the concept of being "*part of*" should be defined and analysed in the most general settings, but in this paper we restrict our attention to direct products and relations (in the standard framework of Zermelo-Fraenkel axioms).

Main motivation for this work was provided by an attempt to define a formal semantics for *tabular expressions* [9, 10, 11]. Tabular expressions (Parnas et al. [1, 12, 14]) are means to represent the complex relations that are used to specify and document software systems. When software engineers discuss a specification using *tabular expressions*, the statements like "this is *a part of* a bigger relation", "this relation is composed of the following *parts*", etc., can be heard very often.

Basic concepts of tabular expressions and their semantics are presented in the next section of this article.

Unfortunately, the only meaning of "being a part of", can so far be only an intuitive one, since the standard algebra of relations lacks the formal concept of being a *part of* concept. The concept of subset is not enough, for instance if $A \subseteq B$ and $D = B \times C$, then A is not a subset of D, but according to standard intuition it is a *part of* D.

One of the biggest advantages of tabular expression technique is the ability to define a relation R that describes the properties of the system specified, as an easy to understand composition[2] of the relations R_α, $\alpha \in I$, where R_α is a *part of* R.

We believe the relation $\sqsubseteq$ considered in this paper can be seen as a special case of Leśniewski's *part of*, intuitively it seems to have most of the required properties.

Our approach is based on the two observations: if $A \subseteq B$ then intuitively A is a part of B, and A is intuitively a part of $A \times B$, i.e. both subset and projection model properly the concept of part of in some particular circumstances. Thus we may try to define the concept of "part of" for relations as a composition of the concept of a subset and a projection. These lead us to the concept of the relation $\sqsubseteq$ (defined first in [10]).

In this paper we will analyse the basic properties of the relation $\sqsubseteq$, as well as some operations that have been influenced by $\sqsubseteq$.

It turns out that the structures generated by the relation $\sqsubseteq$ are quite complex and irregular. To model them we will introduce the concept of a *weak lattice* first. Next we analyse the properties of $\sqsubseteq$ for direct products and then for relations.

2 Tabular Expressions and Their Relational Semantics

Consider the following relation $G \subseteq \mathbf{IN} \times \mathbf{OUT}$, where $\mathbf{IN} = Reals \times Reals$, $\mathbf{OUT} = Reals \times Reals \times Reals$, x_1, x_2 are the variables over $\mathbf{IN}$, y_1, y_2, y_3 are

[2]The word "composition" here means "the act of putting together" (*Oxford English Dictionary*, 1990), not "the" composition of relations that is usually denoted by ";" or "$\circ$" ([2, 13]. In this sense "$\cup$" is a composition.

	$x_2 \le 0$	$x_2 > 0$
$y_1 =$	$x_1 + x_2$	$x_1 - x_2$
$y_2\vert$	$y_2 x_1 - x_2 = y_2^2$	$x_1 + x_2 y_2 = \vert y_2\vert$
$y_3\vert$	$y_3 + x_1 x_2 = \vert y_3\vert^3$	$y_3 = x_1$

Figure 1: The relation G defined by a *vector table*. The symbol "=" after y_1 indicates that the value of y_1 is a function of other variables, the symbol "|" after y_2 and y_3 indicates that the relationship between y_i, $i = 1, 2$, is relational and not functional.

	go sailing	gtb	gtb	pb	garden
Temperature $\in \{hot, cool\}$	$*$	$*$	*hot*	$*$	*cool*
Weather $\in \{sunny, cloudy, rain\}$	$sunny \vee cloudy$	*sunny*	*cloudy*	*rain*	*cloudy*
Windy $\in \{true, false\}$	*true*	*false*	*false*	$*$	*false*

$* =$ *don't care*, gtb = go to the beach, pb = play bridge

Figure 2: The function φ defined by a *decision table*.

variables over **OUT**, and

$$(x_1, x_2)G(y_1, y_2, y_3) \iff \begin{cases} \left.\begin{array}{l} y_1 = x_1 + x_2 \ \wedge \ y_2 x_1 - x_2 = y_2^2 \\ \wedge \ y_3 + x_1 x_2 = |y_3|^3 \end{array}\right\} & \text{if } x_2 \le 0 \\ \left.\begin{array}{l} y_1 = x_1 - x_2 \ \wedge \ x_1 + x_2 + x_2 y_2 = |y_2| \\ \wedge \ y_3 = x_1 \end{array}\right\} & \text{if } x_2 > 0 \end{cases}$$

The relation G is more readable defined by a *tabular expression* in Figure 1. This kind of tabular expression is called a *vector table* and its intuitive meaning is practically self-explained. It reads that if $x_2 \le 0$ then $y_1 = x_1 + x_2$, y_2 must satisfy $y_2 x_1 - x_2 = y_2^2$, y_3 must satisfy $y_3 + x_1 x_2 = |y_3|^3$, and similarly for $x_2 > 0$.
The tabular expression in Figure 2 defines the function

$$\varphi : \text{Temperature} \times \text{Weather} \times \text{Windy} \to \mathit{Activities},$$

where *Activities* = { go sailing, go to the beach, play bridge, garden}. This tabular expression is called a *decision table*, and such tables have been used as a specification tools since fifties [7].
The relation G is a composition of its "atomic" parts, i.e. $G = \bigcirc G_{i,j}$, where $G_{i,j}$, $i = 1, 2$, $j = 1, 2, 3$, are relations defined by expressions in single cells. For instance $G_{1,3} \subseteq IN_{1,3} \times OUT_{1,3}$, where $IN_{1,3} = Reals \times (-\infty, 0\rangle$, $OUT_{1,3} = Reals$, and

$$(x_1, x_2)G_{1,3}y_3 \iff y_3 + x_1 x_2 = |y_3|^3.$$

The relation $G_{1,1}$ is a function $G_{1,1} : IN_{1,1} \rightarrow OUT_{1,1}$, with $IN_{1,1} = IN_{1,3}$ and $OUT_{1,1} = Reals$, and

$$(x_1, x_2)G_{1,1}y_1 \iff y_1 = G_{1,1}(x_1, x_2) = x_1 + x_2.$$

The relations $G_{i,1}$ are functions, which is indicated by the symbol "=" after variable y_1 in the left header. The symbol "|" after y_2 and y_3 indicates that $G_{i,2}$ and $G_{i,3}$ are relations with y_2 and y_3 as a respective range variables.

The function φ is also a composition of "atomic" $\varphi_{i,j}$, $i = 1, ..., 5$, $j = 1, 2, 3$, so $\varphi = \tilde{\bigcirc}\varphi_{i,j}$. For instance $\varphi_{2,2} : \{sunny\} \rightarrow \{\text{go to the beach}\}$, $\varphi_{2,2}(sunny) =$ go to the beach. The domain of $\varphi_{2,2}$ is $\{sunny\}$ rather than $\{sunny, cloudy, rain\}$ since we prefer to deal with total functions.

Of course every $G_{i,j}$ is a part of G, every subset of G is a part of G, every relation defined by a tabular expression derived from the tabular expression that defines G by removing any number of rows and columns, is also a part of G. Similarly for φ. There are many types of tabular expressions, we gave simple examples of only two of them. For all tabular expressions, the global relation/function is defined as a composition of its parts, for each type of tabular expressions, an appropriate composition operation is different. The concept of "part of" is essential for specification techniques based on tabular expressions. For more details the reader is referred to [1, 9, 11, 12].

3 Weak Lattices

Let $(X, \preceq)$ be a poset and let $A \subseteq X$. An element $a \in X$ is called *upper bound*) (*lower bound*) of A iff $\forall x \in A.\ x \preceq a$ ($\forall x \in A.\ a \preceq x$).

Let ub A (lb A) denote the set of all *upper bounds* (*lower bounds*) of A.

A poset $(X, \preceq)$ is called *bounded* if there are $\top, \bot \in X$ such that $\{\top\} = \text{ub}\, X$ and $\{\bot\} = \text{lb}\, X$.

The element $\top$ is called the *top* of X, and the element $\bot$ is called *bottom* of X.

A set $A \subseteq X$ is called a *chain* iff $\forall a, b \in A.\ a \preceq b \vee b \preceq a$.

An element $a \in A$ is a *minimal* (*maximal*) element of A iff $\forall x \in A.\ \neg(x \prec a)$ ($\forall x \in A.\ \neg(a \prec x)$).

Let $\min A$ ($\max A$) denote the set of all *minimal* (*maximal*) elements of A.

The set $\min A$ is *complete* iff $\forall x \in A. \exists a \in \min A.\ a \prec x$.

The set $\max A$ is *complete* iff $\forall x \in A. \exists a \in \max A.\ x \prec a$.

Let cmin A (cmax A) denote the *complete* set of all *minimal* (*maximal*) elements of A.

An element $a \in X$ is called the *least upper bound* (*supremum*) of A, denoted $\sup A$, iff $a \in \text{ub}\, A$ and $\forall x \in \text{ub}\, A.\ a \preceq x$.

An element $a \in X$ is called the *greatest lower bound* (*infimum*) of A, denoted $\inf A$, iff $a \in \mathrm{lb}\, A$ and $\forall x \in \mathrm{lb}\, A.\ x \preceq a$.

A poset $(X, \preceq)$ is called a *lattice* iff for all $a, b \in X$ both $\sup\{a, b\}$ and $\inf\{a, b\}$ do exists.

A poset $(X, \preceq)$ will be called *strongly complete* iff for every chain $A \subseteq X$, $\sup A$ and $\inf A$ do exist.

The concepts introduced above, except perhaps complete sets of minimal/maximal elements, are well known ([2, 13]).

For every set $A \subseteq X$, an element

$$a = \sup \mathrm{cmin}\ \mathrm{ub}\ A,$$

if exists, will be called the *weak supremum*, and denoted by $\mathrm{wsup}\, A$, and an element

$$a = \inf \mathrm{cmax}\ \mathrm{lb}\ A,$$

if exists, will be called the *weak infimum*, and denoted by $\mathrm{winf}\ A$.

Clearly, if $\sup A$ exists then $\sup A = \mathrm{wsup}\ A$, and if $\inf A$ exists the $\inf A = \mathrm{winf}\ A$, but it may happen that $\mathrm{wsup}\, A$ exists but $\sup A$ does not, or $\mathrm{winf}\, A$ exists, but $\inf A$ does not. Figure 3 shows such case.

If $\mathrm{cmin}\ \mathrm{ub}\ A = \{a\}$, then $a = \sup A$, and if $\mathrm{cmax}\ \mathrm{lb}\ A = \{a\}$, then $a = \inf A$. There are two reasons that $\sup A$ does not exists, either $\mathrm{cmin}\ \mathrm{ub}\ A$ does not exist, or it consists of more than one element. If $\mathrm{cmin}\ \mathrm{ub}\ A$ does not exist, neither $\sup A$ nor $\mathrm{wsup}\ A$ exists, if $\mathrm{cmin}\ \mathrm{ub}\ A$ has more than one element, $\mathrm{wsup}\ A$ could be seen as a close approximation of non-existent $\sup A$. Similarly for $\inf A$ and $\mathrm{winf}\ A$.

A poset $(X, \preceq)$ will be called a *weak lattice* if for every $a, b \in X$, both $\mathrm{wsup}\ \{a, b\}$ and $\mathrm{winf}\ \{a, b\}$ do exist. Clearly, every lattice is a weak lattice but not vice versa. The poset from Figure 3 is a weak lattice but not a lattice.

A weak lattice is called a *complete weak lattice* iff for every $A \subseteq X$, both $\mathrm{wsup}\, A$ and $\mathrm{winf}\, A$ do exist. Immediately from the definitions we conclude that if $\mathrm{wsup}\, X$ exists than $\mathrm{wsup}\ X = \sup X = \top$, and dually, if $\mathrm{winf}\ X$ exists, then $\mathrm{winf}\ X = \inf X = \bot$.

Let $a, b \in X$, and let $A \subseteq X$. We define:

$$a \sqcup b = \mathrm{wsup}\ \{a, b\},$$

$$a \sqcap b = \mathrm{winf}\ \{a, b\},$$

$$\sqcup_{a \in A} a = \mathrm{wsup}\ A$$

$$\sqcap_{a \in A} a = \mathrm{winf}\ A.$$

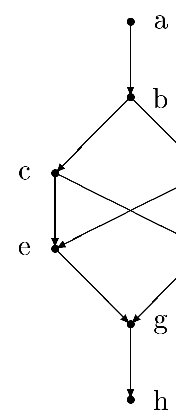

Figure 3: b = wsup $\{e, f\}$, g = winf $\{c, d\}$, sup$\{e, f\}$ does not exist, since cmin ub $\{e, f\} = \{c, d\}$, inf$\{c, d\}$ does not exist, since cmax lb $\{c, d\} = \{e, f\}$. This is a weak lattice but not a lattice.

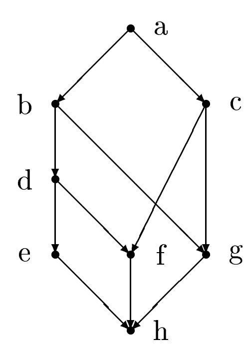

Figure 4: We have $e \sqcup f = d$, $d \sqcup g = b$, so $(e \sqcup f) \sqcup g = b$, but $f \sqcup g = a$, $e \sqcup a = a$, so $e \sqcup (f \sqcup g) = a$, i.e. $(e \sqcup f) \sqcup g \neq e \sqcup (f \sqcup g)$. Note that wsup $\{e, h, g\} = b$. By reversing arrows the same example works for "$\sqcap$".

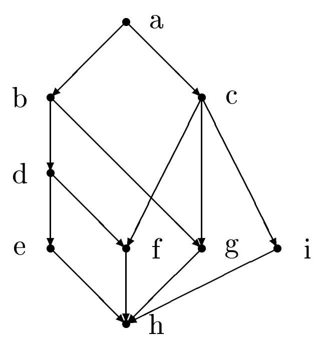

Figure 5: An example of an atomistic weak lattice which is not a lattice. Note that $\sup\{f, g\}$ does not exists, but every element different than h is either an atom, or can be constructed from atoms by using the operator "$\sqcup$".

Unfortunately it might happen that $a \sqcup (b \sqcup c) \neq (a \sqcup b) \sqcup c$, or $a \sqcap (b \sqcap c) \neq (a \sqcap b) \sqcap c$, as in the example in Figure 4.

Let $(X, \preceq)$ be a complete weak lattice. The minimal elements of $X \setminus \{\perp\}$ are called *atoms*.
Let $\mathcal{A}(X)$ be the least set satisfying the following properties:

- every atom of X belongs to $\mathcal{A}(X)$,
- for every $A \subseteq \mathcal{A}(X)$, wsup $A \in \mathcal{A}(X)$.

A complete weak lattice is called *atomistic* iff $X \setminus \{\perp\} = \mathcal{A}(X)$.
Neither the weak lattice from Figure 3 nor from Figure 4 is atomistic. For example for Figure 4, $\mathcal{A}(X) = \{a, b, d, e, f, g\}$, so $c \notin \mathcal{A}(X)$. An example of atomistic weak lattice which is not a lattice is given on Figure 5.

4 Partial Orders defined by *Part Of*

Let T be a universal set of indexes and let $\{D_t \mid t \in T\}$ be an appropriate set of domains. We assume also that the set T is *finite*. From the viewpoint of applications in Software Engineering this is not a restriction at all ([1, 11]). We conjecture the results below are true for any T, but there are some problems with proofs.
For every $I \subseteq T$, let $D_I = \prod_{i \in I} D_i$, where $\prod_{i \in I}$ is a *direct product* over I. In other words, the set D_I is the set of all functions $f : I \to \bigcup_{i \in I} D_i$ such that $\forall i \in I.\ f(i) \in$

D_i. For every function $f : X \to Y$, and every $Z \subseteq X$, the symbol $f|_Z$ will denote the restriction of f to Z. For every function f, $\mathrm{dom}\, f$ will denote the domain of f.

- Let I, J be subsets of T such that $I \subseteq J$, and let $A \subseteq D_I$, $B \subseteq D_J$. We define the relation $\sqsubseteq$ as:

$$A \sqsubseteq B \iff \forall f \in A.\ \exists g \in B.\ f = g|_I.$$

We shall say that A is a *part of* B iff $A \sqsubseteq B$.

Clearly if $I = J$ then $A \sqsubseteq B \iff A \subseteq B$. For example if $A \subseteq X_2 \times X_4$, and $B \subseteq X_1 \times X_2 \times X_3 \times X_4$, then

$$A \sqsubseteq B \iff \forall (x_2, x_4) \in A.\ \exists x_1 \in X_1,\ x_3 \in X_3.\ (x_1, x_2, x_3, x_4) \in B.$$

More precisely we define $\sqsubseteq$ as follows:

$$\sqsubseteq \subseteq \bigcup_{I \subseteq T} \prod_{i \in I} D_i \times \bigcup_{I \subseteq T} \prod_{i \in I} D_i,$$

and

$$A \sqsubseteq B \iff (\exists I, J \subseteq T.\ I \subseteq J \wedge A \subseteq D_I \wedge B \subseteq D_J)\ \wedge\ (\forall f \in A.\ \exists g \in B.\ f = g|_I).$$

From here on we will assume that $D_i \cap D_j = \emptyset$ if $i \neq j$. This assumption allows us to identify every element of $a \in D_i$ with the function $f_a : \{i\} \to D_i$ where $f_a(i) = a$, which makes the notation more consistent and less ambiguous. We do not lose any generality here, moreover, in practical applications each D_i has a different interpretation anyway (for instance: input current, output current; Amperes in both cases but different meaning).

- For every D_I, let $\mathcal{P}_{\sqsubseteq}(D_I) = \{A \mid A \sqsubseteq D_I\}$.

Note that for every $A \in \mathcal{P}_{\sqsubseteq}(D_T)$ and for every $f, g \in A$ we have $\mathrm{dom}\, f = \mathrm{dom}\, g$.

Theorem 4.1 *$(\mathcal{P}_{\sqsubseteq}(D_I), \sqsubseteq)$ is a partial order.* ■

A standard proof is left to the reader, however, in general, $(\mathcal{P}_{\sqsubseteq}(D_I), \sqsubseteq)$ *is NOT a lattice.* To show it let us take $I = \{1, 2\}$ and $D_I = \{a, b\} \times \{1, 2\}$. One may notice that $\{(a, 1), (b, 2)\} \in \mathrm{cmin\ ub}\{\{a, b\}, \{1, 2\}\}$, and $\{(a, 2), (b, 1)\} \in \mathrm{cmin\ ub}\{\{a, b\}, \{1, 2\}\}$,

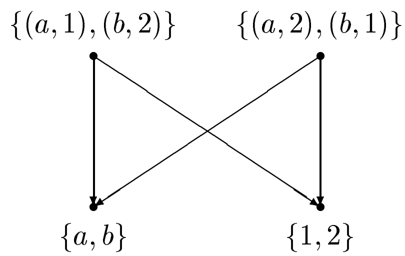

Figure 6: Part of the poset $(\mathcal{P}_{\sqsubseteq}(\{a,b\} \times \{1,2\}), \sqsubseteq)$ illustrating that this poset is not a lattice.

so $\sup\{\{a,b\},\{1,2\}\}$ does not exist. Similarly $\inf\{\{(a,1),(b,2)\},\{(a,2),(b,1)\}\}$ does not exist. Figure 6 illustrates this example.
We will prove that $(\mathcal{P}_{\sqsubseteq}(D_I), \sqsubseteq)$ is a bounded and strongly complete poset and an atomistic weak lattice.

For every $A \in \mathcal{P}_{\sqsubseteq}(D_T)$, let $\tau(A) \subseteq T$, the *index set* of A, be defined as follows: $I = \tau(A) \iff A \subseteq D_I$. In other words, $f \in A \Rightarrow \mathrm{dom}\, f = \tau(A)$.
For every $A \in \mathcal{P}_{\sqsubseteq}(D_T)$ and every $K \subseteq T$, let $A|_K = \{f|_{K\cap\tau(A)} \mid f \in A\}$ if $K \cap \tau(A) \neq \emptyset$, and $A|_K = \emptyset$ if $K \cap \tau(A) = \emptyset$. Clearly $A|_K \subseteq D_{K\cap\tau(A)}$. We will write $A|_i$ instead of $A|_{\{i\}}$ for all $i \in T$.

We may now describe $\sqsubseteq$ is a global manner, namely:

- $A \sqsubseteq B \iff \tau(A) \subseteq \tau(B) \ \wedge \ A \subseteq B|_{\tau(A)}$.

Theorem 4.2 *A poset $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ is bounded and strongly complete.*

Proof. From Theorem 4.1 it follows it is a poset. Clearly $\top = D_T$ and $\bot = \emptyset$, so the poset is bounded. *The next part of the proof uses the assumption that T is finite.*
Let $\mathcal{C} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$ be a nonempty chain, and let $K = \bigcup_{A\in\mathcal{C}} \tau(A)$, $J = \bigcap_{A\in\mathcal{C}} \tau(A)$. Since T is finite, there are $A_0, A_1 \in \mathcal{C}$ such that $\tau(A_0) = J$ and $\tau(A_1) = K$. Clearly for every $A \in \mathcal{C}$, we have $A \sqsubseteq A_0 \ \Rightarrow \ \tau(A) = J$ and $A_1 \sqsubseteq A \ \Rightarrow \ \tau(A) = K$. Define $\mathcal{C}_0 = \{A \in \mathcal{C} \mid A \sqsubseteq A_0\}$, and $\mathcal{C}_1 = \{A \in \mathcal{C} \mid A_1 \sqsubseteq A\}$. Note that if $\sup \mathcal{C}_1$ exists then $\sup \mathcal{C}$ exists and $\sup \mathcal{C}_1 = \sup \mathcal{C}$. Similarly, if $\inf \mathcal{C}_0$ exists then $\inf \mathcal{C}$ exists and $\inf \mathcal{C}_0 = \inf \mathcal{C}$. But for every $A \in \mathcal{C}_1$, $\tau(A) = K$, hence for every $A, B \in \mathcal{C}_1$, $A \sqsubseteq B \iff A \subseteq B$, which means $\bigcup_{A\in\mathcal{C}_1} A = \sup \mathcal{C}_1$. Similarly, $\bigcap_{A\in\mathcal{C}_0} A = \inf \mathcal{C}_0$. Thus $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ is strongly complete. ∎

Before proving that $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ is an atomistic complete weak lattice, we show some simple properties of $\sqsubseteq$.

Lemma 4.3

1. $A \subseteq B \Rightarrow A \sqsubseteq B$
2. $A = B|_{tau(A)} \Rightarrow A \sqsubseteq B$
3. $J \subseteq K \subseteq T \Rightarrow A|_J \sqsubseteq A|_K$, *for all* $A \in \mathcal{P}_{\sqsubseteq}(D_T)$.
4. *Let* $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$ *and for all* $A, B \in \mathcal{A}$, $\tau(A) = \tau(B)$. *Then* $\sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$.
5. *Let* $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$ *and for all* $A, B \in \mathcal{A}$, $\neg(\tau(A) \subseteq \tau(B) \vee \tau(B) \subseteq \tau(A))$. *Then:*

   a. *for all* $A, B \in \mathcal{A}$, $\neg(A \sqsubseteq B \vee B \sqsubseteq A)$.

   b. $\inf \mathcal{A} = \bigcap_{a \in \mathcal{A}} A|_J$, *where* $J = \bigcap_{a \in \mathcal{A}} \tau(A)$.

Proof. (1), (2), (3), (4) and (5a) follow directly from the definition of $\sqsubseteq$. Let us prove (5b). Clearly $\bigcap_{A \in \mathcal{A}} A|_J \sqsubseteq A$ for every $A \in \mathcal{A}$, so $\bigcap_{A \in \mathcal{A}} A|_J \in \mathrm{lb}\, \mathcal{A}$. Let $C \in \mathrm{lb}\, \mathcal{A}$, i.e. $C \sqsubseteq A$ for every $A \in \mathcal{A}$. This means $\tau(C) \subseteq \tau(A)$ for every $A \in \mathcal{A}$, which means $\tau(c) \subseteq J$. Hence for every $A \in \mathcal{A}$, we have $C \subseteq A|_{\tau(C)} \sqsubseteq A|_{\tau(J)}$, i.e. $C \subseteq \bigcap_{A \in \mathcal{A}} A|_{\tau(C)} \sqsubseteq \bigcap_{A \in \mathcal{A}} A|_J$, so $\bigcap_{A \in \mathcal{A}} A|_J = \inf \mathcal{A}$. ■

Theorem 4.4 *A poset* $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ *is an atomistic complete weak lattice, and for each* $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$:

$$\mathrm{wsup}\, \mathcal{A} = \bigcup_{A \in \mathrm{cmin\ ub}\, \mathcal{A}} A,$$

$$\mathrm{winf}\, \mathcal{A} = \bigcap_{A \in \mathrm{cmax\ lb}\, \mathcal{A}} A|_J,$$

where $J = \bigcap_{A \in \mathrm{cmax\ lb}\, \mathcal{A}} \tau(A)$.

Proof. Let $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(T_T)$.
(1)First we prove that $\mathrm{wsup}\, \mathcal{A} = \sup \mathrm{cmin\ ub}\, \mathcal{A}$ exists. Let $C \subseteq \mathrm{ub}\, \mathcal{A}$ be a chain. Reasoning in an almost identical manner as in the proof of Theorem 4.2 (assumption "T is finite" is essential here) we can show that $\inf C$ exists and $\inf C \in \mathrm{ub}\, A$. Hence by (a dual version) of Kuratowski-Zorn Lemma ([13]) we have that $\min \mathrm{ub}\, \mathcal{A}$ exists and for every $A \in \mathrm{ub}\, \mathcal{A}$ there is $B \in \min \mathrm{ub}\, \mathcal{A}$ such that $B \sqsubseteq A$. In other words $\mathrm{cmin\ ub}\, \mathcal{A}$ exists. Let $K = \bigcup_{A \in \mathcal{A}} \tau(A)$. We prove that for every $A \in \mathrm{cmin\ ub}\, \mathcal{A}$, $\tau(A) = K$. Clearly for every $B \in \min \mathrm{ub}\, \mathcal{A}$, $K \subseteq \tau(B)$. Let $B \in \min \mathrm{ub}\, \mathcal{A}$ and $K \neq \tau(B)$. Define $B' = B|_K$. We show that $B' \in \mathrm{ub}\mathcal{A}$. Let $A \in \mathcal{A}$. Then $A \sqsubseteq B$, i.e. $\tau(A) \subseteq \tau(B)$ and $A \subseteq B|_{\tau(A)}$. Since $\tau(A) \subseteq K$ then $A \sqsubseteq B|_K = B'$, so $B' \in \mathrm{ub}\, \mathcal{A}$. But $B' \sqsubseteq B$ and $B' \neq B$, a contradiction since we assumed $B \in \min \mathrm{ub}\, \mathcal{A}$. Thus for every $A \in \mathrm{cmin\ ub}\, \mathcal{A}$, $\tau(A) = K$. From Lemma 4.3(4) we have $\sup \mathrm{cmin\ ub}\, \mathcal{A} =$

$\bigcup_{A\in \text{cmin ub}\,\mathcal{A}} A$. Hence wsup $\mathcal{A}$ exists and wsup $\mathcal{A} = \bigcup_{A\in \text{cmin ub}\,\mathcal{A}} A$,.
(2)Now we prove that winf $\mathcal{A} = \inf \text{cmax lb}\,\mathcal{A}$ exists. Similarly like in (1) above we prove that cmax lb $\mathcal{A}$ exists. Note that cmax lb $\mathcal{A}$ satisfies the conditions of Lemma 4.3(5), so from Lemma 4.3(5b) we have $\inf \text{cmax lb}\,\mathcal{A} = \bigcap_{A\in \text{cmax lb}\,\mathcal{A}} A|_J$, where $J = \bigcap_{A\in cmax\text{lb}\,\mathcal{A}} \tau(A)$. Hence winf $\mathcal{A}$ exists and winf $\mathcal{A} = \bigcap_{A\in \text{cmax lb}\,\mathcal{A}} A|_J$,. Thus $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ is a complete weak lattice.
(3)We now prove that it is also atomistic. Let $A \in \mathcal{P}_{\sqsubseteq}(D_T)$. Clearly $A = \bigcup_{f\in A}\{f\} = \bigsqcup_{f\in A}\{f\}$. For every $f \in \mathcal{F}_{\sqsubseteq}(D_T)$ and every $i \in T$, let f_i be the following function: $f_i : \{i\} \to D_i$ and $f_i(i) = f(i)$. Note that $\{f_i\}$ are atoms of $\mathcal{P}_{\sqsubseteq}(D_T)$, $i \in T$, and that for every $A \in \mathcal{P}_{\sqsubseteq}(D_T)$, and every $f \in A$, we have $f = \bigsqcup_{i\in\tau(A)}\{f_i\}$. Hence $A = \bigsqcup_{f\in A}\bigsqcup_{i\in\tau(A)}\{f_i\}$, which means $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$ is atomistic. ■

The below result that will be used later follows directly from the proof of Theorem 4.4.

Corollary 4.5 *For each* $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$

1. $\min \text{ub}\,\mathcal{A}$ *is complete,*
2. $\max \text{lb}\,\mathcal{A}$ *is complete,*
3. *for every* $A, B \in \text{cmin ub}\,\mathcal{A}$, $\tau(A) = \tau(B)$. ■

5 Operations Generated by ***Part of***

The fundamental principle behind a successful Tabular Expressions ([9, 12]) specification technique is that most of relations may be described as $R = \bigcirc_{i\in I} R_i$, where $\bigcirc$ is an operation, or composition of operations, each R_i is easy to specify. The survey of known tabular expressions used in Software Engineering [1] has shown that in almost all cases we have $R = \hat{\bigcirc}\tilde{\bigcirc} R_\alpha$, where "$\hat{\bigcirc}$" and "$\tilde{\bigcirc}$" are relational versions (see next chapter) of the the operations "$\uplus$", "$\oplus$" , "$\otimes$, and "$\odot$" defined below[3]. The operations "$\oplus$", "$\otimes$, and "$\odot$" have been introduced in the framework of tabular expressions in [10]. The operator $\otimes$ corresponds to the join operator of Codd's relational data-base model [3]. An operator corresponding to $\oplus$ is also known in the framework of cylindric algebras and data bases [4, 8]. The difference between our approach and that of [4, 8] and others, is that our operators were motivated by

[3]The paper [1] does not use those operations per se, as they had not been defined when it was written. It was the opposite way, the relational formulas used in [1] provided motivation and intuition that eventually led to the introduction of the operations "$\uplus$", "$\oplus$", "$\otimes$, and "$\odot$"

our "part of" relation, not by modelling the data base processing. In this section we will analyse those operations for weak lattices of direct products, weak lattices of relations will be discussed in the next section.

Let $A, B \in \mathcal{P}_{\sqsubseteq}(D_T)$ and let $K = \tau(A) \cup \tau(B)$, $J = \tau(A) \cap \tau(B)$. We define the operations "$\uplus$", "$\oplus$", "$\otimes$, and "$\odot$" as follows.

- $A \uplus B = \{f \mid \text{dom}\, f = K \wedge ((f|_{\tau(A)} \in A \wedge f|_{\tau(B)\setminus\tau(A)} \in B|_{\tau(B)\setminus\tau(A)}) \vee ((f|_{\tau(B)} \in B \wedge f|_{\tau(A)\setminus\tau(B)} \in A|_{\tau(A)\setminus\tau(B)}))\}$,
- $A \oplus B = \{f \mid \text{dom}\, f = K \wedge (f|_{\tau(A)} \in A \vee f|_{\tau(B)} \in B)\}$,
- $A \otimes B = \{f \mid \text{dom}\, f = K \wedge (f|_{\tau(A)} \in A \wedge f|_{\tau(B)} \in B)\}$,
- $A \odot B = \{f \mid \text{dom}\, f = J \wedge (\exists g \in A.\ g|_J = f \wedge \exists h \in B.\ h|_J = f)\}$.

Let $A \subseteq D|_{\{1,3,5\}}$, $B \subseteq D|_{\{1,2,4\}}$. Then

$$\begin{aligned} A \uplus B &= \{(x_1, x_2, x_3, x_4, x_5) \mid ((x_1, x_3, x_5) \in A \wedge (x_2, x_4) \in B|_{\{2,4\}}) \\ &\qquad \vee ((x_1, x_2, x_4) \in B \wedge (x_3, x_5) \in B|_{\{3,5\}})\}, \\ A \oplus B &= \{(x_1, x_2, x_3, x_4, x_5) \mid (x_1, x_3, x_5) \in A \vee (x_1, x_2, x_4) \in B\}, \\ A \otimes B &= \{(x_1, x_2, x_3, x_4, x_5) \mid (x_1, x_3, x_5) \in A \wedge (x_1, x_2, x_4) \in B\}, \\ A \odot B &= \{x_1 \mid x_1 \in A|_{\{1\}} \wedge x_1 \in B|_{\{1\}}\}. \end{aligned}$$

If $\tau(A) = \tau(B)$ then $A \uplus B = A \oplus B = A \cup B$, $A \otimes B = A \odot B = A \cap B$.

Let I be some index set, and let $\mathcal{A} = \{A_i \mid A_i \in \mathcal{P}_{\sqsubseteq}(D_T) \wedge i \in I\}$, $K = \bigcup_{i\in I} \tau(A_i)$, $J = \bigcap_{i\in I} \tau(A_i)$.

Let $\text{Comp}_i\mathcal{A}$ be the set of all the *components*[4] of K that are NOT contained in A_i.

For example if $I = \{1, 2\}$, $\tau(A_1) \setminus \tau(A_2) \neq \emptyset$ and $\tau(A_2) \setminus \tau(A_1) \neq \emptyset$ then there are three components of K generated by $\tau(A_1)$ and $\tau(A_2)$, namely $\tau(A_1) \cap \tau(A_2)t$, $\tau(A_1) \setminus \tau(A_2)$, and $\tau(A_2) \setminus \tau(A_1)$, so $\text{Comp}_1\{A_1, A_2\} = \{\tau(A_2) \setminus \tau(A_1)\}$, and $\text{Comp}_2\{A_1, A_2\} = \{\tau(A_1) \setminus \tau(A_2)\}$.

We define the operations "$\biguplus_{i\in I}$", "$\bigoplus_{i\in I}$" , "$\bigotimes_{i\in I}$", and "$\bigodot_{i\in I}$" as:

- $\biguplus_{i\in I} A_i = \{f \mid \text{dom}\, f = K \wedge \exists i \in I.\ (f|_{\tau(A)} \in A \wedge \forall C \in \text{Comp}_i\mathcal{A}.\ f|_C \in \bigcup_{i\neq i} A_j|_C)\}$.

[4]Let X be a set, $X_i \subseteq X$ for all $i \in I$. Define $X_i^0 = X_i$ and $X_i^1 = X \setminus X_i$. A nonempty set $A = \bigcap_{i\in I} X_i^{k_i}$, where $k_i = 0, 1$, is called a *component* of X generated by the sets X_i, $i \in I$. The components are disjoint and cover the entire set X (see [13]).

- $\bigoplus_{i\in I} A_i = \{f \mid \mathrm{dom}\, f = K \ \wedge\ \exists i \in I.\ f|_{\tau(A_i)} \in A_i\}$.
- $\bigotimes_{i\in I} A_i = \{f \mid \mathrm{dom}\, f = K \ \wedge\ \forall i \in I.\ f|_{\tau(A_i)} \in A_i\}$.
- $\bigodot_{i\in I} A_i = \{f \mid \mathrm{dom}\, f = J \ \wedge\ \forall i \in I.\, \exists g \in A_i.\ g|_K = f\}$.

Directly from the definitions we have:

Corollary 5.1

1. $A \uplus B = (A \otimes B|_{\tau(B)\setminus\tau(A)}) \cup (B \otimes A|_{\tau(A)\setminus\tau(B)})$.
2. $A \sqcup B = A \uplus B \sqsubseteq A \oplus B$.
3. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.
5. $A \odot (B \odot C) = (A \odot B) \odot C$,
6. $A \odot B = A|_{\tau(A)\cap\tau(B)} \cap B|_{\tau(A)\cap\tau(B)}$. ∎

We may now formulate the main result of this section.

Theorem 5.2

1. $A \uplus B = A \sqcup B = \mathrm{wsup}\,\{A, B\}$,
2. $\biguplus_{i\in I} A_i = \bigsqcup_{i\in I} A_i = \mathrm{wsup}\,\{A_i \mid i \in I\}$

Proof. Let $\mathcal{A} \subseteq \mathcal{P}_{\sqsubseteq}(D_T)$.
(1)The case (1) is a special case of (2) but we will prove it separately. This will make the proof of case (2) shorter and more intuitive as the methods of proving are based on the same idea. It is obviously true if $\tau(A) = \tau(B)$, since in such a case $A \uplus B = A \cup B = \sup\{A, B\} = \mathrm{wsup}\,\{A, B\}$. Assume that $\tau(A) \neq \tau(B)$. Let $f \in A$ and let $\phi \in B$. We define the function $f^\phi : \tau(A) \cup \tau(B) \to \bigcup_{i\in\tau(A)\cup\tau(B)} D_i$ in the following way:

$$f^\phi(i) = \begin{cases} f(i) & \text{if } i \in \tau(A) \\ \phi(i) & \text{if } i \in \tau(B) \setminus \tau(A) \end{cases}$$

For every $\phi \in B$, let $A^\phi = \{f^\phi \mid f \in A\}$. Similarly, for every $g \in B$, and every $\psi \in A$, we define the function $g^\psi : \tau(A) \cup \tau(B) \to \bigcup_{i\in\tau(A)\cup\tau(B)} D_i$ in the following way:

$$g^\psi(i) = \begin{cases} g(i) & \text{if } i \in \tau(B) \\ \psi(i) & \text{if } i \in \tau(A) \setminus \tau(B) \end{cases}$$

For every $\psi \in$, let $B^\psi = \{g^\psi \mid g \in B\}$. Note that $A^\phi \cup B^\psi \subseteq \prod_{i\in\tau(A)\cup\tau(B)} D_i$.

Obviously, every $A^\phi \cup B^\psi$ is an upper bound of $A \uplus B$. Let C be an upper bound of $A \uplus B$. If $\tau(C) \neq \tau(A) \cup \tau(B)$, then from the definition of $\sqsubseteq$ we have $\neg(C \sqsubseteq A^\phi \cup B^\psi)$, for all ϕ and ψ. Assume that $\tau(C) = \tau(A) \cup \tau(B)$, and that $C \sqsubseteq A^\phi \cup B^\psi$ for some ϕ and ψ. Since $\tau(C) = \tau(A) \cup \tau(B)$, then this is equivalent to $C \subseteq A^\phi \cup B^\psi$. Let $f \in A^\phi \cup B^\psi$ and $f \notin C$. Assume that $f \in A^\phi$ and $f \notin C$. This means that for every $g \in C$, there exists $i \in \tau(A) \cup \tau(B)$ such that $g(i) \neq \phi(i)$. But this contradicts the assumption $C \subseteq A^\phi \cup B^\psi$. Similarly if $f \in B^\psi$. Hence

$$\min \mathrm{ub}\,\{A, B\} = \{A^\phi \cup B^\psi \mid \phi \in B \ \wedge \ \psi \in A\}.$$

From Corollary 4.5(1) it follows that $\min \mathrm{ub}\,\{A, B\}$ is complete, so $\min \mathrm{ub}\,\{A, B\} = \mathrm{cmin\ ub}\,\{A, B\}$. Since $\tau(A^\phi \cup B^\psi) = \tau(A \uplus B) = \tau(A) \cup \tau(B)$, then

$$\sup \mathrm{cmin\ ub}\,\{A, B\} = \bigcup_{\phi\in B \ \wedge \ \psi\in A} (A^\phi \cup B^\psi).$$

But

$$\bigcup_{\phi\in B \ \wedge \ \psi\in A} (A^\phi \cup B^\psi) = A \uplus B$$

just from the definition of "$\uplus$", so we have proven that

$$A \uplus B = A \sqcup B = \mathrm{wsup}\,\{A, B\}.$$

(2)Let $\mathcal{A} = \{A_i \mid A_i \in \mathcal{P}_{\sqsubseteq}(D_T) \wedge i \in I\}$, where I is some index set. Let $K = \bigcup_{i\in I} \tau(A_i)$. A set of functions $\{\phi_C \mid C \in \mathrm{Comp}_i\mathcal{A}\}$, such that $\phi_C \in A_j$, some j, such that $C \subseteq \tau(A_j)$, is called a $\mathcal{A}$-*extension* of A_i. Let Φ be and $\mathcal{A}$-extension of A_i, and $f \in A_i$. We define the function $f^\Phi \in \prod_{i\in K} D_i$ as follows:

$$\forall i \in K.\ f^\Phi(i) = \begin{cases} f(i) & i \in \tau(A_i) \\ \psi_C(i) & i \in C \ \wedge \ \psi_C \in \Phi \end{cases}$$

Let $A_i^\Phi = \{f^\Phi \mid f \in A_i\}$. Clearly $\tau(A_i^\Phi) = K$. Consider the set

$$B^{\Phi_i} = \bigcup_{i\in I} A_i^{\Phi_i},$$

where Φ_i is an $\mathcal{A}$-extension of A_i. It may happen that $B^\Phi = B^\Psi$ even if $\mathcal{A}$-extensions are different, i.e. $\Phi \neq \Psi$. Reasoning in the similar manner as in the proof of (1), one may show that

$$\min \mathrm{ub}\,\mathcal{A} = \{B^{\Phi_i} \mid \Phi_i \text{ is an } \mathcal{A}\text{-extension of } A_i, \text{ and } i \in I\}.$$

From Corollary 4.5(1) it follows that $\min \mathrm{ub}\mathcal{A}$ is complete, so $\min \mathrm{ub}\mathcal{A} = \mathrm{cmin}\ \mathrm{ub}\mathcal{A}$. Since for all Φ_i, $\tau(B^{\Phi_i}) = K$, then

$$\sup\{B^{\Phi_i} \mid \Phi_i \text{ is an } \mathcal{A}\text{-extension of } A_i, \text{ and } i \in I\} = \bigcup_{\text{All } \Phi_i,\ i \in I} B^{\Phi_i}.$$

From the definition of "$\biguplus_{i\in I}$", we have that

$$\bigcup_{\text{All } \Phi_i,\ i \in I} B^{\Phi_i} = \biguplus_{i\in I} A_i,$$

which means

$$\mathrm{wsup}\,\mathcal{A} = \bigsqcup_{i\in I} A_i = \biguplus_{i\in I} A_i.$$

■

Unfortunately, it may happen that $A \uplus (B \uplus C) \neq (A \uplus B) \uplus C$. Consider $T = \{1,2\}$ and $D_T = \{a,b\} \times \{1,2\}$ and $A = \{a\}$, $B = \{1\}$ and $C = \{(b,2)\}$. We have $A \uplus (B \uplus C) = \{a\} \uplus (\{1\} \uplus \{(b,2)\}) = \{a\} \uplus \{(b,1),(b,2)\} = \{(a,1),(a,2),(b,1),(b,2)\}$, while $(A \uplus B) \uplus C = (\{a\} \uplus (\{1\}) \uplus \{(b,2)\} = \{(a,1)\} \uplus \{(b,2)\} = \{(a,1),(b,2)\}$.

We do not have any operational definition of "$\sqcap$", in the style of "$\uplus$", and the properties of "$\sqcap$" seem to occasionally be counter-intuitive. Consider again $D_T = \{a,b\} \times \{1,2\}$. One may prove that $\{(a,1),(b,1),(b,2)\} \sqcap \{(a,2),(b,1),(b,2)\} = \{b\}$, NOT equal to $\{(b,1),(b,2)\}$, as one might expect.

Lemma 5.3 *$A \sqcap B \sqsubseteq A \odot B$.*

Proof. By Theorem 4.4, we have $A \sqcap B = \mathrm{winf}\,\{A,B\} = \bigcap_{C \in \mathrm{cmax\ lb}\,\{A,B\}} C|_J$, where $J = \bigcap_{C \in \mathrm{cmax\ lb}\,\{A,B\}} \tau(C)$. Clearly each $C \in \mathrm{cmax}\ \{A,B\}$ satisfies $C \sqsubseteq A$ and $C \sqsubseteq B$, i.e. $\tau(C) \subseteq \tau(A) \cap \tau(B)$. Hence $J \subseteq \tau(A) \cap \tau(B)$. This means that $C|_J \sqsubseteq A|_{\tau(A)\cap\tau(B)}$, $C|_J \sqsubseteq B|_{\tau(A)\cap\tau(B)}$, and consequently $C|_J \sqsubseteq A|_{\tau(A)\cap\tau(B)} \cap B|_{\tau(A)\cap\tau(B)} = A \odot B$. Thus each $C \in \mathrm{cmax}\ \{A,B\}$ satisfies $C|_J \sqsubseteq A \odot B$, i.e $\bigcap_{C \in \mathrm{cmax\ lb}\,\{A,B\}} C|_J \sqsubseteq A \odot B$. ■

Lemma 5.4 *If $\tau(A) = \tau(B)$ then $A \otimes B = A \odot B = A \cap B \in \mathrm{cmax\ lb}\,\{A,B\}$.*

Proof. Suppose that there is C such that $A \cap B \sqsubseteq C \sqsubseteq A$, $A \cap B \sqsubseteq C \sqsubseteq B$, and $C \neq A \cap B$. Since $\tau(A) = \tau(B)$, we have $\exists C \neq A \cap B.\ (A \cap B \subseteq C \subseteq A\ \wedge\ A \cap B \subseteq C \subseteq B$, a contradiction. ■

Usually $A \odot B \neq A \sqcap B$. For instance if $D_T = \{a,b\} \times \{1,2\}$, then $\{(a,1),(b,1),(b,2)\} \odot \{(a,2),(b,1),(b,2)\} = \{(b,1),(b,2)\}$, while
$\{(a,1),(b,1),(b,2)\} \sqcap \{(a,2),(b,1),(b,2)\} = \{b\}$.

We know very little about the algebraic and lattice properties of the operator "$\otimes$", which is very important from the application point of view, despite the fact that it has been used any analysed in the relational data bases for years [3, 4, 8]. There are cases that $A \otimes B \sqsubseteq A$, there are cases when $A \sqsubseteq A \otimes B$ and there are cases that $\neg(A \otimes B \sqsubseteq A \ \vee \ A \sqsubseteq A \otimes B)$. The interesting property is that "$\oplus$" and "$\otimes$" obey distributivity laws.

Lemma 5.5 ([11])

1. *$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.*
2. *$A \oplus (B \otimes C) = (A \oplus B) \otimes (A \oplus C)$.* ■

Some properties of "$\sqsubseteq$", "$\oplus$", and "$\otimes$ in the framework of tabular expressions but from the viewpoint of relational and cylindric algebras were analysed in [5, 11]

6 Weak Lattice of Relations

Let $T_{\mathrm{left}} \subseteq T$ and $T_{\mathrm{right}} \subseteq T$ be universal set of indexes satisfying $T_{\mathrm{left}} \cap T_{\mathrm{right}} = \emptyset$ and $T_{\mathrm{left}} \cup T_{\mathrm{right}} = T$, where T is our universal and finite set of indexes. Consider the product $D_T = \prod_{i \in T} D_i$. It could also be interpreted as:

$$D_T = D_{T_{\mathrm{left}}} \times D_{T_{\mathrm{right}}}.$$

A set $R \sqsubseteq D_T = D_{T_{\mathrm{left}}} \times D_{T_{\mathrm{right}}}$ is called a *relation* iff $\tau(R) \cap T_{\mathrm{left}} \neq \emptyset$ and $\tau(R) \cap T_{\mathrm{right}} \neq \emptyset$.
If R is a relation than

$$R \subseteq D_{\tau(R) \cap T_{\mathrm{left}}} \times D_{\tau(R) \cap T_{\mathrm{right}}}$$

which is a standard definition of a relation over product domains ([2, 11]). The set $D_{\tau(R) \cap T_{\mathrm{left}}}$ is a *domain* of R, and the set $D_{\tau(R) \cap T_{\mathrm{right}}}$ is a *co-domain* of R. Define

$$\mathcal{R}_{\sqsubseteq}(D_{T_{\mathrm{left}}} \times D_{T_{\mathrm{right}}}) = \{R \mid R \sqsubseteq D_T \ \wedge \ \tau(R) \cap T_{\mathrm{left}} \neq \emptyset \ \wedge \ \tau(R) \cap T_{\mathrm{right}} \neq \emptyset\} \cup \{\emptyset\}.$$

From the viewpoint of applications ([11]) the properties of the poset $(\mathcal{R}_{\sqsubseteq}(D_{T_{\mathrm{left}}} \times D_{T_{\mathrm{right}}}), \sqsubseteq)$ are more important than those of $(\mathcal{P}_{\sqsubseteq}(D_T), \sqsubseteq)$. Fortunately the properties of these two posets are almost identical. Directly from the definitions we have.

Corollary 6.1 *Let* $\mathcal{Q} = \{R \mid R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$ *be a nonempty set of relations.*

1. $\bigsqcup_{R\in\mathcal{Q}} R = \biguplus_{R\in\mathcal{Q}} R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$.
2. $\bigoplus_{R\in\mathcal{Q}} R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$.
3. $\bigotimes_{R\in\mathcal{Q}} R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$. ■

In general it may happen that $R_1, R_2 \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$ but $R_1 \odot R_2 \notin \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$, or $R_1 \sqcap R_2 \notin \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$.
Let $T_{\text{left}} = \{1\}$, $T_{\text{right}} = \{2,3\}$, $D_1 = \{a,b\}$, $D_2 = \{1,2\}$, $D_3 = \{\alpha,\beta\}$, $R_1 = \{(a,1)\} \subseteq D_1 \times D_2$, $R_2 = \{(a,\alpha)\} \subseteq D_1 \times D_3$. Hence $R_1, R_2 \in \mathcal{R}_{\sqsubseteq}(D_{\{1\}} \times D_{\{2,3\}})$, but $R_1 \odot R_2 = \{a\} \subseteq D_1$, so $R_1 \odot R_2 \notin \mathcal{R}_{\sqsubseteq}(D_{\{1\}} \times D_{\{2,3\}})$.
Let $T_{\text{left}} = \{1\}$, $T_{\text{right}} = \{2\}$, $D_1 = \{a,b\}$, $D_2 = \{1,2\}$,
$R_1 = \{(a,1),(b,1),(b,2)\}$, $R_2 = \{(a,2),(b,1),(b,2)\}$. Hence $R_1, R_2 \in \mathcal{R}_{\sqsubseteq}(D_{\{1\}} \times D_{\{2\}})$, but $R_1 \sqcap R_2 = \{b\} \subseteq D_1$, so $R_1 \sqcap R_2 \notin \mathcal{R}_{\sqsubseteq}(D_{\{1\}} \times D_{\{2\}})$.

Let $\mathcal{Q} = \{R \mid R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$ be a set of relations. We define the operations "$\hat{\bigodot}_{R\in\mathcal{Q}} R$" and "$\hat{\sqcap}_{R\in\mathcal{Q}} R$" as follows:

$$\hat{\sqcap}_{R\in\mathcal{Q}} R = \begin{cases} \sqcap_{R\in\mathcal{Q}} R & \text{if } \sqcap_{R\in\mathcal{Q}} R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}}) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\hat{\bigodot}_{R\in\mathcal{Q}} R = \begin{cases} \bigodot_{R\in\mathcal{Q}} R & \text{if } \bigodot_{R\in\mathcal{Q}} R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}}) \\ \emptyset & \text{otherwise} \end{cases}$$

Directly from the above definition we have the corollary.

Corollary 6.2 *Consider the poset* $(\mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}}), \sqsubseteq)$.
Let $\mathcal{Q} = \{R \mid R \in \mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}})$ *be a set of relations.*
We have $\text{winf}\, \mathcal{Q} = \hat{\sqcap}_{R\in\mathcal{Q}} R$. ■

In general we can state the following conclusion.

Conclusion 6.3 *All the results of the two previous sections, i.e. Theorem 4.1, Theorem 4.2, Lemma 4.3, Theorem 4.4, Corollary 4.5, Corollary 5.1, Theorem 5.2, Lemma 5.3, Lemma 5.4 and Lemma 5.5, hold for the poset* $(\mathcal{R}_{\sqsubseteq}(D_{T_{\text{left}}} \times D_{T_{\text{right}}}), \sqsubseteq)$, *when "*$\sqcap$*" is replaced by "*$\hat{\sqcap}$*", and "*$\odot$*" by "*$\hat{\odot}$*".* ■

7 Final Comment

The approach presented here is an orthogonal to the standard relational algebra. In the standard relational algebra [2] the basic operations are composition-like operations (angelic rules, demonic rules, etc., but the operator ";" [2] or "$\circ$" [13] is a primary concept) and co-domain of one relation becomes domain of another one. The explicit structure of domains and co-domains does not play the primary role in the standard relational algebra. One may say that the operations of standard relational algebra are mainly horizontal. In our approach we always compose domains with domains and co-domains with co-domains, and we assume a specific structure of both domains and co-domains, they are both direct products. In this sense our approach may be called vertical. These two approaches are not completely foreign. It was shown in [5, 11] that the relation "$\sqsubseteq$" and operations "$\oplus$", $\otimes$" can entirely be expresses in terms of traditional relational and cylindric algebras.

The results obtained are surprisingly unpleasant. The poset created by the relation "$\sqsubseteq$" does not have very regular properties. The most intuitive operations "$\oplus$" and "$\odot$" do not correspond to weak supremum and weak infimum. The operator "$\uplus$" which defines weak supremum is useful. It had not been used in [1, 11], but in some cases it would make a description of semantic for some forms of tabular expressions easier. Unfortunately lack of associativity makes its behaviour not easy to predict, and its use is very error prone. Due to a lack of an operational definition, the operator "$\sqcap$" is usually difficult to compute. The operator $\otimes$, which in principle is the *join* operator from relational data bases [3]) does not seem to fit well in the framework generated by the relation "$\sqsubseteq$". This might suggest that perhaps we should define different relations of type "part of" to model different aspects of compositionality.

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