What does the partial order "mean"?

- Domain theory developed by Dana Scott and Gordon Plotkin in the late '60s
- use partial order to represent (informally):
    - approximates
    - carries more information than
    - better or more defined

the fixed point is then the limit of a chain of ever better approximations.

Partially Ordered Sets

A binary relation $\subseteq$ [(1)] on a set D is a *partial order* if and only if (iff) it is:

- reflexive: $\forall d \in D$, $d \subseteq d$
- transitive: $a,b,c \in D$, $a \subseteq b$ and $b \subseteq c \Rightarrow a \subseteq c$
- anti-symmetric: $a,b \in D$, $a \subseteq b$ and $b \subseteq a \Rightarrow a=b$

The pair $(D, \subseteq)$ is called a *partially ordered set*

or *Poset*.

1. This is also denoted $\leq$ in some texts, and can be stated as "less than or equal to". This is perhaps a better symbol, but in both Neilsen(1) and Pitts(1), the subset symbol is used, and this is helpful later when examining the domain of partial functions.

Least Element (⊥)

The element d is a *least element* of $S \subseteq D$ if:

$d \subseteq x \ \forall \ x \in S$

anti-symmetric $\Rightarrow$ least element unique

The least element of an entire Poset is also called *bottom* and is represented by the symbol $\perp$.

Chains and Least Upper Bounds

A *countable, increasing chain* is a sequence of elements in set D such that $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$

(this will be called a chain, countable and increasing will be implicit)

An *upper bound* d of a chain satisfies: $\forall n \in N, \ d_n \sqsubseteq d$

If it exists, the *least upper bound* (*lub*) of a chain satisfies:

$$\bigcup_{n \geq 0} d_n \sqsubseteq d$$

for any upper bound d of the chain.

Domains and CPOs

A *chain complete poset* (*cpo*) is a poset in which all countable, increasing chains have least upper bounds.

A *domain* is a cpo with a least element $\perp$.

Note: any finite poset is a chain complete poset, but not necessarily a domain (may not have a $\perp$).

example: Boolean = {true, false} is not a domain, but

$\text{Boolean}_\perp = \{\text{true, false}, \perp\}$ where

$b \sqsubseteq b' \Rightarrow b=\perp$ or $b=b'$ is called a flat domain.

Domain of Partial Functions

The set of partial functions $f: X \rightarrow Y$ and partial order $f \subseteq g$

$$\text{domain}(f) \subseteq \text{domain}(g),\ x \in \text{domain}(f) \Rightarrow f(x) = g(x)$$

form a domain, with $\perp$ the completely undefined function.

All increasing chains $f0 \subseteq f1 \subseteq f2 \subseteq \ldots$ are bounded by

least upper bound $\cup_{n\geq 0} f$ where

$$\text{domain}(\cup_{n\geq 0} f) = \cup_{n\geq 0}\ \text{domain}(f_n)$$

$$f(x) = f_n(x) \text{ for } x \in \text{domain}(f_n) \text{ for some } n$$

Why is this domain important?

Functions

A function f:D→E is *monotone* if it satisfies:

$$\forall d,d' \in D,\ d \subseteq d' \Rightarrow f(d) \subseteq f(d')$$

A function f:D→E is *continuous* if it satisfies:

$$f \text{ is monotone and } f(\cup_{n\geq 0} d_n) = \cup_{n\geq 0} f(d_n)$$

A function f:D→E is *strict* if $f(\bot) = \bot$

Lemma: a monotone function f is continuous if and only if

$$f(\cup_{n\geq 0} d_n) \subseteq \cup_{n\geq 0} f(d_n)$$

because monotonicity $\Rightarrow \cup_{n\geq 0} f(d_n) \subseteq f(\cup_{n\geq 0} d_n)$ and

anti-symmetry of $\subseteq \Rightarrow f(\cup_{n\geq 0} d_n) = \cup_{n\geq 0} f(d_n)$

Pre-fixed Points

An element $d \in D$ is *pre-fix point* of $f:D \rightarrow D$ if $f(d) \subseteq d$.

If a *least pre-fix point* written *fix(f),* exists, it satisfies:

$$1)\ f(\mathit{fix}(f)) \subseteq \mathit{fix}(f)$$

$$2)\ \forall d \in D,\ f(d) \subseteq d \Rightarrow \mathit{fix}(f) \subseteq d$$

fix(f) is unique

<u>Proposition(Kleene's):</u> For a monotone $f:D \rightarrow D$ with a least pre-fixed point, *fix(f)* is a least fixed point of f.

Proof: by monotonicity of f and (1), $f(f(\mathit{fix}(f))) \subseteq f(\mathit{fix}(f))$.

Let $d = f(\mathit{fix}(f))$, then by (2) and above $\mathit{fix}(f) \subseteq f(\mathit{fix}(f))$. (3)

So, by (1), (3) and anti-symmetry, $\mathit{fix}(f) = f(\mathit{fix}(f))$.

Tarski's Fixed Point Theorem

Let $f{:}D\rightarrow D$ be continuous for domain D. Then:

- f has a least pre-fixed point $\boldsymbol{fix(f)} = \cup_{n\geq 0} \mathbf{f}^{\mathbf{n}}(\perp)$
- $fix(f)$ is a fixed point of f, e.g. $f(fix(f)) = fix(f)$, and is therefore the least fixed point of f.

Proof:

By def'n of domain, there is a $\perp \in D$. By monotonicity of f,

$$f^{n}(\perp) \subseteq f^{n+1}(\perp) \Rightarrow f^{n+1}(\perp) \subseteq f^{n+2}(\perp)$$

so f^{n} is a chain. Since a domain is a cpo, it has an upper bound,

$$fix(f) = \cup_{n\geq 0} f^{n}(\perp)$$

and $f(fix(f)) = f(\cup_{n\geq 0} f^{n}(\perp)) = \cup_{n\geq 0} f(f^{n}(\perp))$ (by continuity)

$$= \cup_{n\geq 0} f^{n+1}(\perp) = \cup_{\tilde{n}\geq 0} f^{\tilde{n}}(\perp) = fix(f)$$

because you can drop any finite number of terms from the beginning of the chain without affecting the upper bound.

Proof by Definition

Tarski's Theorem

- allows denotational semantics for recursive features
- you must still:
  - define the underlying set and show it has a bottom
  - define a partial ordering
  - prove the least upper bound exists for all chains
    - remember all chains on a finite domain are bounded
  - define the function for a class of statement
  - prove it is continuous

The While Statement

Define f so

$$[[\text{while B do C}]] = f_{[B][C]}([[\text{while B do C}]])$$

where for b:State→Boolean and w,c:State→State

$$f_{b,c}(w) = \lambda s \in \text{State. if}(\ b(s), w(c(s)), s)$$

Solve for w as a fixed point of f, e.g. $w = f_{b,c}(w)$; define

$$[[\text{while B do C}]] = \mathit{fix}(f_{[B][C]})$$

But first we need to verify the conditions for Tarski's theorem....

While cont...

Domain: partial functions f:State$\rightarrow$State (see prev. slide)

– verify partial order and that $f_0 \subseteq f_1 \subseteq \ldots$ is a chain

Function: $f_{b,c}(w) = \lambda s \in \text{State}.\ \text{if}(\ b(s), w(c(s)), s)$

Must show f is continuous, but if so

- $fix(f_{b,c}) = \cup_{n\geq 0} f^n(\bot)$ and is a least fixed point of f
- first define chain $w_0 = \bot$; $w_{n+1} = f(w_n)$
- chain must have a limit $\cup_{n\geq 0} w_n$ as this is a domain

While cont...

$$f(w_i)s = w_i(c(s)) \text{ for } b(s) \text{ true; } s \text{ for } b(s) \text{ false}$$

Obvious for b(s) false, so only consider true..

$$w_i \subseteq w_j \Rightarrow f(w_i) = w_i(c(s)) \subseteq w_j(c(s)) = f(w_j) \Rightarrow f \text{ monotone}$$

$$f \text{ monotone} + f(\cup_{n\geq 0} w_n) \subseteq \cup_{n\geq 0} f(w_n) \Rightarrow f \text{ continuous } (*)$$

$$f(\cup_{n\geq 0} w_n)s = \cup_{n\geq 0} w_n(c(s)) \subseteq \cup_{n\geq 0} w_{n+1}(c(s)) \text{ by def'n of lub}$$

$$= \cup_{n\geq 0}(f(w_n))(c(s)) \text{ by def'n of w}$$

so f is continuous, and therefore $fix(f) = f(\cup_{n\geq 0} w_n)$ exists and is the least fixed point of f by Tarski's Theorem.

Where does this lead?

- establish a set of useful domains to represent the virtual environment of a programming language
  - e.g. $Boolean_{\perp}$, $Naturals_{\perp}$
- learn some additional definitions and theorems to support the construction of these domains and continuous functions on them
  - e.g. composition preserves continuity, functions of multiple arguments are continuous if continuous in their arguments, etc.

Bibliography

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