Domain Theory

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for CAS 706, March, 2005

- a branch of order theory which looks at special types of ordered sets called domains
- used to specify denotational semantics
- formalizes intuitive ideas about approximation

Partially Ordered Sets

A partial order, $\sqsubseteq$, defined on a set P, is:

- reflexive: For all $x \in P$, $x \sqsubseteq x$.
- transitive: For $x,y,z \in P$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$
- antisymmetric: For $x,y \in P$, if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.

The pair $(P, \sqsubseteq)$ is then a partially ordered set.

Orderings of Partial Functions

Let $\mathrm{Pfn}(A \rightarrow B)$ be the set of all (partial and total) functions from A to B.

Then we can order $\mathrm{Pfn}(A \rightarrow B)$ such that, for $f,g \in \mathrm{Pfn}(A \rightarrow B)$, $f \sqsubseteq g$ iff $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$.

In this way, if we have a partial function f and a total function g, where $f,g \in \mathrm{Pfn}(A \rightarrow B)$, and $f \sqsubseteq g$, we can think of f as approximating g because, where f is defined, it behaves exactly as g; it is just defined over fewer elements than g.

Chains and Antichains

A partially ordered set P is a chain if for all $x,y \in P$, $x \sqsubseteq y$ or $y \sqsubseteq x$.

P is an antichain if, for any $x,y \in P$, $x \sqsubseteq y$ only if $x = y$.

Any subset of a chain is also a chain. Likewise with antichains.

Any set can be given the antichain ordering.

Top and Bottom

If P is a partially ordered set, then its bottom element, $\bot$, if it exists, has the property that $\bot \sqsubseteq x$ for all $x \in P$. P's top element, $\top$, is defined as $x \sqsubseteq \top$.

$\top$ represents a contradictory element, whereas $\bot$ represents a complete lack of information.

All finite chains have a top and bottom element.

In domain theory, we typically want a bottom element, so we can form $P_\bot$ by "lifting" a set P by giving it the antichain order, and then taking $P \cup \{\bot\}$, where $\bot \notin P$, and updating the order so that $\bot \sqsubseteq x$ for all $x \in P$. $P_\bot$ is then flat.

Least Upper Bounds and Directed Sets

Let P be an ordered set and $S \subseteq P$. Then $x \in P$ is an upper bound of S if $y \sqsubseteq x$ for all $y \in S$. x is a least upper bound, $\sqcup S$ if $x \sqsubseteq y$ for all upper bounds y.

A non-empty subset S of an ordered set P is directed if for every pair of elements $x,y \in S$, there exists a $z \in S$ such that z is an upper bound of $\{x,y\}$.

This is equivalent to saying that S is directed iff every finite subset of S has an upper bound in S.

Complete Partially Ordered Sets

A partially ordered set P is complete if each directed subset $D \subseteq P$ has a least upper bound and if P contains a bottom element.

An equivalent definition is that P is complete iff each chain $D \subseteq P$ has a least upper bound in P.

The fact that a CPO needs a least upper bound does not need to be an explicit part of the second definition because the empty subset is (vacuously) a chain, and its least upper bound will be a bottom, whereas a directed subset, by definition, is non-empty.

A CPO without a bottom is called a pre-CPO.

Monotone and Continuous Functions

A function f: $A \rightarrow B$, where A and B are ordered, is monotone if, for $x,y \in A$, $x \sqsubseteq y$ in A implies $f(x) \sqsubseteq f(y)$ in B.

A function f: $A \rightarrow B$ is continuous if A and B are pre-CPOs and, for every directed set C in A, the subset f(C) of B is directed, and $f(\sqcup C) = \sqcup f(C)$.

Lemma: All continuous functions are monotone.
Proof: Given a continuous function $f(P \rightarrow Q)$, take $x,y \in P$, $x \sqsubseteq y$. Then $\{x,y\}$ is a directed set, and we know $\sqcup\{x,y\} = y$, and therefore $f(\sqcup\{x,y\}) = f(y)$.
We also know that $f(x) \sqsubseteq \sqcup f(\{x,y\})$ because $x \sqsubseteq y$.
Since f is continuous, we then get $f(x) \sqsubseteq \sqcup f(\{x,y\}) = f(\sqcup \{x,y\}) = f(y)$.

CPOs of Partial Functions

Let A be a non-empty set. Then $f \in Pfn(A \to A)$ can be converted into a total function by mapping every element in $A \backslash \{dom(f)\}$ to the bottom element of $A\perp$.

We get $f\perp(x) = f(x)$ if $x \in dom(f)$ and $f\perp(x) = \perp$ otherwise.

We can then use the partial function $V: f \mapsto f\perp$ to show that both $Pfn(A \to A)$ and $Tfn(A \to A\perp)$ are CPOs by showing that V is an order-isomorphism ($x \sqsubseteq y$ iff $V(x) \sqsubseteq V(y)$).