Design and Selection of Sequential Programming Languages

SFWR ENG 3E03, Fall 2003

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Lecture 1

Semantics

Syntax and Semantics

Syntax — *Shape* of PL constructs

- What are the **tokens** of the language? — **Lexical** syntax, “word level”
- How are programs built from tokens? — Mostly use **Context-Free Grammars** (CFG) or **Backus-Naur-Form** (BNF) to describe **syntax** at the “sentence level”

“Static semantics”: aspects of program structure that are checked at compile time, but cannot be captured by CFGs ($\rightarrow$ context-sensitive syntax):

– Scopes of names

– Typing

Semantics — *Meaning* of PL constructs

Three major approaches:

– **Axiomatic semantics:** $\{p\}$ `Prog` $\{q\}$

– **Denotational semantics:** `Prog` *denotes* a *mathematical* function $[\![$`Prog`$]\!]$

– **Operational semantics:** state transitions of an abstract machine

Simple Semantic Domains

From the textbook:

> A semantic domain is any set whose properties and operations are independently well-understood and upon which the functions that define the semantics of a language are ultimately based.

Primitive domains: $\mathrm{B} = \{\mathsf{True}, \mathsf{False}\}$, $\mathbb{N}$, $\mathbb{Z}$, Char, $\mathsf{seq}\,\mathit{Char}$, Ident

Domains for Program States:

- **Locations** are usually natural numbers: $\mathit{Loc} = \mathbb{N}$
- **Values** are, in a simple context, integers: $\mathit{Val}_0 = \mathbb{Z}$
- **Memory states** can be considered as partial functions: $\mathit{Mem}_0 = \mathbb{N} \nrightarrow \mathit{Val}_0$
- **Simple environments** are partial functions, too: $\mathit{Env}_0 = \mathit{Ident} \nrightarrow \mathit{Loc}$
- A simple **state** is pair: $\mathit{State}_0 = \mathit{Env}_0 \times \mathit{Mem}_0$
- A **simple store** directly maps identifiers to values: $\mathit{Store}_0 = \mathit{Ident} \nrightarrow \mathit{Val}_0$

Relation Overriding

Given $Q, R : \mathcal{A} \leftrightarrow \mathcal{B}$.

The relation $Q \oplus R$ relates everything in the domain of R to the same objects as R does, and everything else in the domain of Q to the same objects as Q does.

$$Q \oplus R = \{(x, y) : Q \mid x \notin \mathsf{dom}\ R\} \cup R$$

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- ***Haskell:***
  addListToFM :: *Ord key* ⇒ *FiniteMap k v* → [(*k* , *v*)] → *FiniteMap k v*

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- If Q and R are both partial functions, then $Q \oplus R$ is a partial function, too.
- $\oplus$ is used to model
  - writing into memory or store locations
  - insertion into environments (*shadowing* previous bindings)

Operational Semantics

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— Evaluating expression e starting in store σ produces value v $\sigma(e) \Rightarrow v$

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$$\frac{premise}{conclusion} \quad \text{or} \quad \frac{premise_1 \quad \ldots \quad premise_n}{conclusion}$$

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Example rule — **addition:**

$$\frac{\sigma(e_1) \Rightarrow v_1 \qquad \sigma(e_2) \Rightarrow v_2}{\sigma(e_1 + e_2) \Rightarrow v_1 + v_2}$$

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(The left “ + ” is **syntax,** the right “ + ” is a mathematical operation on numbers.)

Mechanized Operational Semantics: Interpreter

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— expression evaluation: $eval : State_1 \times Expr \nrightarrow Val_1$

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Two **interpreter functions** (assuming **deterministic** semantics):

$$evalExpr :: Expression \rightarrow State1 \rightarrow Maybe\ Value1$$
$$interpStmt :: Statement \rightarrow State1 \rightarrow Maybe\ State1$$

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evalExpr   :: Expression -> State1 -> Maybe Value1
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```

```
data Value1 = ValInt Int | ValBool Bool
type State1 = FiniteMap Variable Value1 -- even simpler than State_0
```

Interpreter: Expression Evaluation

$$\mathit{evalExpr} :: \mathit{Expression} \rightarrow \mathit{State1} \rightarrow \mathit{Maybe}\ \mathit{Value1}$$

$$\begin{aligned}\textbf{data}\ \mathit{Value1} &= \mathit{ValInt}\ \mathit{Int}\\ &\mid \mathit{ValBool}\ \mathit{Bool}\end{aligned}$$

$\textbf{type}\ \mathit{State1} = \mathit{FiniteMap}\ \mathit{Variable}\ \mathit{Value1}$ -- even simpler than $State_0$

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$$evalExpr\ (Var\ v)\ s =$$

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$(Just\ (ValInt\ v1), Just\ (ValInt\ v2)) \to Just\ (ValInt\ (v1 + v2))$

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case $(evalExpr\ e1\ s, evalExpr\ e2\ s)$ **of**
$(Just\ (ValInt\ v1), Just\ (ValInt\ v2)) \rightarrow Just\ (ValInt\ (v1 + v2))$
$\_ \rightarrow Nothing$

Interpreter: Expression Evaluation (*Maybe* Monad)

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$\quad | ValBool\ Bool$

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$evalExpr\ (Value\ (LitBool\ b))\ s = Just\ (ValBool\ b)$
$evalExpr\ (Binary\ (MkArithOp\ Plus)\ e1\ e2)\ s =$ **do**
$\quad ValInt\ v1 \leftarrow evalExpr\ e1\ s$
$\quad ValInt\ v2 \leftarrow evalExpr\ e2\ s$
$\quad Just\ (ValInt\ (v1 + v2))$

Assignment

$$\frac{\sigma(e) \Rightarrow v}{\sigma(x := e) \Rightarrow \sigma \oplus \{x \mapsto v\}}$$

For example:

- Assume $\sigma_1 = \{x \mapsto 39, y \mapsto 7\}$
- Then:

$$\sigma_1(x := x + 3) \Rightarrow$$

Assignment

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For example:

- Assume $\sigma_1 = \{x \mapsto 39, y \mapsto 7\}$
- Then:

$$\cfrac{\cfrac{\sigma_1(x) \Rightarrow 39 \qquad\qquad \sigma_1(3) \Rightarrow 3}{\sigma_1(x + 3) \Rightarrow}}{\sigma_1(x := x + 3) \Rightarrow}$$

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For example:

- Assume $\sigma_1 = \{x \mapsto 39, y \mapsto 7\}$
- Then:

$$\frac{\dfrac{\sigma_1(x) \Rightarrow 39 \qquad \sigma_1(3) \Rightarrow 3}{\sigma_1(x + 3) \Rightarrow 42}}{\sigma_1(x := x + 3) \Rightarrow}$$

Assignment

$$\frac{\sigma(e) \Rightarrow v}{\sigma(x := e) \Rightarrow \sigma \oplus \{x \mapsto v\}}$$

For example:

- Assume $\sigma_1 = \{x \mapsto 39, y \mapsto 7\}$
- Then:

$$\frac{\dfrac{\sigma_1(x) \Rightarrow 39 \qquad \sigma_1(3) \Rightarrow 3}{\sigma_1(x + 3) \Rightarrow 42}}{\sigma_1(x := x + 3) \Rightarrow \{x \mapsto 42, y \mapsto 7\}}$$

Assignment

$$\frac{\sigma(e) \Rightarrow v}{\sigma(x := e) \Rightarrow \sigma \oplus \{x \mapsto v\}}$$

For example:

- Assume $\sigma_1 = \{x \mapsto 39, y \mapsto 7\}$
- Then:

$$\frac{\dfrac{\sigma_1(x) \Rightarrow 39 \qquad \sigma_1(3) \Rightarrow 3}{\sigma_1(x + 3) \Rightarrow 42}}{\sigma_1(x := x + 3) \Rightarrow \{x \mapsto 42, y \mapsto 7\}}$$

since $\sigma_1 \oplus \{x \mapsto 42\} = \{x \mapsto 42, y \mapsto 7\}$

Interpreter: Assignment

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$$\mathit{interpStmt} :: \mathit{Statement} \to \mathit{State1} \to \mathit{Maybe}\ \mathit{State1}$$

data *Value1* = *ValInt Int*
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interpStmt (*Assignment var e*) *s* =

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$$\textbf{data}\ \mathit{Value1} = \mathit{ValInt}\ \mathit{Int}$$
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$$\frac{\sigma(e) \Rightarrow v}{\sigma(x := e) \Rightarrow \sigma \oplus \{x \mapsto v\}}$$

$$\mathit{interpStmt}\ (\mathit{Assignment}\ \mathit{var}\ e)\ s = \textbf{case}\ \mathit{evalExpr}\ e\ s\ \textbf{of}$$
$$\quad \mathit{Just}\ \mathit{val} \to \mathit{Just}\ (\mathit{addToFM}\ s\ \mathit{var}\ \mathit{val})$$
$$\quad \mathit{Nothing} \to \mathit{Nothing}$$

Interpreter: Assignment

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data *Value1* = *ValInt Int*
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$$\frac{\sigma(e) \Rightarrow v}{\sigma(x := e) \Rightarrow \sigma \oplus \{x \mapsto v\}}$$

interpStmt (*Assignment var e*) *s* = **case** *evalExpr e s* **of**
  Just val → *Just* (*addToFM s var val*)
  Nothing → *Nothing*

(Using the *Maybe* monad:)

interpStmt (*Assignment var e*) *s* = **do**
  val ← *evalExpr e s* **of**
  Just (*addToFM s var val*)

Sequencing, Conditionals, Loops

$$\frac{\sigma_1(s_1) \Rightarrow \sigma_2 \qquad \sigma_2(s_2) \Rightarrow \sigma_3}{\sigma_1(s_1;\ s_2) \Rightarrow \sigma_3}$$

$$\frac{\sigma(b) \Rightarrow \textsf{True} \qquad \sigma(s_1) \Rightarrow \sigma_1}{\sigma(\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) \Rightarrow \sigma_1}$$

$$\frac{\sigma(b) \Rightarrow \textsf{False} \qquad \sigma(s_2) \Rightarrow \sigma_2}{\sigma(\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) \Rightarrow \sigma_2}$$

$$\frac{\sigma(b) \Rightarrow \textsf{True} \qquad \sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}$$

$$\frac{\sigma(b) \Rightarrow \textsf{False}}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma}$$

Loop Example

$$P \equiv \textbf{while } x < 50 \textbf{ do } x := 2 * x \textbf{ od}$$

$$\{x \mapsto 7\}(P) \Rightarrow$$

$$\frac{\sigma(b) \Rightarrow \text{True} \qquad \sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2} \qquad \frac{\sigma(b) \Rightarrow \text{False}}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma}$$

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$$\frac{\{x \mapsto 7\}\,(x < 50) \Rightarrow \text{True} \qquad \{x \mapsto 7\}\,(x := 2 * x) \Rightarrow \{x \mapsto 14\}}{\{x \mapsto 7\}(P) \Rightarrow}$$

$$\frac{\sigma(b) \Rightarrow \text{True} \qquad \sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2} \qquad \frac{\sigma(b) \Rightarrow \text{False}}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma}$$

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Sequencing, Conditionals, Loops

$$\frac{\sigma_1(s_1) \Rightarrow \sigma_2 \qquad \sigma_2(s_2) \Rightarrow \sigma_3}{\sigma_1(s_1;\ s_2) \Rightarrow \sigma_3}$$

$$\frac{\sigma(b) \Rightarrow \textsf{True} \qquad \sigma(s_1) \Rightarrow \sigma_1}{\sigma(\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) \Rightarrow \sigma_1}$$

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- *repeat* { ... } *until* (...)
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– **Translation into core language** — ***derived*** features

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Additional Language Features

- Output: **print** (e)
- Input: **read** (e)
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- Prove "conservative extension": mapping from old states to new is injective and preserves transitions.

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Assume a new statement “ **print** (e)”

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- Check determinism, add to interpreter.

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- **Checked Exceptions** which must be declared with a `throws` clause in a method declaration:

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- `RuntimeException`: Abnormal runtime events which need not be declared with a `throws` clause:

  ArithmeticExc., *ClassCastExc*., *IllegalArgumentExc*., *IndexOutOfBoundsExc*., *NullPointerExc*., …

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Try-Catch-Finally Statement

try {
  try body
}
catch (*Exception*$_1$ *var*$_1$) {
  catch$_1$ *body*
}
catch (*Exception*$_2$ *var*$_2$) {
  catch$_2$ *body*
}
…
catch (*Exception*$_n$ *var*$_n$) {
  catch$_n$ *body*
}
finally {
  finally body
}

Try-Catch-Finally Example

```
import java.io.*;
class Read1 {
  public static void main(String[] args) {
    BufferedReader in =
     new BufferedReader(new InputStreamReader(System.in));
    try { System.out.println("How old are you?");
            String inputLine = in.readLine();
            int age = Integer.parseInt(inputLine);
            age++;
            System.out.println("Next year, you'll be " + age);
    }
    catch (IOException exception)
    { System.out.println("Input/output error " + exception); }
    catch (NumberFormatException exception)
    { System.out.println("Input was not a number"); }
    finally { if (in ≠ null) { try { in.close(); }
                                    catch (IOException exception) {}  } }
}}
```

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- Capture it and execute no code (ignore the exception) — ***bad idea!***
- Do **not capture** it (let it propagate up) — *may need to declare!*

Exceptions — Example

```
class Simulate4 {
  private static void println(String s)
   {System.out.println(s);}
  public static int _q = 0;
  public static void main(String[] a)
  { int s = g (2);
    println("* "+s+"  "+_q);
  }
  public static int f(int k, int m) {
    println("f("+k+","+m+")");
    _q += m;
    int r = g(k) + _q;
    println("f("+k+","+m+")="+r);
    return r;
  }
```

```
public static int g(int n) {
    println("g(" + n + ")");
    int t = 3 * n;
    if ( t < 10 ) {
      try { t = (f (n + 1, _q));
          }
      catch (Exception e) {
        println("g: caught exception!");
        _q += n;
      }
    }
    t = t / _q;
    println("g( " + n + " )= " + t);
    return t;
  }
}
```

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Two additional if **rules (no exceptions in expression evaluation yet):**

$$\frac{\sigma(b) \Rightarrow \textsf{True} \qquad \sigma(s_1) \stackrel{!}{\Rightarrow} (\sigma_{1,} x)}{\sigma(\textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2\ \textbf{fi}) \stackrel{!}{\Rightarrow} (\sigma_1, x)} \qquad \frac{\sigma(b) \Rightarrow \textsf{False} \qquad \sigma(s_2) \stackrel{!}{\Rightarrow} (\sigma_{2,} x)}{\sigma(\textbf{if}\ b\ \textbf{then}\ s_1\ \textbf{else}\ s_2\ \textbf{fi}) \stackrel{!}{\Rightarrow} (\sigma_2, x)}$$

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Exceptions in Expression Evaluation

Exercise

Another Loop Example

$$P \quad \equiv \quad \textbf{while } x \neq 0 \textbf{ do } s := s + x \textbf{ ; } x := x - 1 \textbf{ od}$$

$$\{s \mapsto 0, x \mapsto -4\}(P) \Rightarrow$$

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This is not a direct proof of non-termination!

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Note: For an expression *e*, we write "$[\![e]\!]_E$" instead of "$M_{\mathsf{Expr}}(e)$".

Relational Denotational Statement Semantics

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This uses properties of <u>mathematical object</u> found as denotational semantics of a statement.

Denotational Semantics is Compositional

$$\mathit{evalExpr} :: \mathit{Expression} \to (\mathit{State1} \to \mathit{Maybe}\ \mathit{Value1})$$

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We can reflect these parentheses in the definition:

$$\mathit{evalExpr}\ (\mathit{Var}\ v) = \lambda\ s \rightarrow \mathit{lookupFM}\ s\ v$$
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\mathit{evalExpr}\ (\mathit{Binary}\ (\mathit{MkArithOp}\ \mathit{Plus})\ e1\ e2) = \lambda\ s \to \\
\quad \mathbf{case}\ ((\mathit{evalExpr}\ e1)\ s, (\mathit{evalExpr}\ e2)\ s)\ \mathbf{of} \\
\qquad (\mathit{Just}\ (\mathit{ValInt}\ v1), \mathit{Just}\ (\mathit{ValInt}\ v2)) \to \mathit{Just}\ (\mathit{ValInt}\ (v1 + v2)) \\
\qquad \_ \qquad\qquad\qquad\qquad\qquad\qquad\quad \to \mathit{Nothing}
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evalExpr\ (Value\ lit) = const\ (Just\ (litToVal\ lit)) \\
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\quad \textbf{case}\ ((evalExpr\ e1)\ s, (evalExpr\ e2)\ s)\ \textbf{of} \\
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Translating the last case back into mathematical notation:

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$$[\![e_1 + e_2]\!]_E \quad := \quad \lambda\, s \;\to\; [\![e_1]\!]_E\ s + [\![e_2]\!]_E\ s$$

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evalExpr :: *Expression* → (*State1* → *Maybe Value1*)

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evalExpr (Var v)       = λ s → lookupFM s v
evalExpr (Value lit) = const (Just (litToVal lit))
evalExpr (Binary (MkArithOp Plus) e1 e2) = λ s →
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    (Just (ValInt v1), Just (ValInt v2)) → Just (ValInt (v1 + v2))
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Compositional Semantics

The semantics of each syntactic construct is defined in terms of the semantics of its constituents.

Sequencing, Conditionals, Loops in Operational Semantics

$$\frac{\sigma_1(s_1) \Rightarrow \sigma_2 \qquad\qquad \sigma_2(s_2) \Rightarrow \sigma_3}{\sigma_1(s_1;\ s_2) \Rightarrow \sigma_3}$$

$$\frac{\sigma(b) \Rightarrow \text{True} \qquad\qquad \sigma(s_1) \Rightarrow \sigma_1}{\sigma(\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) \Rightarrow \sigma_1} \qquad \frac{\sigma(b) \Rightarrow \text{False} \qquad\qquad \sigma(s_2) \Rightarrow \sigma_2}{\sigma(\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) \Rightarrow \sigma_2}$$

$$\frac{\sigma(b) \Rightarrow \text{False}}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma}$$

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The last operational semantics rule here is **<u>not</u> compositional!**

Interpreter: Sequencing

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This corresponds to a special case of our Jay ASTs:

interpStmt (*MkBlock* [*stmt1* , *stmt2*]) =

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General case:

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interpStmt (*MkBlock* [*stmt1*, *stmt2*]) = λ *s* → **case** (*interpStmt stmt1*) *s* **of**
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  Nothing → *Nothing*

General case:

interpStmt (*MkBlock stmts*) = λ *s* → *interpBlock stmts s*

interpBlock :: [*Statement*] → (*State1* → *Maybe State1*)
interpBlock [ ] = *Just*
interpBlock (*stmt* : *stmts*) =

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```
interpStmt (MkBlock [stmt1, stmt2]) = λ s → case (interpStmt stmt1) s of
  Just s1 → (interpStmt stmt2) s1
  Nothing → Nothing
```

General case:

```
interpStmt (MkBlock stmts) = λ s → interpBlock stmts s


interpBlock :: [Statement] → (State1 → Maybe State1)
interpBlock [ ] = Just
interpBlock (stmt : stmts) = λ s → case interpStmt stmt s of
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Interpreter: Sequencing

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Interpreter: Loops

$$\frac{\sigma(b) \Rightarrow \mathsf{True} \qquad \sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}) \Rightarrow \sigma_2}{\sigma(\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}) \Rightarrow \sigma_2}$$

$$\frac{\sigma(b) \Rightarrow \mathsf{False}}{\sigma(\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}) \Rightarrow \sigma}$$

Interpreter: Loops

$$\frac{\sigma(b) \Rightarrow \text{True} \qquad \sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}{\sigma(\textbf{while } b \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_2}$$

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$\mathit{interpStmt}\ (\mathit{Loop}\ \mathit{cond}\ \mathit{body}) = \lambda\ s \rightarrow \textbf{case}\ (\mathit{evalExpr}\ \mathit{cond})\ s\ \textbf{of}$

$\quad \mathit{Just}\ (\mathit{ValBool}\ \textbf{False}) \rightarrow$

$\quad \mathit{Just}\ (\mathit{ValBool}\ \textbf{True}) \rightarrow$

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interpStmt (*Loop cond body*) = λ *s* → **case** (*evalExpr cond*) *s* **of**
  Just (*ValBool* **False**) → *Just s*
  Just (*ValBool* **True**) →

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```
interpStmt (Loop cond body) = λ s → case (evalExpr cond) s of
  Just (ValBool False) → Just s
  Just (ValBool True) → case (interpStmt body) s of
    Just s1 →
```

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    Nothing → Nothing
```

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\quad\quad \mathit{Just}\ s1 \rightarrow (\mathit{interpStmt}\ (\mathit{Loop}\ \mathit{cond}\ \mathit{body}))\ s1 \\
\quad\quad \mathit{Nothing} \rightarrow \mathit{Nothing} \\
\quad \mathit{Just}\ (\mathit{ValInt}\ i) \rightarrow \mathit{Nothing} \\
\quad \mathit{Nothing} \rightarrow \mathit{Nothing}
\end{array}
$$

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This is **not compositional**

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“*interpStmt* (*Loop cond body*)”

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```

This is **not compositional**, but **recursive:**

"*interpStmt (Loop cond body)*" occurs also on the right-hand side.

Recursive *Definitions*?

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Translating the Haskell definition back into the mathematical notation we obtain something of the following shape:

$$[\![\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}]\!]_S = F(\ [\![\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}]\!]_S\ )$$

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"$[\![\textbf{while}\ b\ \textbf{do}\ s\ \textbf{od}]\!]_{\mathrm{S}}$" usually has **many** solutions!

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Start (Haskell):

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Reverse β-reduction to isolate the RHS occurrence of *fact*:

$$\mathit{fact} = (\lambda\ f \to \lambda\ n \to \textbf{if}\ n \equiv 0\ \textbf{then}\ 1\ \textbf{else}\ n * f\ (n-1))\ \mathit{fact}$$

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Defining (via an explicit definition)

$$\tau = \lambda\ f \to \lambda\ n \to \textbf{if}\ n \equiv 0\ \textbf{then}\ 1\ \textbf{else}\ n * f\ (n-1)$$

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Defining (via an explicit definition)

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we recognise a **fixedpoint equation** (stating that *fact* is a fix edpointof τ):

$$fact = \tau\ fact$$

Fixedpoint Approximation

For the functional τ associated with the definition of the factorial function *fact*, we observe:

$$\tau^0 \perp \quad = \quad \perp \quad = \quad \{\}$$

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$$
\begin{array}{lllll}
\tau^0 \; \bot & = & \bot & = & \{\} \\
\tau^1 \; \bot & = & \tau \; \bot & = & \{0 \mapsto 1\}
\end{array}
$$

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For the functional τ associated with the definition of the factorial function *fact*, we observe:

$$
\begin{aligned}
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\tau^2\ \bot &= \tau\ (\tau\ \bot\ ) &&= \{0 \mapsto 1,\ 1 \mapsto 1\}
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\tau^3 \perp & = & \tau\,(\tau\,(\tau \perp)) & = \{0 \mapsto 1, 1 \mapsto 1, 2 \mapsto 2\} \\
\tau^4 \perp & = & \tau\,(\tau\,(\tau\,(\tau \perp))) & = \{0 \mapsto 1, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 6\}
\end{array}
$$

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\tau^5\ \bot & = & \tau\,(\tau\,(\tau\,(\tau\,(\tau\ \bot\,)))) & = & \{0\mapsto 1,\ 1\mapsto 1,\ 2\mapsto 2,\ 3\mapsto 6,\ 4\mapsto 24\}
\end{array}$$

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\end{array}$$

In addition:

$$\perp \sqsubseteq \tau \perp \sqsubseteq \tau^2 \perp \sqsubseteq \tau^3 \perp \sqsubseteq \tau^4 \perp \sqsubseteq \tau^5 \perp \sqsubseteq \ldots \sqsubseteq \tau^{1,000,000} \perp \sqsubseteq \ldots$$

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Iterated application of τ yields better and better **finite approximations**!

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\tau^4\ \bot & = & \tau\,(\tau\,(\tau\,(\tau\ \bot\,))) & = & \{0\mapsto 1,\ 1\mapsto 1,\ 2\mapsto 2,\ 3\mapsto 6\} \\
\tau^5\ \bot & = & \tau\,(\tau\,(\tau\,(\tau\,(\tau\ \bot\,)))) & = & \{0\mapsto 1,\ 1\mapsto 1,\ 2\mapsto 2,\ 3\mapsto 6,\ 4\mapsto 24\}
\end{array}
$$

In addition:

$$\bot\ \sqsubseteq \tau\ \bot\ \sqsubseteq \tau^2\ \bot\ \sqsubseteq \tau^3\ \bot\ \sqsubseteq \tau^4\ \bot\ \sqsubseteq \tau^5\ \bot\ \sqsubseteq \ldots \sqsubseteq \tau^{1{,}000{,}000}\ \bot\ \sqsubseteq \ldots$$

Iterated application of τ yields better and better **finite approximations**!

The union of all these approximations **is** the factorial function.

For partial functions, the least upper bound of an ascending chain is given by set-theoretic union over all elements of the chain.

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- This least upper bound is the **least fixedpoint** of τ
- We write " $Y\ \tau$ " for the **least fixedpoint** of τ.

while-Loop Semantics

$$[\![\_]\!]_S \quad : \; Stmt \to (State \nrightarrow State)$$

For $p : Stmt$ and $e : Expr$:

$$[\![\textbf{while } e \textbf{ do } p]\!]_S$$

$$= \mathbf{Y}\left(\lambda f : State \nrightarrow State \bullet \lambda s : State \bullet \begin{cases} f([\![p]\!]_S(s)) & \text{if } [\![e]\!]_E(s) = \mathsf{True} \\ s & \text{if } [\![e]\!]_E(s) = \mathsf{False} \\ \bot & \text{otherwise} \end{cases} \right)$$

Example Statement Semantics

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&= \mathbf{Y}(\lambda f : State \mathrel{+\!\!\!\to} State \bullet f) \\
&= \bot
\end{aligned}
$$

For $k : \mathbb{N}$, we have:

$$[\![\textbf{while}\ n > 0\ \textbf{do}\ (r \mathbin{\textbf{:=}} n * r\ \textbf{;}\ n \mathbin{\textbf{:=}} n - 1)]\!]_S(\{n \mapsto k, r \mapsto 1\}) = \{n \mapsto 0, r \mapsto k!\}$$

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- Some programming languages have a "**more mathematical semantics**": mathematical rules of reasoning can be applied directly to program constructs
- **Programming is a mathematical activity**

Semantics with Exceptions — Simple Statements

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$$[\![\textbf{try} \ s_1 \ \textbf{catch(} \ i \ \textbf{)} \ s_2]\!]_S \ (s) \ = \begin{cases} t & \text{if } [\![s_1]\!]_S(s) = Left \ t \\ [\![s_2]\!]_S(t \oplus \{i \mapsto e\}) & \text{if } [\![s_1]\!]_S(s) = Right \ (t, e) \\ \bot & \text{if } s \notin \mathsf{dom} \ [\![s_1]\!]_S \end{cases}$$

Semantics with Exceptions — Expressions

$$Expr \to (Store \to (Val + Num))$$

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$$[\![\textbf{if } b \textbf{ then } s_1 \textbf{ else } s_2]\!]_S(s) = \begin{cases} [\![s_1]\!]_S(s) & \text{if } [\![b]\!]_E(s) = Left\ \mathsf{True} \\ [\![s_2]\!]_S(s) & \text{if } [\![b]\!]_E(s) = Left\ \mathsf{False} \\ Right\ (s, exc_{\mathbb{B}}) & \text{if } [\![b]\!]_E(s) = Left\ n \wedge n \in Num \\ Right\ (s, exc) & \text{if } [\![b]\!]_E(s) = Right\ exc \end{cases}$$

while-Semantics with Exceptions

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Exercise!

Output

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$\mathit{State} \rightarrow \mathit{State}$		(**new** statement semantics)

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In case of program errors or nontermination, **previous output is not lost**!

$$[\![\mathbf{print}\ e]\!]_S = \lambda\,(s, ns) : State \bullet \begin{cases} (s, n{:}ns) & \text{if } n = [\![e]\!]_E(s) \in Num \\ (\perp, ns) & \text{otherwise} \end{cases}$$

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Note: statement semantics here is ***oversimplified*** — fixpoint construction in $State \rightarrow State$ does not work, except with Haskell-like list domains.

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$$[\![\textbf{read } v]\!]_S =$$

$$\lambda\,(s, outs, ins) : State \bullet \begin{cases} (s \oplus \{v \mapsto in\}, outs, ins') & \text{if } ins = in{:}ins' \\ (\perp, outs, ins) & \text{if } ins = [] \end{cases}$$

Scope

Nested scopes with shadowing of identifiers are modelled as **stacks** (lists) **of environments**:

Env	$= Id \nrightarrow SVal^{\perp}$	**environments** (with $\perp$ for uninit. var.)
$Store$	$= [Env]$	**stores**
$State$	$= Store^{\perp} \times [Num] \times [Num]$	**states including I/O**

Records

Semantic Domains: Only **storable values** change:

$SVal$	$= Bool + Num + (Id \mathrel{+\!\!\!\!\twoheadrightarrow} SVal)$	storable values
$State$	$= Id \mathrel{+\!\!\!\!\rightarrow} SVal$	(simple) stores
$State \mathrel{+\!\!\!\!\rightarrow} SVal$		(expression semantics)
$State \mathrel{+\!\!\!\!\rightarrow} State$		(statement semantics)

New record field expressions:

$$[\![e.\, f]\!]_E = \lambda\, s : State \bullet ([\![e]\!]_E\,(s))\; f$$

New record construction expressions (not in C or Oberon, but e.g. in Ada):

$$[\![\mathbf{record}(f_1 = e_1, \ldots, f_n = e_n)]\!]_E = \lambda\, s : State \bullet \{ f_1 \mapsto [\![e_1]\!]_E(s), \ldots, f_n \mapsto [\![e_n]\!]_E(s) \}$$

New record field assignment statements:

$$[\![r.\, f \mathbin{:=} e]\!]_S = \lambda\, s : State \bullet s \oplus \{ r \mapsto ((s\ r) \oplus \{ f \mapsto [\![e]\!]_E(s) \}) \}$$

Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics Semantics

Lecture 2

Axiomatic Semantics

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Derivation of judgements written as “Hoare triples”

$$\{P\}S\{Q\}$$

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Axiomatic Semantics

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(“terminates” means “terminates without run-time error”)

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— “S **is totally correct with respect to** P and Q”

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Axiomatic Semantics vs. Operational Semantics

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Unless explicitly mentioned, we read " $\{P\}\ S\ \{Q\}$ " as meaning **partial correctness**.

Derivation Rules for Sequencing, Conditionals, Loops

Logical consequence:

$$\frac{P \Rightarrow P' \qquad \{P'\}S\{Q'\} \qquad Q' \Rightarrow Q}{\{P\}S\{Q\}}$$

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while-Loop:

$$\frac{\{INV \wedge b\}S\{INV\}}{\{INV\}\textbf{while } b \textbf{ do } S \textbf{ od}\{INV \wedge \neg b\}}$$

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$$\{P[x \setminus e]\}x := e\{P\}$$

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Considering this axiom schema as a way to ***calculate*** a precondition from assignment and postcondition, it calculates the **weakest precondition** that completes a valid Hoare triple.

Example Verification

$$\{\text{True}\}k := 0\textbf{;}\ s := 0\textbf{; while}\ k \neq n\ \textbf{do}\ k := k + 1\textbf{;}\ s := s + k\ \textbf{od}\{s = \sum_{i=1}^{n} i\}$$

Example Annotated Program

$$\{\textsf{True}\} \;\Rightarrow\; \{0 = \sum_{i=1}^{0} i\}$$

$$k := 0\mathbf{;} \qquad \{0 = \sum_{i=1}^{k} i\}$$

$$s := 0\mathbf{;} \qquad \{s = \sum_{i=1}^{k} i\}$$

while $k \neq n$

do $\{s = \sum_{i=1}^{k} i \wedge k \neq n\} \Rightarrow \{s + k + 1 = \sum_{i=1}^{k+1} i\}$

$$k := k + 1\mathbf{;} \qquad \{s + k = \sum_{i=1}^{k} i\}$$

$$s := s + k \qquad \{s = \sum_{i=1}^{k} i\}$$

od

$$\{s = \sum_{i=1}^{k} i \wedge k = n\} \;\Rightarrow\; \{s = \sum_{i=1}^{n} i\}$$

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$$\wedge\ \{s + k + 1 = \sum_{i=1}^{k+1} i\} k := k + 1 \{s + k = \sum_{i=1}^{k} i\}$$

$$\wedge\ \text{True}$$

Example Verification (ctd.)

$$\Leftarrow (\text{True} \Rightarrow 0 = \sum_{i=1}^{0} i\ ) \quad \wedge \quad \{0 = \sum_{i=1}^{0} i\}k := 0\{0 = \sum_{i=1}^{k} i\}$$

$$\wedge\ \text{True}$$

$$\wedge\ \{s = \sum_{i=1}^{k} i \wedge k \neq n\}k := k + 1\{s + k = \sum_{i=1}^{k} i\}$$

$$\wedge\ \{s + k = \sum_{i=1}^{k} i\}s := s + k\{s = \sum_{i=1}^{k} i\}$$

$$\Leftarrow \text{True}$$

$$\wedge\ s = \sum_{i=1}^{k} i \wedge k \neq n \Rightarrow s + k + 1 = \sum_{i=1}^{k+1} i$$

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Given a loop "**while** b **do** S **od**" and a postcondition Q, use the consequence rule to strengthen Q to Q', such that

- $Q' \Rightarrow Q$ (strengthening)
- Q' involves all auxiliary variables — **generalisation!**
- Q' is of shape $INV \wedge \neg b$

Simultaneous Assignments

$$\{P[x_1 \setminus e_1, \ldots, x_n \setminus e_n]\}(x_1, \ldots, x_n) := (e_1, \ldots e_n)\{P\}$$

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Simultaneous assignments

– shorten code

– save auxiliary variables (for example for swapping)

– make proofs easier

– **require simultaneous substitution**

Example Problems (with Simultaneous Assignments)

$$\{n \geq 0\} \quad (y, a, b) := (0, 1, 1) \,\textbf{;}$$
$$\textbf{while } y \neq n \textbf{ do } (y, a, b) := (y + 1, b, a + b) \textbf{ od} \quad \{a = fib_n\}$$

Given an n-element C-like array s, prove partial correctness:

$$\{\mathsf{True}\}$$
$$(i, a) := (0, 0) \,\textbf{;}$$
$$\textbf{while } i \neq n$$
$$\textbf{do} \quad \textbf{if } x = s[i]$$
$$\textbf{then } (i, a) := (i + 1, a + 1)$$
$$\textbf{fi od}$$
$$\{a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < n\}\}$$

What does this program do?

Semantics and Language Design

Semantics and Language Design

- We **use the rules** of axiomatic semantics to prove properties of programs

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- Such rule sets are **specifications** of **language implementations**
- We **define the rules** for language features and extensions
- We **justify the rules** against different presentations of the defined features
- We **derive the rules** e.g. from source-to-source translations

Fibonacci

$\{n \geq 0\}$ $(y, a, b) := (0, 1, 1)$ **;**

while $y \neq n$ **do** $(y, a, b) := (y + 1, b, a + b)$ **od** $\{a = fib_n\}$

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$\Leftarrow \langle$ (right consequence) $\rangle$

$\{n \geq 0\}\ P\ \{a = fib_y \wedge b = fib_{y+1} \wedge y = n\}$

$\wedge\ (a = fib_y \wedge b = fib_{y+1} \wedge y = n \Rightarrow a = fib_n)$

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$\Leftarrow \langle$ (sequence , logic) $\rangle$

$$\{n \geq 0\}\ (y, a, b) := (0, 1, 1)\ \{a = fib_y \wedge b = fib_{y+1}\} \wedge$$
$$\{a = fib_y \wedge b = fib_{y+1}\}\textbf{while } y \neq n \textbf{ do } A \textbf{ od}\{a = fib_y \wedge b = fib_{y+1} \wedge y = n\}$$
$$\wedge \text{ True}$$

Fibonacci (ctd.)

$\Leftarrow \langle$ (left consequence , **while**) $\rangle$

$$(n \geq 0 \Rightarrow 1 = fib_0 \wedge 1 = fib_{0+1})$$
$$\wedge \{1 = fib_0 \wedge 1 = fib_{0+1}\}\ (y, a, b) := (0, 1, 1)\ \{a = fib_y \wedge b = fib_{y+1}\}$$
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$\{a = fib_y \wedge b = fib_{y+1}\}$

$\Leftarrow \langle$ (arithmetic , assignment , left consequence) $\rangle$

True $\wedge$ True

$\wedge (a = fib_y \wedge b = fib_{y+1} \wedge y \neq n \Rightarrow b = fib_{y+1} \wedge a + b = fib_{(y+1)+1})$

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Array Traversal

Given an n-element C-like array s, prove partial correctness:

$\{\textsf{True}\}$

$(i, a) := (0, 0)$ **;**

while $i \neq n$

do **if** $x = s[i]$

then $(i, a) := (i + 1, a + 1)$

fi od

$\{a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < n\}\}$

$\{\textsf{True}\}\ P\ \{a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < n\}\}$

$\Leftarrow \langle$ (right consequence) $\rangle$

$\{\textsf{True}\}\ Init\ \mathbf{;}\ W\ \{a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i\} \wedge i = n\}$

$\wedge ((a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i\} \wedge i = n) \Rightarrow Post)$

$\Leftarrow \langle$ (sequence , logic) $\rangle$

$$\{\mathsf{True}\}\ (i,a) := (0,0)\ \{a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\}\ \}$$

$$\wedge\ \{a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\}\ \}\ W$$

$$\{a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\} \wedge i = n\}$$

$$\wedge\ \mathsf{True}$$

$$\Leftarrow \langle\ (\text{ left consequence },\ \ \textbf{while}\ )\ \rangle$$

$$(\mathsf{True} \Rightarrow 0 = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < 0\})$$

$$\wedge\ \{0 = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < 0\}\ \}\ (i,a) := (0,0)$$

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$$\Leftarrow \langle\ (\text{ logic and arithmetic },\ \ \text{assignment },\ \ \text{conditional }) \ \rangle$$

$$\mathsf{True} \wedge \mathsf{True}$$

$$\wedge\ \{a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\} \wedge i \ne n \wedge x = s[i]\}$$

$$(i,a) := (i+1, a+1)\ \{a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\}\ \}$$

$$\wedge\ ((a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\} \wedge i \ne n \wedge x \ne s[i]) \Rightarrow a = \#\{j : \mathbb{N} \ \mid s[j] = x \wedge 0 \le j < i\})$$

$\Leftarrow \langle$ (left consequence , logic) $\rangle$

$$((a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i\} \wedge i \neq n \wedge x = s[i])$$
$$\Rightarrow a + 1 = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i + 1)$$
$$\wedge \{a + 1 = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i + 1\}\} \; (i, a) := (i + 1, a + 1)$$
$$\{a = \#\{j : \mathbb{N} \mid s[j] = x \wedge 0 \leq j < i\} \}$$

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What does this program do?

Integer Square Root

$\{n \geq 0\}$

$(a, b) := (0, n + 1)$ **;**

while $a + 1 \neq b$

do $d := (a + b)/2$ **;**

if $d * d \leq n$

then $a := d$

else $b := d$

fi

od

$\{a^2 \leq n < (a + 1)^2\}$

All variables and expressions are of type **integer**.

Exercise 11.1(a, b)

$$\dfrac{\dfrac{\dfrac{\text{True}}{x \geq -5 \;\Rightarrow\; x \geq -6}\,(\text{arith.})}{x \geq -5 \;\Rightarrow\; 5 - x \leq 11}\,(\text{arith.}) \qquad \dfrac{\text{True}}{\{5 - x \leq 11\}\; z := 5 - x\; \{z \leq 11\}}\,(\text{assign.})}{\{x \geq -5\}\; z := 5 - x\; \{z \leq 11\}}\,(\text{conseq.})$$

Exercise 11.1(a, b)

$$\dfrac{\dfrac{\dfrac{\textsf{True}}{x \geq -5 \Rightarrow x \geq -6}(\text{arith.})}{x \geq -5 \Rightarrow 5 - x \leq 11}(\text{arith.}) \qquad \dfrac{\textsf{True}}{\{5 - x \leq 11\}\ z := 5 - x\ \{z \leq 11\}}(\text{assign.})}{\{x \geq -5\}\ z := 5 - x\ \{z \leq 11\}}(\text{conseq.})$$

$$\{x \geq -5\}\ z := 5 - x\ \{z \leq 11 \wedge x \geq -7\}$$

$\Leftarrow$ ⟨ (left consequence) ⟩

$$(x \geq -5 \Rightarrow 5 - x \leq 11 \wedge x \geq -7)$$
$$\wedge \{5 - x \leq 11 \wedge x \geq -7\}\ z := 5 - x\ \{z \leq 11 \wedge x \geq -7\}$$

$\Leftarrow$ ⟨ (arithmetic , assignment) ⟩

$$(x \geq -5 \Rightarrow x \geq -6 \wedge x \geq -7) \wedge \textsf{True}$$

$\Leftarrow$ ⟨ (arithmetic) ⟩

True

Exercise 11.1(c)

$$\{x \geq -5\}\ z := 5 - x\ \{z \leq 11 \wedge x \geq -3\}$$

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Using operational semantics, we can prove a **counterexample:**

$$\{x \mapsto -5\}(z := 5 - x) \Rightarrow$$

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This last state clearly does not satisfy $\{z \leq 11 \wedge x \geq -3\}$

Exercise 11.1(d)

$$\{x \geq -5\}\ z := 5 - x\ \mathbf{;}\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$$

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$$\{x \geq -5\}\ z := 5 - x\ \mathbf{;}\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$$

$\Leftarrow$ $\langle$ sequence rule $\rangle$

$$\{x \geq -5\}\ z := 5 - x\ \{z \leq 11\ \wedge\ x + 2 \geq -3\}$$

$$\wedge\ \{z \leq 11\ \wedge\ x + 2 \geq -3\}\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$$

Exercise 11.1(d)

$$\{x \geq -5\}\ z := 5 - x\ ;\ x := x + 2\ \{z \leq 11 \land x \geq -3\}$$

$\Leftarrow$ $\langle$ sequence rule $\rangle$

$$\{x \geq -5\}\ z := 5 - x\ \{z \leq 11 \ \land\ x + 2 \geq -3\}$$

$$\land\ \{z \leq 11 \ \land\ x + 2 \geq -3\}\ x := x + 2\ \{z \leq 11 \land x \geq -3\}$$

$\Leftarrow$ $\langle$ left consequence , assignment $\rangle$

$$(x \geq -5 \Rightarrow (5 - x \leq 11 \ \land\ x + 2 \geq -3))$$

$$\land\ \{5 - x \leq 11 \ \land\ x + 2 \geq -3\}\ z := 5 - x\ \{z \leq 11 \ \land\ x + 2 \geq -3\}$$

$$\land\ \mathsf{True}$$

Exercise 11.1(d)

$\{x \geq -5\}\ z := 5 - x\ ;\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$

$\Leftarrow$ ⟨ sequence rule ⟩

$\{x \geq -5\}\ z := 5 - x\ \{z \leq 11 \wedge x + 2 \geq -3\}$

$\wedge\ \{z \leq 11 \wedge x + 2 \geq -3\}\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$

$\Leftarrow$ ⟨ left consequence , assignment ⟩

$(x \geq -5 \Rightarrow (5 - x \leq 11 \wedge x + 2 \geq -3))$

$\wedge\ \{5 - x \leq 11 \wedge x + 2 \geq -3\}\ z := 5 - x\ \{z \leq 11 \wedge x + 2 \geq -3\}$

$\wedge$ True

$\Leftarrow$ ⟨ logic and arithmetic , assignment ⟩

$(x \geq -5 \Rightarrow x \geq -6) \wedge (x \geq -5 \Rightarrow x \geq -5) \wedge$ True

Exercise 11.1(d)

$\{x \geq -5\}\ z := 5 - x\ ;\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$

$\Leftarrow \quad \langle$ sequence rule $\rangle$

$\quad \{x \geq -5\}\ z := 5 - x\ \{z \leq 11\ \wedge\ x + 2 \geq -3\}$

$\quad \wedge \{z \leq 11\ \wedge\ x + 2 \geq -3\}\ x := x + 2\ \{z \leq 11 \wedge x \geq -3\}$

$\Leftarrow \quad \langle$ left consequence , assignment $\rangle$

$\quad (x \geq -5 \Rightarrow (5 - x \leq 11\ \wedge\ x + 2 \geq -3))$

$\quad \wedge \{5 - x \leq 11\ \wedge\ x + 2 \geq -3\}\ z := 5 - x\ \{z \leq 11\ \wedge\ x + 2 \geq -3\}$

$\quad \wedge$ True

$\Leftarrow \quad \langle$ logic and arithmetic , assignment $\rangle$

$\quad (x \geq -5 \Rightarrow x \geq -6) \wedge (x \geq -5 \Rightarrow x \geq -5) \wedge$ True

$\Leftarrow \quad \langle$ arithmetic $\rangle$

$\quad$ True

Exercise 11.1(e)

$$\{x \geq -5\}\ z := 5 - x\ ;\ x := x + z\ \{z \leq 11 \wedge x = 2\}$$

Exercise 11.1(e)

$$\{x \geq -5\}\ z := 5 - x\ ;\ x := x + z\ \{z \leq 11 \wedge x = 2\}$$

We again use operational semantics (expression evaluation not shown) to prove a **counterexample:**

$$\sigma_1 = \{x \mapsto 0\}$$
$$\sigma_2 = \{x \mapsto 0, z \mapsto 5\}$$
$$\sigma_3 = \{x \mapsto 5, z \mapsto 5\}$$

$$\dfrac{\dfrac{\sigma_1(5 - x) \Rightarrow 5}{\sigma_1(z := 5 - x) \Rightarrow \sigma_2} \qquad \dfrac{\sigma_2(x + z) \Rightarrow 5}{\sigma_2(x := x + z) \Rightarrow \sigma_3}}{\sigma_1(z := 5 - x\ ;\ x := x + z) \Rightarrow \sigma_3}$$

Although $\sigma_1 = \{x \mapsto 0\}$ satisfies the precondition $\{x \geq -5\}$, the final state $\sigma_3 = \{x \mapsto 5, z \mapsto 5\}$ does not satisfy the postcondition $\{z \leq 11 \wedge x = 2\}$.

Exercise 11.1(f)

$$\{z = abs(x)\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ z := -z\ \textbf{fi}\ \{xz = -x^2\}$$

Exercise 11.1(f)

one-sided conditional:

$\{z = abs(x)\}$ **if** $x \geq 0$ **then** $z := -z$ **fi** $\{xz = -x^2\}$

Exercise 11.1(f)

Rule for one-sided conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{fi}\ \{Q\}}$$

$\{z = abs(x)\}$ **if** $x \geq 0$ **then** $z := -z$ **fi** $\{xz = -x^2\}$

Exercise 11.1(f)

Rule for one-sided conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{fi}\ \{Q\}}$$

$$\{z = abs(x)\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ z := -z\ \textbf{fi}\ \{xz = -x^2\}$$

$$\Leftarrow \quad \langle \text{ one-sided conditional } \rangle$$

$$\{z = abs(x) \wedge x \geq 0\}\ z := -z\ \{xz = -x^2\}$$

$$\wedge\ (z = abs(x) \wedge x < 0 \Rightarrow xz = -x^2)$$

Exercise 11.1(f)

Rule for one-sided conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{fi}\ \{Q\}}$$

$$\{z = abs(x)\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ z := -z\ \textbf{fi}\ \{xz = -x^2\}$$

$$\begin{aligned}
\Leftarrow \quad & \langle \text{ one-sided conditional } \rangle \\
& \{z = abs(x) \wedge x \geq 0\}\ z := -z\ \{xz = -x^2\} \\
& \wedge (z = abs(x) \wedge x < 0 \Rightarrow xz = -x^2) \\
\Leftarrow \quad & \langle \text{ left consequence , arithmetic } \rangle \\
& (z = abs(x) \wedge x \geq 0 \Rightarrow x \cdot (-z) = -x^2) \\
& \wedge \{x \cdot (-z) = -x^2\}\ z := -z\ \{xz = -x^2\} \\
& \wedge (z = -x \Rightarrow xz = -x^2)
\end{aligned}$$

Exercise 11.1(f)

Rule for one-sided conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{fi}\ \{Q\}}$$

$$\{z = abs(x)\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ z := -z\ \textbf{fi}\ \{xz = -x^2\}$$

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$$\{z = abs(x) \wedge x \geq 0\}\ z := -z\ \{xz = -x^2\}$$

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$$\Leftarrow \quad \langle \text{ left consequence , arithmetic } \rangle$$

$$(z = abs(x) \wedge x \geq 0 \Rightarrow x \cdot (-z) = -x^2)$$

$$\wedge\ \{x \cdot (-z) = -x^2\}\ z := -z\ \{xz = -x^2\}$$

$$\wedge\ (z = -x \Rightarrow xz = -x^2)$$

$$\Leftarrow \quad \langle \text{ arithmetic , assignment , arithmetic } \rangle$$

$$(z = x \Rightarrow x \cdot (-z) = -x^2) \wedge \mathsf{True} \wedge \mathsf{True}$$

Exercise 11.1(f)

Rule for one-sided conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{fi}\ \{Q\}}$$

$\{z = abs(x)\}$ **if** $x \geq 0$ **then** $z := -z$ **fi** $\{xz = -x^2\}$

$\Leftarrow \quad \langle$ one-sided conditional $\rangle$

$\quad \{z = abs(x) \wedge x \geq 0\}\ z := -z\ \{xz = -x^2\}$

$\quad \wedge (z = abs(x) \wedge x < 0 \Rightarrow xz = -x^2)$

$\Leftarrow \quad \langle$ left consequence , arithmetic $\rangle$

$\quad (z = abs(x) \wedge x \geq 0 \Rightarrow x \cdot (-z) = -x^2)$

$\quad \wedge \{x \cdot (-z) = -x^2\}\ z := -z\ \{xz = -x^2\}$

$\quad \wedge (z = -x \Rightarrow xz = -x^2)$

$\Leftarrow \quad \langle$ arithmetic , assignment , arithmetic $\rangle$

$\quad (z = x \Rightarrow x \cdot (-z) = -x^2) \wedge \mathsf{True} \wedge \mathsf{True}$

$\Leftarrow \quad \langle$ arithmetic $\rangle$

$\quad$ True

Exercise 11.1(g)

$\{z = 0\}$ **if** $x = 0$ **then** $w := \mathsf{True}$ **else** $z := 1/x$ **fi** $\{\neg w \rightarrow xz = 1\}$

Exercise 11.1(g)

$$\{z = 0\}\ \textbf{if}\ x = 0\ \textbf{then}\ w := \mathsf{True}\ \textbf{else}\ z := 1/x\ \textbf{fi}\ \{\neg w \rightarrow xz = 1\}$$

$$\Leftarrow\ \langle\ \text{conditional}\ \rangle$$

$$\{z = 0 \wedge x = 0\}\ w := \mathsf{True}\ \{\neg w \rightarrow xz = 1\}$$

$$\wedge\ \{z = 0 \wedge x \neq 0\}\ z := 1/x\ \{\neg w \rightarrow xz = 1\}$$

Exercise 11.1(g)

$\{z = 0\}$ **if** $x = 0$ **then** $w :=$ True **else** $z := 1/x$ **fi** $\{\neg w \to xz = 1\}$

$\Leftarrow$ ⟨ conditional ⟩

$\{z = 0 \wedge x = 0\}\ w :=$ True $\{\neg w \to xz = 1\}$

$\wedge \{z = 0 \wedge x \neq 0\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow$ ⟨ left consequence , left consequence ⟩

$(z = 0 \wedge x = 0 \Rightarrow (\neg\text{True} \to xz = 1))$

$\wedge \{\neg\text{True} \to xz = 1\}\ w := \text{True}\ \{\neg w \to xz = 1\}$

$\wedge (z = 0 \wedge x \neq 0 \Rightarrow (\neg w \to x \cdot (1/x) = 1))$

$\wedge \{\neg w \to x \cdot (1/x) = 1\}\ z := 1/x\ \{\neg w \to xz = 1\}$

Exercise 11.1(g)

$\{z = 0\}$ **if** $x = 0$ **then** $w := \mathsf{True}$ **else** $z := 1/x$ **fi** $\{\neg w \to xz = 1\}$

$\Leftarrow$ ⟨ conditional ⟩

$\{z = 0 \wedge x = 0\}\ w := \mathsf{True}\ \{\neg w \to xz = 1\}$

$\wedge\ \{z = 0 \wedge x \neq 0\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow$ ⟨ left consequence , left consequence ⟩

$(z = 0 \wedge x = 0 \Rightarrow (\neg\mathsf{True} \to xz = 1))$

$\wedge\ \{\neg\mathsf{True} \to xz = 1\}\ w := \mathsf{True}\ \{\neg w \to xz = 1\}$

$\wedge\ (z = 0 \wedge x \neq 0 \Rightarrow (\neg w \to x \cdot (1/x) = 1))$

$\wedge\ \{\neg w \to x \cdot (1/x) = 1\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow$ ⟨ logic , assignment , logic , assignment ⟩

$(z = 0 \wedge x = 0 \Rightarrow (\mathsf{False} \to xz = 1)) \wedge \mathsf{True} \wedge (x \neq 0 \Rightarrow x \cdot (1/x) = 1)$

Exercise 11.1(g)

$\{z = 0\}$ **if** $x = 0$ **then** $w := \mathsf{True}$ **else** $z := 1/x$ **fi** $\{\neg w \to xz = 1\}$

$\Leftarrow \langle$ conditional $\rangle$

$\{z = 0 \wedge x = 0\}\ w := \mathsf{True}\ \{\neg w \to xz = 1\}$

$\wedge\ \{z = 0 \wedge x \neq 0\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow \langle$ left consequence , left consequence $\rangle$

$(z = 0 \wedge x = 0 \Rightarrow (\neg \mathsf{True} \to xz = 1))$

$\wedge\ \{\neg \mathsf{True} \to xz = 1\}\ w := \mathsf{True}\ \{\neg w \to xz = 1\}$

$\wedge\ (z = 0 \wedge x \neq 0 \Rightarrow (\neg w \to x \cdot (1/x) = 1))$

$\wedge\ \{\neg w \to x \cdot (1/x) = 1\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow \langle$ logic , assignment , logic , assignment $\rangle$

$(z = 0 \wedge x = 0 \Rightarrow (\mathsf{False} \to xz = 1)) \wedge \mathsf{True} \wedge (x \neq 0 \Rightarrow x \cdot (1/x) = 1)$

$\Leftarrow \langle$ *ex falso quodlibet* , arithmetic $\rangle$

$(z = 0 \wedge x = 0 \Rightarrow \mathsf{True}) \wedge \mathsf{True}$

Exercise 11.1(g)

$\{z = 0\}$ **if** $x = 0$ **then** $w :=$ True **else** $z := 1/x$ **fi** $\{\neg w \to xz = 1\}$

$\Leftarrow \langle$ conditional $\rangle$

$\{z = 0 \land x = 0\}\ w :=$ True $\{\neg w \to xz = 1\}$

$\land\ \{z = 0 \land x \neq 0\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow \langle$ left consequence , left consequence $\rangle$

$(z = 0 \land x = 0 \Rightarrow (\neg \text{True} \to xz = 1))$

$\land\ \{\neg \text{True} \to xz = 1\}\ w :=$ True $\{\neg w \to xz = 1\}$

$\land\ (z = 0 \land x \neq 0 \Rightarrow (\neg w \to x \cdot (1/x) = 1))$

$\land\ \{\neg w \to x \cdot (1/x) = 1\}\ z := 1/x\ \{\neg w \to xz = 1\}$

$\Leftarrow \langle$ logic , assignment , logic , assignment $\rangle$

$(z = 0 \land x = 0 \Rightarrow (\text{False} \to xz = 1)) \land \text{True} \land (x \neq 0 \Rightarrow x \cdot (1/x) = 1)$

$\Leftarrow \langle$ *ex falso quodlibet* , arithmetic $\rangle$

$(z = 0 \land x = 0 \Rightarrow \text{True}) \land \text{True}$

$\Leftarrow \langle$ logic $\rangle$

True

repeat … until …

Operational Semantics:

repeat … **until** …

Operational Semantics:

$$\frac{\sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(b) \Rightarrow \mathsf{True}}{\sigma(\textbf{repeat } s \textbf{ until } b) \Rightarrow \sigma_1}$$

repeat ... **until** ...

Operational Semantics:

$$\frac{\sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(b) \Rightarrow \textsf{True}}{\sigma(\textbf{repeat } s \textbf{ until } b) \Rightarrow \sigma_1}$$

$$\frac{\sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(b) \Rightarrow \textsf{False} \qquad \sigma_1(\textbf{repeat } s \textbf{ until } b) \Rightarrow \sigma_2}{\sigma(\textbf{repeat } s \textbf{ until } b) \Rightarrow \sigma_2}$$

repeat … **until** …

Operational Semantics:

$$\frac{\sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(b) \Rightarrow \mathsf{True}}{\sigma(\textbf{repeat}\ s\ \textbf{until}\ b) \Rightarrow \sigma_1}$$

$$\frac{\sigma(s) \Rightarrow \sigma_1 \qquad \sigma_1(b) \Rightarrow \mathsf{False} \qquad \sigma_1(\textbf{repeat}\ s\ \textbf{until}\ b) \Rightarrow \sigma_2}{\sigma(\textbf{repeat}\ s\ \textbf{until}\ b) \Rightarrow \sigma_2}$$

Axiomatic Semantics:

$$\frac{\{INV\}\ S\ \{INV\}}{\{INV\}\ \textbf{repeat}\ s\ \textbf{until}\ b\ \{INV \wedge b\}}$$

Proper for-Loops — Operational Semantics

Proper **for**-loop: Number of iterations determined at loop entrance:

- the upper limit cannot be changed
- the loop variable cannot be advanced or reset

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***Proper* for**-loop: Number of iterations determined at loop entrance:

- the upper limit cannot be changed
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Operational Semantics:

$$\frac{\sigma(beg) \Rightarrow b \qquad \sigma(end) \Rightarrow e \qquad \sigma_b = \sigma \qquad \begin{array}{l} \forall i : \mathbb{N} \mid b \le i \le e \bullet \\ (\sigma_i \oplus \{v \mapsto i\})(s) \Rightarrow \sigma_{i+1} \end{array}}{\sigma(\textbf{for } v := beg \textbf{ to } end \textbf{ do } s \textbf{ od}) \Rightarrow \sigma_{\max(b,e+1)}}$$

Proper for-Loops — Operational Semantics

***Proper* for**-loop: Number of iterations determined at loop entrance:

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- This resets the loop variable at the beginning of each iteration
- A static test can prevent assignments to the loop variable

Exercise 11.2

$\{\text{True}\}$

$(i, j, s) := (0, 0, 0)$ **;**

while $i \neq n$ **do**

  if $i = j$

  then $(i, j, s) := (i + 1, 0, s + 1)$

  else $(j, s) := (j + 1, s + 2)$

  fi

 od

$\{s = n^2 + 2j\}$

Exercise 11.2 — Proof

$\{\mathsf{True}\}\ (i, j, s) := (0, 0, 0)$ **; while** $i \neq n$ **do** B **od** $\{s = n^2 + 2j\}$

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$\{\mathsf{True}\}\ (i, j, s) := (0, 0, 0)$ **; while** $i \neq n$ **do** B **od** $\{s = n^2 + 2j\}$

$\Leftarrow$ ⟨ right consequence ⟩

$\{\mathsf{True}\}\ (i, j, s) := (0, 0, 0)$ **; while** $i \neq n$ **do** B **od** $\{s = i^2 + 2j \wedge i = n\}$

$\wedge\ (s = i^2 + 2j \wedge i = n \Rightarrow s = n^2 + 2j)$

Exercise 11.2 — Proof

$\{\textsf{True}\}\ (i, j, s) := (0, 0, 0)\ \textbf{; while}\ i \neq n\ \textbf{do}\ B\ \textbf{od}\ \{s = n^2 + 2j\}$

$\Leftarrow\quad \langle$ right consequence $\rangle$

$\{\textsf{True}\}\ (i, j, s) := (0, 0, 0)\ \textbf{; while}\ i \neq n\ \textbf{do}\ B\ \textbf{od}\ \{s = i^2 + 2j \wedge i = n\}$

$\wedge\ (s = i^2 + 2j \wedge i = n \Rightarrow s = n^2 + 2j)$

$\Leftarrow\quad \langle$ sequence , logic $\rangle$

$\{\textsf{True}\}\ (i, j, s) := (0, 0, 0)\ \{s = i^2 + 2j\}$

$\wedge\ \{s = i^2 + 2j\}\ \textbf{while}\ i \neq n\ \textbf{do}\ B\ \textbf{od}\ \{s = i^2 + 2j \wedge i = n\}$

$\wedge\ \textsf{True}$

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$\{\mathsf{True}\}\ (i,j,s) := (0,0,0)\ \mathbf{;\ while}\ i \neq n\ \mathbf{do}\ B\ \mathbf{od}\ \{s = n^2 + 2j\}$

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$\wedge\ (s = i^2 + 2j \wedge i = n \Rightarrow s = n^2 + 2j)$

$\Leftarrow\ \langle$ sequence , logic $\rangle$

$\{\mathsf{True}\}\ (i,j,s) := (0,0,0)\ \{s = i^2 + 2j\}$

$\wedge\ \{s = i^2 + 2j\}\ \mathbf{while}\ i \neq n\ \mathbf{do}\ B\ \mathbf{od}\ \{s = i^2 + 2j \wedge i = n\}$

$\wedge\ \mathsf{True}$

$\Leftarrow\ \langle$ left consequence , **while** -rule $\rangle$

$(\mathsf{True} \Rightarrow 0 = 0^2 + 2 \cdot 0)$

$\wedge\ \{0 = 0^2 + 2 \cdot 0\}\ (i,j,s) := (0,0,0)\ \{s = i^2 + 2j\}$

$\wedge\ \{s = i^2 + 2j \wedge i \neq n\}\ \mathbf{if}\ i = j\ \mathbf{then}\ (i,j,s) := (i+1, 0, s+1)$

$\mathbf{else}\ (j,s) := (j+1, s+2)\ \mathbf{fi}\ \{s = i^2 + 2j\}$

Exercise 11.2 — Proof (ctd.)

$\Leftarrow$ $\langle$ arithmetic , assignment , conditional $\rangle$

$\mathsf{True} \wedge \mathsf{True}$

$\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i = j\}\ (i, j, s) := (i + 1, 0, s + 1)\ \{s = i^2 + 2j\}$

$\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i \neq j\}\ (j, s) := (j + 1, s + 2)\ \{s = i^2 + 2j\}$

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Exercise 11.2 — Proof (ctd.)

$\Leftarrow$ $\langle$ arithmetic , assignment , conditional $\rangle$

$$\begin{aligned}
&\mathsf{True} \wedge \mathsf{True} \\
&\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i = j\}\ (i, j, s) := (i + 1, 0, s + 1)\ \{s = i^2 + 2j\} \\
&\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i \neq j\}\ (j, s) := (j + 1, s + 2)\ \{s = i^2 + 2j\}
\end{aligned}$$

$\Leftarrow$ $\langle$ left consequence , left consequence $\rangle$

$$\begin{aligned}
&(s = i^2 + 2j \wedge i \neq n \wedge i = j \Rightarrow s + 1 = (i + 1)^2 + 2 \cdot 0) \\
&\wedge \{s + 1 = (i + 1)^2 + 2 \cdot 0\}\ (i, j, s) := (i + 1, 0, s + 1)\ \{s = i^2 + 2j\} \\
&\wedge (s = i^2 + 2j \wedge i \neq n \wedge i \neq j \Rightarrow s + 2 = i^2 + 2 \cdot (j + 1)) \\
&\wedge \{s + 2 = i^2 + 2 \cdot (j + 1)\}\ (j, s) := (j + 1, s + 2)\ \{s = i^2 + 2j\}
\end{aligned}$$

Exercise 11.2 — Proof (ctd.)

$\Leftarrow$ $\langle$ arithmetic , assignment , conditional $\rangle$

$$\mathsf{True} \wedge \mathsf{True}$$
$$\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i = j\}\ (i, j, s) := (i + 1, 0, s + 1)\ \{s = i^2 + 2j\}$$
$$\wedge \{s = i^2 + 2j \wedge i \neq n \wedge i \neq j\}\ (j, s) := (j + 1, s + 2)\ \{s = i^2 + 2j\}$$

$\Leftarrow$ $\langle$ left consequence , left consequence $\rangle$

$$(s = i^2 + 2j \wedge i \neq n \wedge i = j \Rightarrow s + 1 = (i + 1)^2 + 2 \cdot 0)$$
$$\wedge \{s + 1 = (i + 1)^2 + 2 \cdot 0\}\ (i, j, s) := (i + 1, 0, s + 1)\ \{s = i^2 + 2j\}$$
$$\wedge (s = i^2 + 2j \wedge i \neq n \wedge i \neq j \Rightarrow s + 2 = i^2 + 2 \cdot (j + 1))$$
$$\wedge \{s + 2 = i^2 + 2 \cdot (j + 1)\}\ (j, s) := (j + 1, s + 2)\ \{s = i^2 + 2j\}$$

$\Leftarrow$ $\langle$ arithmetic , assignment , arithmetic , assignment $\rangle$

$$\mathsf{True} \wedge \mathsf{True} \wedge \mathsf{True} \wedge \mathsf{True}$$

Exercise 11.2 — Stronger Postcondition

$\{\mathsf{True}\}$

$(i, j, s) := (0, 0, 0)$ **;**

while $i \neq n$ **do**

  if $i = j$

  then $(i, j, s) := (i + 1, 0, s + 1)$

  else $(j, s) := (j + 1, s + 2)$

  fi

od

$\{s = n^2\}$

Exercise 11.2 — Stronger Postcondition

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$(i, j, s) := (0, 0, 0)$ **;**

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  fi

od

$\{s = n^2\}$

Challenge: How can you prove this?

Exercise 11.2 — Stronger Postcondition

$\{\mathsf{True}\}$

$(i, j, s) := (0, 0, 0)$ **;**

while $i \neq n$ **do**

  if $i = j$

  then $(i, j, s) := (i + 1, 0, s + 1)$

  else $(j, s) := (j + 1, s + 2)$

  fi

od

$\{s = n^2\}$

Challenge: How can you prove this?
Can you reformulate the program to make this easier?

Operational and Axiomatic Semantics

- **Operational Semantics** proves statements of the following shape:

$$\mathit{State}_1 \; (\; \mathit{Statement} \;) \;\Rightarrow\; \mathit{State}_2$$

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Operational and Axiomatic Semantics

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- **Axiomatic Semantics** proves statements of the following shape:

  $$\{ \; Precondition \; \} \; Statement \; \{ \; Postcondition \; \}$$

  Proofs operate on **conditions on states** instead of states: **abstracting** away from states

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- **Operational Semantics** proves statements of the following shape:

  $$State_1 \ (\ Statement\ ) \ \Rightarrow\ State_2$$

  For some combinations of *Statement* and $State_1$, no such $State_2$ exists …

  Proofs about **a single execution path**: (counter-)examples rather than verification of specifications

- **Axiomatic Semantics** proves statements of the following shape:

  $$\{\ Precondition\ \}\ Statement\ \{\ Postcondition\ \}$$

  Proofs operate on **conditions on states** instead of states: **abstracting** away from states

  For different properties, different proofs ***along the same program structure.***

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  In the textbook, denotational semantics appears mostly as a reorganisation of operational semantics.

  In general, the denotational semantics is **far more abstract** than operational semantics, and employs advanced concepts from discrete mathematics.

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For **simple** semantics of imperative programs, **sets of partial functions** $A \nrightarrow B$ can be used as domains:

– the subset ordering $\subseteq$ serves as definedness ordering:

$$\forall f, g : A \nrightarrow B \bullet f \sqsubseteq g :\Leftrightarrow f \subseteq g$$

– the empty function $\emptyset : A \nrightarrow B$ is the a least element of $A \nrightarrow B$.

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$$\begin{aligned}&\mathit{Left} :: a \to \mathit{Either}\ a\ b\\ &\mathit{Right} :: b \to \mathit{Either}\ a\ b\end{aligned}$$

and allows pattern matching:

```
valShow :: Either Integer Bool → String
valShow (Left  i) = "int: "  ++ show i
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In mathematical use, *Left* and *Right* are frequently not mentioned.

Semantic Functions

$[\![\_]\!]_E$	$: \; \mathit{Expr} \to (\mathit{State} \nrightarrow \mathit{Val})$	expression semantics
$[\![\_]\!]_S$	$: \; \mathit{Stmt} \to (\mathit{State} \nrightarrow \mathit{State})$	statement semantics

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Textbook:

M	:	$(\mathit{Expr} \times \mathit{State}) \nrightarrow \mathit{Val}$	expression semantics
M	:	$(\mathit{Stmt} \times \mathit{State}) \nrightarrow \mathit{State}$	statement semantics
$\mathit{ApplyBinary}$	:	$(\mathit{Op} \times \mathit{Val} \times \mathit{Val}) \nrightarrow \mathit{Val}$	operator semantics

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$ApplyBinary$	:	$(Op \times Val \times Val) \nrightarrow Val$	operator semantics

- No clean separation between syntax and semantics
- Undefinedness ordering less obvious

Expression Semantics

$$Expr ::= Id \mid Num \mid Bool \mid Expr\ Op\ Expr$$

$$[\![\_]\!]_E \quad : \ Expr \to (State \nrightarrow Val)$$

Assuming $s : State$, i.e., $s : Id \nrightarrow SVal$, we define:

for $v : Id$: $\quad [\![v]\!]_E\ (s) = s(v)$

— **undefined** if $s(v)$ is undefined!

for $n : Num$: $\quad [\![n]\!]_E\ (s) = n$

for $b : Bool$: $\quad [\![b]\!]_E\ (s) = b$

for $e_1, e_2 : Expr; op : Op$: $\quad [\![e_1\ op\ e_2]\!]_E\ (s) = [\![op]\!]_O([\![e_1]\!]_E\ (s), [\![e_2]\!]_E\ (s))$

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Where clear from the context, we write $[\![e]\!]$ instead of $[\![e]\!]_{\mathrm{E}}$.

Expression Semantics — Examples

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Writing “ $\bot$ ” here is short-hand for indicating **undefined** terms.

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$$[\![x \&\& y]\!](s_1) = [\![x]\!](s_1) \wedge [\![y]\!](s_1) = s_1(x) \wedge s_1(y) = 5 \wedge 42 = \bot$$

wrong type!

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Statement Semantics

$$[\![\_]\!]_{\mathrm{S}} \quad : \ \mathit{Stmt} \to (\mathit{State} \nrightarrow \mathit{State})$$

For $s : \mathit{State}$, i.e., $s : \mathit{Id} \nrightarrow \mathit{SVal}$, and $p, p_1, p_2 : \mathit{Stmt}$ and $e : \mathit{Expr}$ and $v : \mathit{Id}$:

$$[\![\mathbf{skip}]\!]_{\mathrm{S}}\ (s) = s$$

$$[\![v \mathbf{:=} e]\!]_{\mathrm{S}}\ (s) = s \oplus \{v \mapsto [\![e]\!]_{\mathrm{E}}\ (s)\}$$

— **undefined** if $[\![e]\!]_{\mathrm{E}}\ (s)$ is undefined!

$$[\![p_1\ \mathbf{;}\ p_2]\!]_{\mathrm{S}} = [\![p_2]\!]_{\mathrm{S}} \circ [\![p_1]\!]_{\mathrm{S}}$$

$$[\![\mathbf{if}\ e\ \mathbf{then}\ p_1\ \mathbf{else}\ p_2]\!]_{\mathrm{S}}\ (s) = \begin{cases} [\![p_1]\!]_{\mathrm{S}}\ (s) & \text{if } [\![e]\!]_{\mathrm{E}}\ (s) = \text{True} \\ [\![p_2]\!]_{\mathrm{S}}\ (s) & \text{if } [\![e]\!]_{\mathrm{E}}\ (s) = \text{False} \\ \textbf{undefined} & \text{otherwise} \end{cases}$$

Relating Simple Denotational and Operational Semantics

Simple Denotational		Simple Operational
$[\![e]\!]_{\mathrm{E}}\,(\sigma_1) = v$	$\Leftrightarrow$	$\sigma_1(e) \Rightarrow v$
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- $[\![e]\!]_{\mathrm{E}}$ and $[\![s]\!]_{\mathrm{S}}$ are **explicit functions**
- These can be considered as results of **function abstraction** from the operational semantics of *e* and *s*.

λ-Calculus (Textbook 8.1) — Motivation for the λ-Notation

The usual way to define functions:

$$f(x) = 2 * x - 3$$

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λ-abstraction **binds** a variable (here: x). Application of a λ-abstraction to an argument is **reduced** to the body of the abstraction with the bound variable replaced by the argument.

λ-Terms

Now the formal definition of **untyped λ-terms**: An untyped λ-term is either

- a **variable** $x, y, z, \ldots,$ or
- a **function application** $(M)\ N$ of one untyped λ-term F (the function) to another A (the argument), or
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The λ-calculus was intended by its inventor, **Alonzo Church** (1903–1995), as a foundation of mathematics based on functions instead of on sets.

Free Variables

The set $FV(M)$ of the variables occurring **free** in the λ-term M is defined inductively over the construction of λ-terms (this is called: *structural induction*):

- $FV(x) = \{x\}$
- $FV(\lambda\, x \bullet M) = FV(M) \setminus \{x\}$
- $FV(M\ N) = FV(M) \cup FV(N)$

Variable Replacement (auxiliary concept)

$M[x \setminus y]$ denotes the term resulting from M by replacing all **free** occurrences of variable x with variable y:

- $v[x \setminus y] = \begin{cases} y & \text{if } v = x \\ v & \text{if } v \neq x \end{cases}$
- $(M\ N)[x \setminus y] = M[x \setminus y]\ N[x \setminus y]$
- $(\lambda\ v \bullet M)[x \setminus y] = \begin{cases} \lambda\ v \bullet M & \text{if } v = x \\ \lambda\ v \bullet (M[x \setminus y]) & \text{if } v \neq x \end{cases}$

— Variable replacement is **only** used in the definition of α-conversion.

α-Conversion

If $y \notin FV(M)$, and if there is no λ-binding for y in M, then the following **renaming of a bound variable** is defined:

$$\lambda\, x \bullet M \;\equiv_\alpha\; \lambda\, y \bullet M[x \setminus y]$$

This can also be applied in any context $C[\ ]$ (a context is a term with exactly one occurrence of the "hole" "[ ]"):

$$C[\ \lambda\, x \bullet M\ ] \;\equiv_\alpha\; C[\ \lambda\, y \bullet M[x \setminus y]\ ]$$

α-Conversion	**=**	**renaming of bound variables**

Substitution

Substitution is replacement of free variables by terms:

- $v[x \setminus t] = \begin{cases} t & \text{if } v = x \\ v & \text{if } v \neq x \end{cases}$

- $(M\ N)[x \setminus t] = M[x \setminus t]\ N[x \setminus t]$

- $(\lambda\, v \bullet M)[x \setminus t] = \begin{cases} \lambda\, v \bullet M & \text{if } v = x \lor x \notin FV(M) \\ \lambda\, v \bullet (M[x \setminus t]) & \text{if } v \neq x \land x \in FV(M) \land v \notin FV(t) \\ \textbf{\textit{not permitted!}} & \text{if } v \neq x \land x \in FV(M) \land v \in FV(t) \end{cases}$

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β-Reduction

The central reduction rule of λ-calculus:

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$$\begin{aligned}
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& \equiv_\alpha & (\lambda\, u \bullet x\, (u\, x))[x \setminus (\lambda\, y \bullet z\, y)] \\
& = & \lambda\, u \bullet ((\lambda\, y \bullet z\, y)\, (u\, (\lambda\, y \bullet z\, y))) \\
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Reduction Strategies

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Important properties:

- Leftmost-outermost strategy (**Haskell**, Miranda, Clean):
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Important properties:

- Leftmost-outermost strategy (**Haskell**, Miranda, Clean):
  - **call by name, lazy evaluation**
  - terminates if possible
  - non-strict
- Leftmost-innermost strategy (**OCaml**, SML, LISP, Scheme):
  - **call by value, eager evaluation**
  - easier to implement
  - **strict:** for all f we have $f(\perp) = \perp$

The Fixedpoint Combinator Y

Church's fix edpointcombinator "Y":

$$Y \;=\; \lambda f \bullet (\lambda x \bullet f\ (x\ x))(\lambda x \bullet f\ (x\ x))$$

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All general fix edpointcombinators involve **self-application** like "$x\ x$" — this possible in the untyped λ-calculus, but not in most typed systems.

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- We write "Y" also as fix edpointcombinator in a mathematical context
- More precisely, we let "$Y\ F$" denote the **least fixedpoint** of F
- Other notations: "$\mu\ F$", or "fix F"

Parameter Passing

Parameter passing has two sides — assume a parameterized subprogram *P*:

- **Formal Parameters:** The names used to refer to the parameters in the definition of *P*.
- **Actual Parameters:** The expressions supplied to *P* as instances of its parameters for the purpose of creating an *incarnation* of *P*.

If several actual parameters are supplied to a subprogram defined with several formal parameters, how is a **correspondence** established?

- **Positional correspondence:** *n*-th actual parameter instantiates *n*-th formal parameter
- **Explicit parameter labels:** allow arbitrary order as far as labels are supplied (often position al fallback).

Ada: `Sub( Y => B, X => 27 );`

OCaml: `sub ~y:b ~x:27`

Labelled Arguments in OCaml

`#let f ~x ~y = x - y;;` `val f : x:int -> y:int -> int = <fun>` — x and y are **labels**

`#let x = 3 and y = 2 in f ~x ~y;;` — x and y are **labelled arguments** `- : int = 1`

`#let x = 3 and y = 2 in f ~y ~x;;` — labelled arguments may be **commuted** `- : int = 1`

`#let f ~x:x1 ~y:y1 = x1 - y1;;` — x1 and y1 are **formal parameters**

`val f : x:int -> y:int -> int = <fun>` — x and y are **labels**

`#f ~x:3 ~y:2;;` `- : int = 1` `#f ~y:2 ~x:3;;` `- : int = 1`

`#f 3 2;;` — labels can be omitted if **all** arguments are supplied! `- : int = 1`

Typical Application of Labelled Arguments in OCaml

```
ListLabels.fold_right : f:('a -> 'b -> 'b) -> 'a list -> init:'b -> 'b
val list_predsplit : f:('a -> bool) -> 'a list -> 'a list * 'a list
let list_predsplit ~f = ListLabels.fold_right  ~init:([],[])
      ~f:(fun x (xs,ys) -> if f x then (x :: xs,ys) else (xs,x::ys));;
val of_psp : ((f * 'a) list * f) -> 'a t let of_psp (ps,tl) =   let d = singleton tl in
  List.fold_right ~init:d ps    ~f:(fun p d -> let _ = fe_onto_start p d in d);;
    ...  (List.fold_right ns ~init:[] ~f:(fun n res ->
              (try (match snd (Nodemap.find attrmap ~key:n) with
                      None -> res                    | Some (Left h) -> (n,h) :: res
                    | Some (Right q) -> res                    )
               with Not_found -> res          )));;
```

Optional Arguments in OCaml

```
#let bump ?(step = 1) x = x + step;;
val bump : ?step:int -> int -> int = <fun>
```

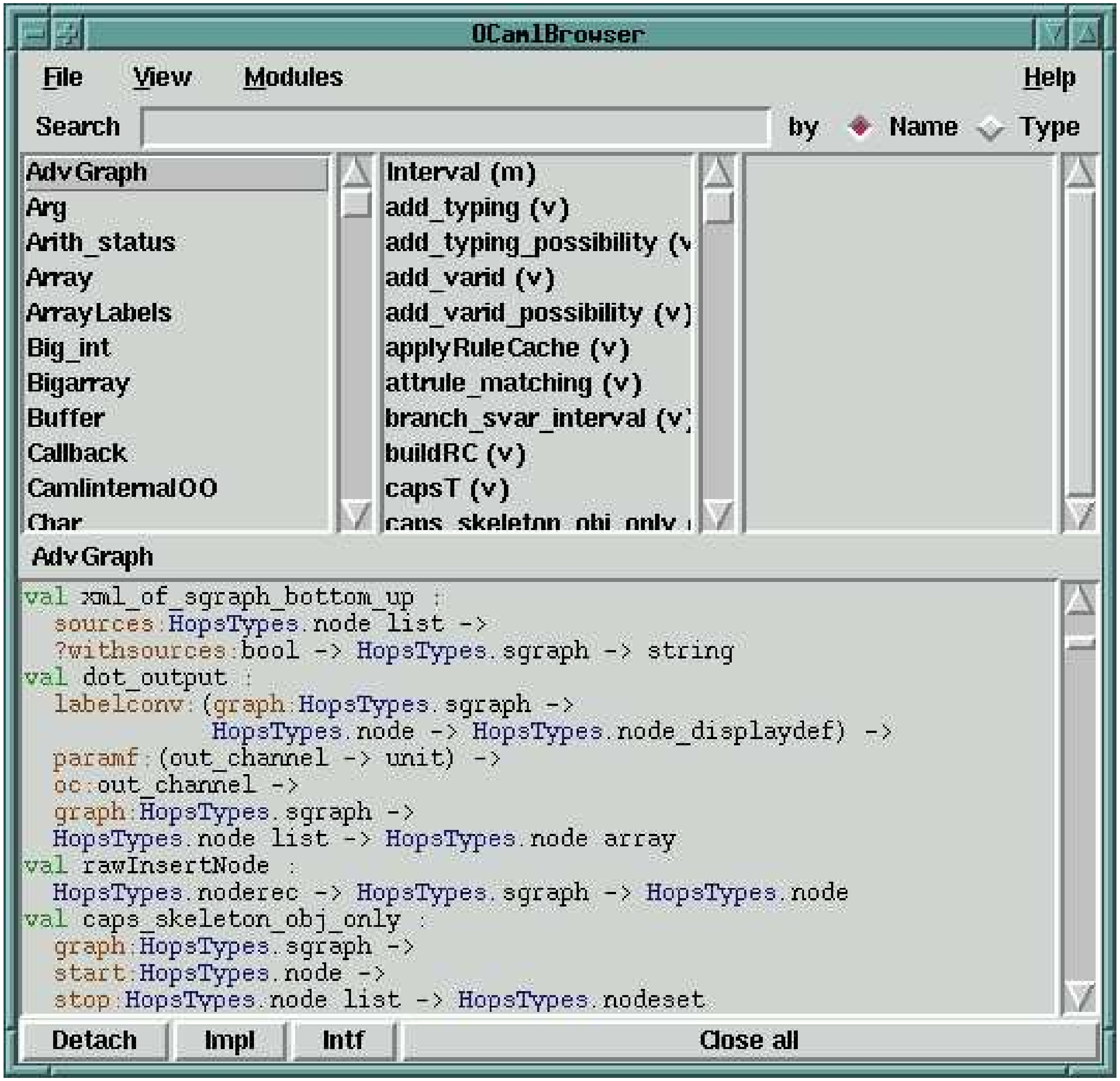
— optional arguments have a label and a default value.

```
#bump 2;;
- : int = 3
```

```
#bump ~step:3 2;;
- : int = 5
```

A function taking some optional arguments must also take at least one non-labeled argument. (Important for partial applications)

OCamlBrowser



Evaluation Aspects of Parameter Passing

- **Call by value:** Actual parameter expression is evaluated to a **value** before instantiation of formal parameter.
  - leftmost-innermost strategy in λ-calculus
  - OCaml, C, Java, Oberon, Ada, …
  - **strict** function call semantics: undefined arguments **always** produce undefined results

```
# const 4 (3 / 0);;
Exception: Division_by_zero.
```

- **Call by name:** Formal parameter f is instantiated with unevaluated actual parameter expression e — several occurrences of f result in several copies of e.
  - leftmost-outermost strategy in λ-calculus

- **Lazy evaluation:** call by name with **sharing** instead of copying: e is never duplicated; all occurrences of f are instantiated with references to single e, which is evaluated at most once.

- graph reduction implementation of leftmost-outermost
- most Haskell implementations
- **non-strict** function call semantics: undefined arguments **need not** produce undefined results

```
Prelude> const 4 (3 / 0)
4
```

Storage Aspects of Parameter Passing

- **Call by constant value:** value available as local constant
- **Call by copy** (call by value): value copied into local variable
- **Call by reference:** actual parameter needs to be a reference, or define a reference; this is used as value of formal parameter.
- **Call by value-result:** actual parameter needs to be a reference, or define a reference *r*; local variable *l* is initialized from r-value of *r*, and on subprogram exit, *r* is overwritten with contents of *l*.
- **Call by result:** Similar, but *l* starts out uninitialized.

Call by value, Call by reference, and Scoping

```
MODULE Scope1;
IMPORT Out;
VAR n : INTEGER;
PROCEDURE B(VAR x : INTEGER;
            z : INTEGER);
  VAR hv : INTEGER;
  BEGIN
    IF z = 0
    THEN x := 0
    ELSE hv := x;
         B(x, z-1);
         x := x+hv  END;
  END B;
```

```
PROCEDURE A(y,x: INTEGER;
            VAR result: INTEGER);
  BEGIN
    IF x = 0
    THEN result:=1;
    ELSE A(y, x-1, result);
         B(result, y)       END;
  END A;
BEGIN
  n := 0;
  A(2,1,n);
  Out.Int(n,0); Out.Ln
END Scope1.
```

Call by value, Call by reference, and Scoping

```
MODULE Scope1;
IMPORT Out;
VAR n : INTEGER;
PROCEDURE B(VAR x : INTEGER;
            z : INTEGER);
  VAR hv : INTEGER;
  BEGIN
    IF z = 0
    THEN x := 0
    ELSE hv := x;
         B(x, z-1);
         x := x+hv  END;
  END B;
```

```
PROCEDURE A(y,x: INTEGER;
            VAR result: INTEGER);
  BEGIN
    IF x = 0
    THEN result:=1;
    ELSE A(y, x-1, result);
         B(result, y)      END;
  END A;
BEGIN
  n := 0;
  A(2,1,n);
  Out.Int(n,0); Out.Ln
END Scope1.
```

$$\{\mathtt{x} = a\} \quad \mathtt{B(x,z)} \quad \{\mathtt{x} = a * \mathtt{z}\}$$

$$\{\mathsf{True}\} \quad \mathtt{A(y,x,result)} \quad \{\mathtt{result} = \mathtt{y}^{\mathtt{x}}\}$$

Parameter Passing

- Formal parameters — actual parameters
- **Correspondence aspects:** by position, by label, optional arguments
- **Evaluation aspects:** call by value, call by name, lazy evaluation
- **Storage aspects:** call by constant value, call by copy, call by reference, call by value-result, call by result

X

X

X

X