Domain Theory

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Denotational Semantics

- Express meaning of terms by mathematical objects
- Constructing a model for λ-calculus
- Need a domain of individuals, and associating a function with each lambda term.

Problems

- Domain must consist of all constants `S`, and functions over them `S → S`, ...
- $S \simeq [S \to S]$
- Simple example:
  $\lambda x . x x$
- Associated functions cannot be total

Domains

- Need to define structures to describe domains of computation
- In typed λ-calculus, domains are models for types
- Representation of partial/incomplete data
- {Dana Scott, 1970}

Order Theory

- Partially-ordered sets:
  - $Reflexive: x \leq x$
  - $Transitive: x \leq y \wedge y \leq z \Rightarrow x \leq z$
  - $Antisymmetric: x \leq y \wedge y \leq x \Rightarrow x = y$
- Subsets may have upper and lower bounds, maximal and minimal elements
  - $ub(A) = \{x \in P \mid \forall_{a \in A} a \leq x\}$
  - a is maximal in set A if there is no other elements above it

More Order Theory

- Suprema and infima:
  - x is sup(A) (join) if it is the minimal element of the upper bound on A.
  - inf (meet) is maximal of lower bound
- Lattice: if sup and inf exist for every pair of elements
- Complete lattice: if they exist for any subset

Fixed Points

- A function on partial ordered sets is *monotone* if:

$$\forall_{x,\,y \in P} \quad x \leq y \Rightarrow f(x) \leq f(y)$$

- Theorem: If L is a complete lattice, then every monotone map from L to L has a fixed point.

Directed-Complete Partial Orders

- Subset A is *directed*, if it's non-empty, and every pair of elements in A has an upper bound in A.
- A partially-ordered set where every directed subset has a supremum is called a directed-complete partial order.

Approximation

- `x` approximates `y` (`x << y`) if for any directed subset A of D,

  $$y \leq \sup(A) \Rightarrow \exists_{a \in A} x \leq a$$

- `x` is compact if `x << x`

- A subset B of D is a basis if for every x, the set of elements of B that approximate x contains a directed subset with sup = x

Continuity

- A function `f: D → E` is continuous if it is monotone and for each directed subset of D:

  $$f(\sup A) = \sup f(A)$$

- A *continuous domain* is a directed-complete partial order that has a basis. An *algebraic domain* has a basis of compact elements. It is *ω-continuous* if the basis is countable.

More continuity

- Continuous functions have a fixed point
- Continuity implied computability
- Scott's work on “computationally feasible”: infinite objects are limits of their finite approximations.

Topology of Domains

- A subset C of domain D is *closed* if it's closed under suprema of directed subsets, and:

  $x \in D \Rightarrow \forall_{y \leq x}\, y \in D$

- A subset is *open* if its complement is closed.

- $x \leq y \Leftrightarrow x \in \overline{y}$

References

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