

Lab 6 - Sequences, Sums and Mathematical Induction

Computer Science 1FC3

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Abstract

This lab will demonstrate how to work with sequences and sums using Maple. We also show how to use Maple to assist in working out proofs of various mathematical assertions using mathematical induction.

1 Sequences

In mathematics, a sequence is a list of objects arranged in a “linear” fashion, such that *the order of the members* is well defined and significant. For example, (X,Y,Z) is a sequence of letters that differs from (Y,Z,X) , as the ordering does matter. Sequences can be *finite*, as in this example, or *infinite*, such as the sequence of all even positive integers $(2,4,6,\dots)$.

The members of a sequence are also called its *terms*, and the number of terms (possibly infinite) is called the *length* of the sequence. Formally, a sequence can be defined as follows:

Definition 1:

A finite sequence with terms in a set S is a function from $\{1, 2, \dots, n\}$ to S for some $n \geq 0$. An infinite sequence in S is a function from $\{1, 2, \dots\}$ (the set of natural numbers) to S . **(End of Definition.)**

As you can see, sequences are very closely related to *mathematical function*. (Do you see why?) In this course, we are more interested in infinite sequences. By definition, for each sequence, we can always find a function from the set of natural numbers to its terms, and hence we can use function definition in Maple to define sequences as shown in the following example:

Example 1:

Example the sequence $\{a_n\}$, where $a_n = 1/n$, can be defined in Maple as:

```
an := n -> 1/n;
```

Example the sequence $\{b_n\}$, where $b_n = 17 \cdot (5^n)$, can be defined in Maple as:

```
an := n -> 17*(5^n);
```

Remark. Notice the notation $\{a_n\}$ to describe sequence where a_n represents an individual term of the sequence $\{a_n\}$. This is not to be confused with the notation of a singleton (one element set). **(End of Remark.)**

Exercise 1:

Define the following sequences using Maple:

1. The sequence $\{c_n\}$, where $c_n = n!$
2. The sequence $\{d_n\}$, where $d_n = (1 - n)^2 + (1 - n)^3$

2 Sums

First of all, we want to review how to use the summation notation \sum through the following set of exercises.

Exercise 2:

1. Define the following sum using \sum notation:

$$1! + 2! + \dots + n!$$

2. Rewrite the \sum notation in part 1. but indexing from 5.
3. Simplify the sum of the following three \sum 's by reindexing the summation notation.

$$\sum_{k=1}^n k^k + \sum_{k=0}^{n-1} [(k+1)! - (k+1)^{k+1}] + \sum_{k=0}^n k!$$

4. Simplify the sum of the following two \sum 's by reindexing the summation notation.

$$\sum_{k=1}^{\infty} k^k + \sum_{k=0}^{\infty} (k! - (k+1)^{k+1})$$

We can also symbolically compute the sum in Maple using command **sum** as in the following example.

Example 2:

The sum $\sum_{k=1}^n k(1/2^k)!$ can be defined in Maple as:

```
sum(k*(1/2*k)!, k=1..n);
```

or in a functional form (partial sum) as following:

```
sn := n -> sum(k*(1/2*k)!, k=1..n);
```

You can try to play with it a bit:

```
sn(a);
sn(a+12);
sn(10);
sn(100);
```

Notice that $sn(100)$ will give you something ridiculous like:

$$\begin{aligned} & 35 \frac{35}{2}! + 49 \frac{49}{2}! + 89 \frac{89}{2}! + 47 \frac{47}{2}! + 37 \frac{37}{2}! + 23 (23/2)! + \\ & 31022375065747645604484860329386064221265194400339722239999999999998 + (1/2)! + 3 (3/2)! + \\ & 5 (5/2)! + 7 (7/2)! + 9 (9/2)! + 11 (11/2)! + 13 (13/2)! + 15 (15/2)! + 17 (17/2)! + 19 (19/2)! + 21 (21/2)! + \\ & 25 \frac{25}{2}! + 27 \frac{27}{2}! + 29 \frac{29}{2}! + 31 \frac{31}{2}! + 33 \frac{33}{2}! + 39 \frac{39}{2}! + 41 \frac{41}{2}! + 43 \frac{43}{2}! + 45 \frac{45}{2}! + 51 \frac{51}{2}! + 53 \frac{53}{2}! + 55 \frac{55}{2}! + 57 \frac{57}{2}! + \\ & 59 \frac{59}{2}! + 61 \frac{61}{2}! + 63 \frac{63}{2}! + 65 \frac{65}{2}! + 67 \frac{67}{2}! + 69 \frac{69}{2}! + 71 \frac{71}{2}! + 73 \frac{73}{2}! + 75 \frac{75}{2}! + 77 \frac{77}{2}! + 79 \frac{79}{2}! + 81 \frac{81}{2}! + 83 \frac{83}{2}! + \\ & 85 \frac{85}{2}! + 87 \frac{87}{2}! + 91 \frac{91}{2}! + 93 \frac{93}{2}! + 95 \frac{95}{2}! + 97 \frac{97}{2}! + 99 \frac{99}{2}! \end{aligned}$$

Remark. You can also define infinite sum quite easily with Maple. For example, $\sum_{k=0}^{\infty} (k!)^{-1}$ can be defined in Maple as:

```
sum((1/k!), k=0..infinity);
```

Check what Maple returns. Have you seen why we study Maple yet?(**End of Remark.**)

3 Mathematical Induction

3.1 Simple Induction

Let's review how Simple Induction works ("A picture is worth a thousand words!"):



Exercise 3:

Knowing that you are studying in the famous CS1FC3 course, if someone asks you to convince him/her that Simple Induction actually works, how are you going to do that? (Hint: well-ordering.)

A normal Simple Induction proof will look like in the following example:

Example 3:

Use the Principle of Mathematical Induction to prove that $\sum_{k=1}^n (2k + 3) = n(n + 4)$ for all $n \geq 1$.

Solution. Assume

$$P(n) : 5 + 7 + 9 + \dots + (2n + 3) = n(n + 4)$$

Base Step: $P(1)$ states that $2 \cdot 1 + 3 = 1(1 + 4)$, which is true since both sides equal 5.

Inductive Hypothesis: Suppose $P(k)$ is true; that is

$$5 + 7 + 9 + \dots + (2k + 3) = k(k + 4) \quad (*)$$

Inductive Step: $P(k) \rightarrow P(k + 1)$: We want to show $P(k + 1) : 5 + 7 + 9 + \dots + (2k + 3) + (2k + 5) = (k + 1)(k + 5)$. To do so, we add $2k + 3$ to both sides of (*) and simplify:

$$\begin{aligned} LHS &= 5 + 7 + 9 + \dots + (2k + 3) + (2k + 5) = k(k + 4) + (2k + 5) \\ &= k^2 + 6k + 5 = (k + 1)(k + 5) = RHS \end{aligned}$$

which proves $P(k + 1)$. Thus, $P(k) \rightarrow P(k + 1)$ is true.

Therefore, by the Principle of Mathematical Induction, $5 + 7 + 9 + \dots + (2n + 3) = n(n + 4)$ is true for all $n \geq 1$.
(End of Solution.)

You can use Maple's ability to manipulate expressions symbolically to help to construct an inductive proof. Here is how an interactive proof of the formula above, by mathematical induction, can be carried out in Maple.

Example 4:

The general term of the sum is

$$\text{term} := k \rightarrow 2*k+3;$$

while the right hand side of the formula is

$$\text{formula} := n \rightarrow n * (n + 4);$$

We can check the basis step for the induction; here the base case is that in which $n = 1$.

$$\begin{aligned} &\text{term}(1); \\ &\text{formula}(1); \end{aligned}$$

The results agree, so the basis step is established.

For the inductive step, we suppose the formula to be valid for $n = k$.

```
indhyp := formula(k);
```

To sum $k + 1$ terms, we compute the value of LHS

```
LHS:=indhyp + term(k+1);
```

Finally, the formula for $n = k + 1$ is

```
RHS:=formula(k+1);
```

To check if the results agree, we simplify the difference of LHS and RHS symbolically using **simplify** Maple command.

```
simplify(LHS-RHS);
```

So the inductive step is verified. The formula now follows by mathematical induction. Thus, you can see that, while Maple is not (yet) able to construct proofs entirely on its own, it is a very effective tool to use in an interactive proof construction or **symbolic computation**. In fact, the ability of Maple doing symbolic computation is extremely useful in practice!

3.2 Strong Induction

Why is this called Strong Induction? You might need one or every previous case to prove the k^{th} case. This is like saying that each domino is bigger than the previous one, so the *combined weight of all the previous dominoes* is needed to push over the next one. A Proof by Strong Induction only differs from a proof by Simple Induction in the Inductive Hypothesis and Inductive Step:

1. **Inductive Hypothesis:** We state that we are assuming the hypothesis $P(i)$ to be true *for all* $i \leq k$.
2. **Inductive Step:** We want to show that the hypothesis is true for the next case $k + 1$, i.e $P(k + 1)$ is true, but this approach can now take advantage of the *stronger assumptions* stated in our Inductive Hypothesis.

In computer science, an interesting example of a proof by Strong Induction can be found in the proof of correctness for Quicksort algorithm. Please read Rosen for more examples on how Strong Induction works!

4 Problem Set

1. Use the Principle of Mathematical Induction to prove that $\sum_{k=0}^n 5^k = \frac{5^{n+1}-1}{4}$ for all $n \geq 0$.
2. Redo 1. with the help of Maple.
3. Using Maple to find a formula of $\sum_{k=1}^n = k^2$. And then prove it using Principle of Mathematical Induction. Remember to use **simplify** and **factor** to get a “nicer” formula! To read document on how to use **simplify** and **factor** in Maple, use command:

```
?simplify  
?factor
```

4. Using Maple to find a formula of $\sum_{k=1}^n = k^2 + k^3$. And then prove it using Principle of Mathematical Induction with the help of Maple.