Learning from Invisible Mathematics

Jacques Carette

McMaster University

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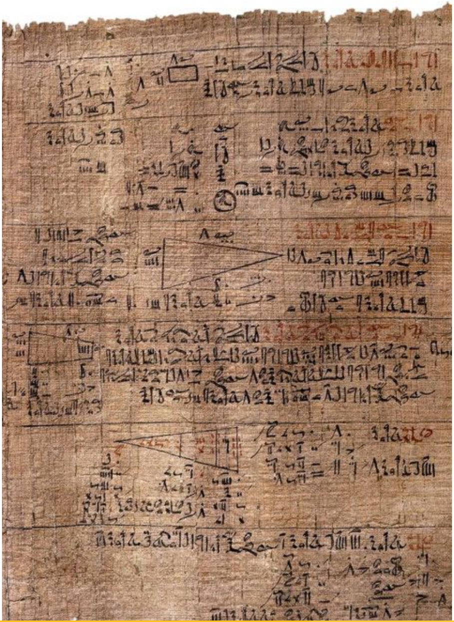
Invisible Mathematics



Definition

Invisible Mathematics is that part of paper mathematics which is usually not written down.

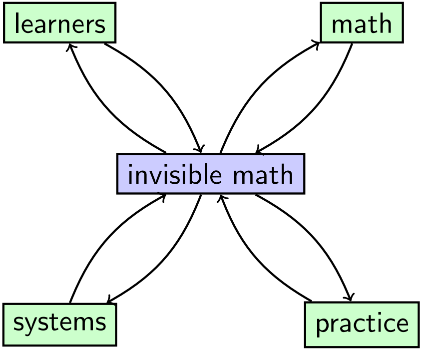
- Elided
- Unseen

Learning From

Learning From

LEAN

Maple



Learning From

Assia Mahboubi,

Claudio Sacerdotti Coen,

Freek Wiedijk,

James Davenport,

Jeremy Avigad,

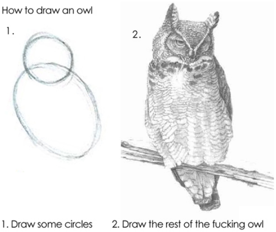
Johan Commelin,

Josef Urban,

Mario Carneiro,

Makarius Wenzel.

Example: Elaboration



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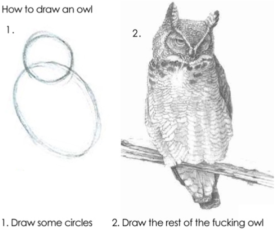
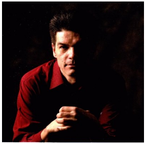
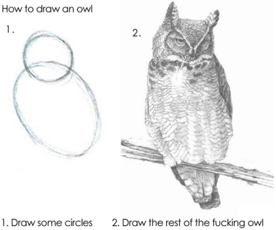
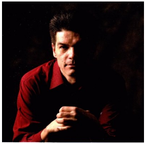




Example: Elaboration

If f is linear then $f(2 \cdot x + y) = 2 \cdot f(x) + f(y)$.





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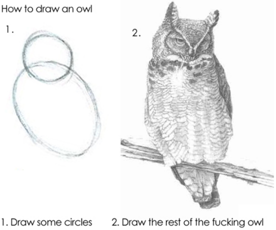

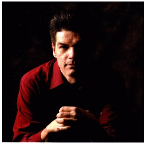

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Let U, V be vector spaces, and if $f : U \to V$ is linear then $\forall x, y : U.\ f(2 \cdot x + y) = 2 \cdot f(x) + f(y)$.

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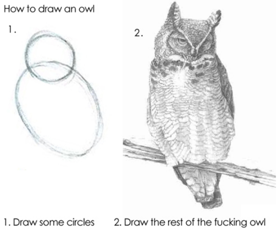

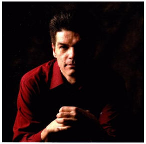

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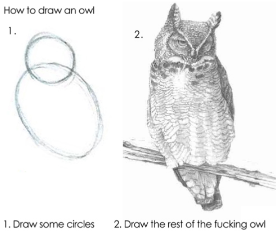

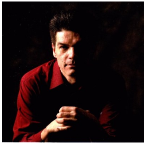

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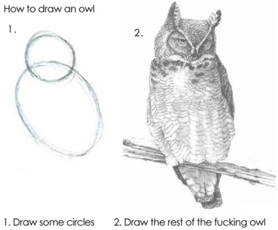

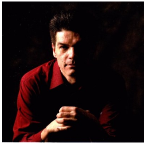

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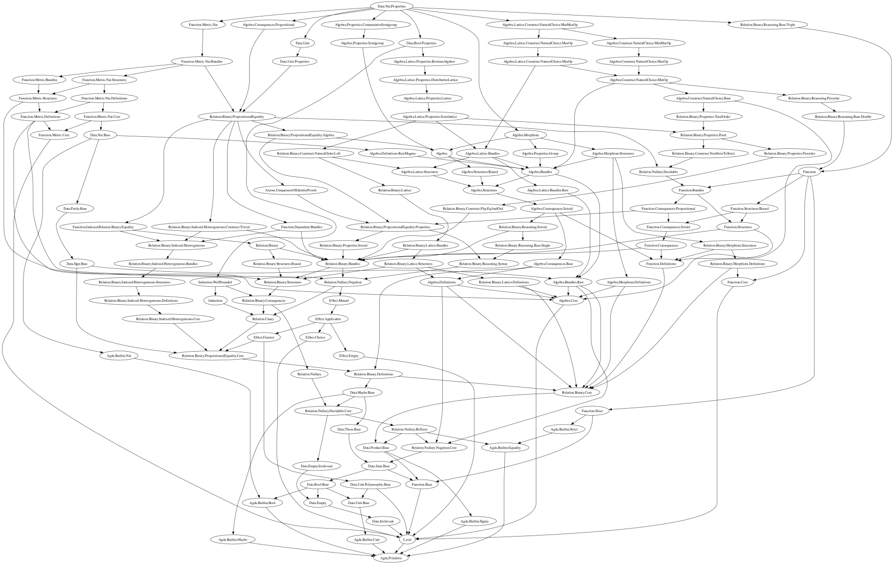
Let K be a field and U, V be vector spaces over K, and if $f : |U| \to |V|$ is linear then $\forall x, y : |U|.\ f(2 \cdot_U x +_U y) = 2 \cdot_V f(x) +_V f(y)$.

Example: Dependencies



Example: Dependencies

```
module Data.Nat.Properties where

open import Axiom.UniquenessOfIdentityProofs using (module Decidable⇒UIP)
open import Algebra.Bundles using (Magma; Semigroup; CommutativeSemigroup;
  CommutativeMonoid; Monoid; Semiring; CommutativeSemiring; CommutativeSemiringWithoutOne)
open import Algebra.Definitions.RawMagma using (_,_)
open import Algebra.Morphism
open import Algebra.Consequences.Propositional
  using (comm+cancelˡ⇒cancelʳ; comm∧distrʳ⇒distrˡ; comm∧distrˡ⇒distrʳ)
open import Algebra.Construct.NaturalChoice.Base
  using (MinOperator; MaxOperator)
import Algebra.Construct.NaturalChoice.MinMaxOp as MinMaxOp
import Algebra.Lattice.Construct.NaturalChoice.MinMaxOp as LatticeMinMaxOp
import Algebra.Properties.CommutativeSemigroup as CommSemigroupProperties
open import Data.Bool.Base using (Bool; false; true; T)
open import Data.Bool.Properties using (T?)
open import Data.Nat.Base
open import Data.Product.Base using (∃; _×_; _,_)
open import Data.Sum.Base as Sum using (inj₁; inj₂; _⊎_; [_,_]′)
open import Data.Unit.Base using (tt)
open import Function.Base using (_∘_; flip; _$_; id; _∘′_; _$′_)
open import Function.Bundles using (_↣_)
open import Function.Metric.Nat using (TriangleInequality; IsProtoMetric; IsPreMetric;
  IsQuasiSemiMetric; IsSemiMetric; IsMetric; PreMetric; QuasiSemiMetric;
  SemiMetric; Metric)
open import Level using (0ℓ)
open import Relation.Unary as U using (Pred)
open import Relation.Binary.Core
  using (_⇒_; _Preserves_⟶_; _Preserves₂_⟶_⟶_)
open import Relation.Binary
open import Relation.Binary.Consequences using (flip-Connex)
open import Relation.Binary.PropositionalEquality
open import Relation.Nullary hiding (Irrelevant)
open import Relation.Nullary.Decidable using (True; via-injection; map′; recompute)
open import Relation.Nullary.Negation.Core using (¬_; contradiction)
open import Relation.Nullary.Reflects using (fromEquivalence)

open import Algebra.Definitions {A = ℕ} _≡_
  hiding (LeftCancellative; RightCancellative; Cancellative)
open import Algebra.Definitions
  using (LeftCancellative; RightCancellative; Cancellative)
open import Algebra.Structures {A = ℕ} _≡_
```

Example: Names in Scope

Abbreviation	*Definition*
BP	band pass
BSA	bovine serum albumin
chl *a*	chlorophyll *a*
CHO	carbohydrate
DMEM	Dulbecco's Modified Eagle's Medium
FCS	fetal calf serum
FITC	fluorecein isothiocyanate (green fluorochrome)
FLS	forward light scatter
HAB(s)	harmful algal bloom(s)
GAM	goat anti-mouse
IgG_1	Immunoglobulin protein, subclass G_1
IMBS	Immuno-magnetic bead separation
LP	long pass
MAb	monoclonal antibody or antiserum
MMP	mouse myloma protein
MPC	magnetic particle concentrator
NGS	normal goat serum
NSS	normal sheep serum
PBS	phosphate buffered saline
PE	phycoerythrin (orange fluorochrome)
RFU	relative fluorescence unit
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let R be a Ring.

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open import Algebra
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let R be a Ring.

Agda:

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(Paper) math:
let R be a Symmetric Rig Category.

Agda:

```
open import Categories.Category.RigCat
open RigCategory R
```

Example: Context and Semantics

What does $\frac{x}{x}$ mean?

Algebra: $\frac{x}{x} =_{\mathbb{Q}(x)} 1$

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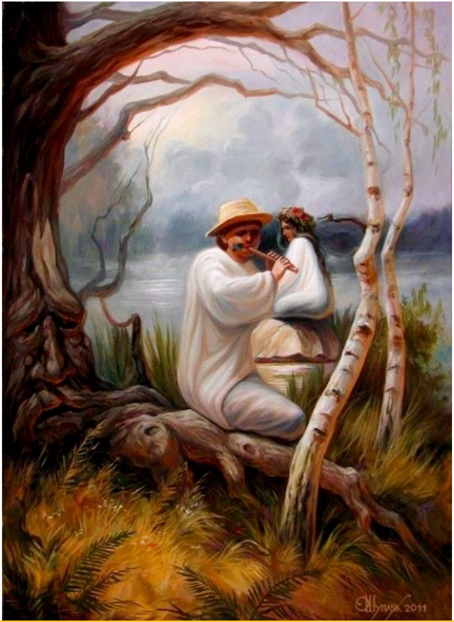
What does $\frac{x}{x}$ mean?

Algebra: $\frac{x}{x} =_{\mathbb{Q}(x)} 1$

Analysis: $\frac{x}{x}$ *denotes*

$$\lambda x. \begin{cases} 1 & x \neq 0 \\ \bot & x = 0 \end{cases}$$

Example: Ambiguous Questions



solve(y^2-x^2,y) ?

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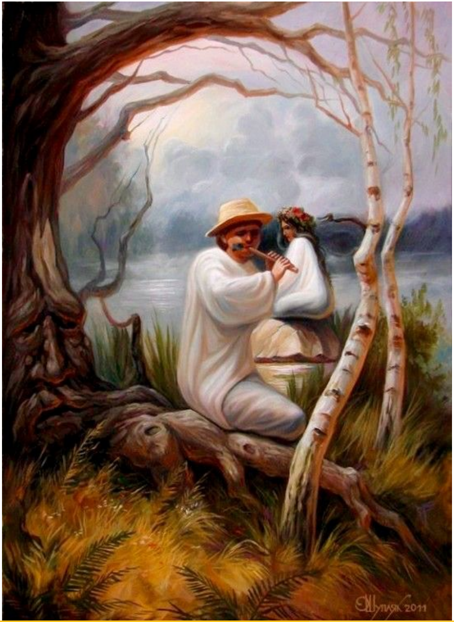

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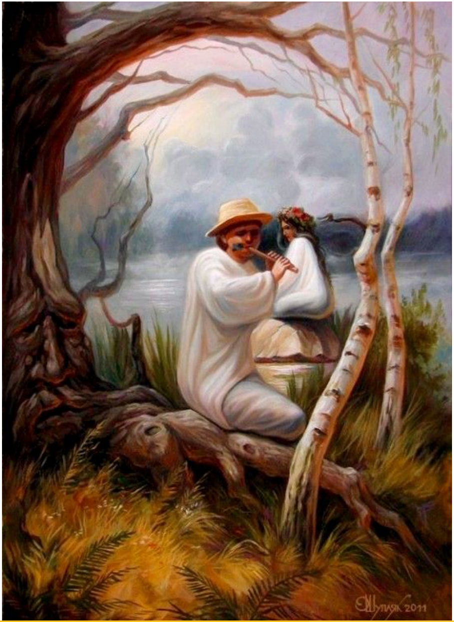

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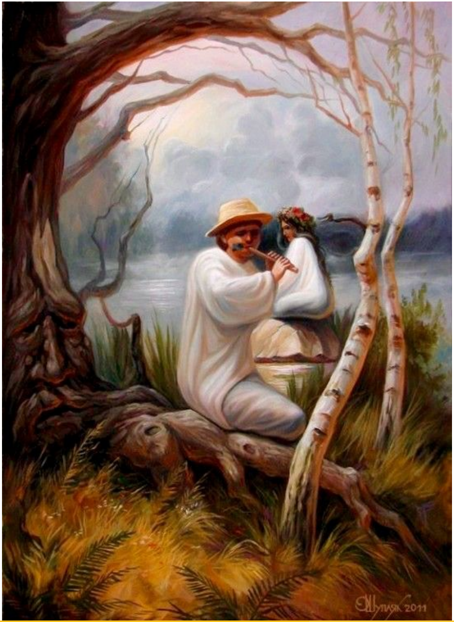

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- y function of x $\Rightarrow$
  Let $\xi : \mathbb{R} \to \{-1, 1\}$ be an arbitrary function;
  $y(x) = \xi(x)\, x$ is a solution.
- y continuous function of x $\Rightarrow$
  $y(x) = \pm x$ and $\pm|x|$ are solutions.

Example: Extension and Intension

II.—ON DENOTING.

By Bertrand Russell.

By a "denoting phrase" I mean a phrase such as any one of the following: a man, some man, any man, every man, all men, the present King of England, the present King of France, the centre of mass of the Solar System at the first instant of the twentieth century, the revolution of the earth round the sun, the revolution of the sun round the earth. Thus a phrase is denoting solely in virtue of its *form*. We may distinguish three cases: (1) A phrase may be denoting, and yet not denote anything; *e.g.*, "the present King of France". (2) A phrase may denote one definite object; *e.g.*, "the present King of England" denotes a certain man. (3) A phrase may denote ambiguously; *e.g.*, "a man" denotes not many men, but an ambiguous man. The interpretation of such phrases is a matter of considerable difficulty; indeed, it is very hard to frame any theory not susceptible of formal refutation. All the difficulties with which I am acquainted are met, so far as I can discover, by the theory which I am about to explain.

The subject of denoting is of very great importance, not only in logic and mathematics, but also in theory of knowledge. For example, we know that the centre of mass of the Solar System at a definite instant is some definite point, and we can affirm a number of propositions about it; but we have no immediate *acquaintance* with this point, which is only known to us by description. The distinction between *acquaintance* and *knowledge about* is the distinction between the things we have presentations of, and the things we only reach by means of denoting phrases. It often happens that we know that a certain phrase denotes unambiguously, although we have no acquaintance with what it denotes; this occurs in the above case of the centre of mass. In perception we have acquaintance with the objects of perception, and in thought we have acquaintance with objects of a more abstract logical character; but we do not necessarily have acquaintance with the objects denoted by phrases composed

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Theorem 2.31. Integration by Parts. *Let u and v be differentiable functions, then*

$$\int u\,dv = uv - \int v\,du,$$

where

$u = f(x)$ and $v = g(x)$ so that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

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Example2.32. Product of a Linear Function and Logarithm.

Evaluate $\int x \ln x\,dx$.

▾ Solution

Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\begin{aligned}\int x \ln x\,dx &= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx \\ &= \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.\end{aligned}$$

Example: Notations for Humans

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$$ax^2 + bx + c$$

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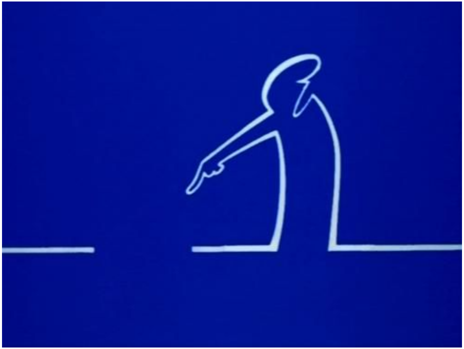
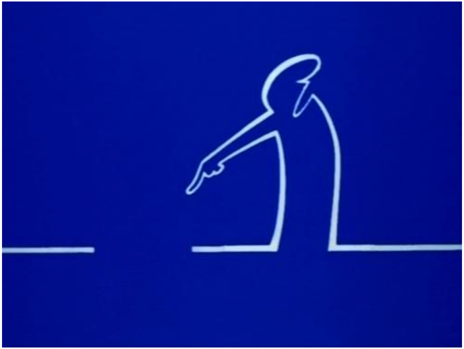
Example: Ad Hoc to Methodological



Limits

Textbooks:
Dozens of *tricks* for dealing.

Example: Ad Hoc to Methodological



Limits

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Dozens of *tricks* for dealing.

Computation: Multi-series

Proof: filters

Continued Zoo

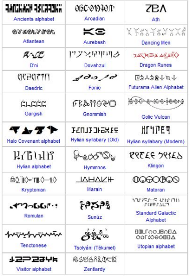
- bundling / staging
- coercions
- overloading
- standing assumptions
- equality vs equivalence
- canonical morphism
- proofs vs vague hints
- definedness conditions
- name binding
- deep dependence between def'ns
- implicit β
- interop of equiv. signatures
- underlying use of universal algebra

Lessons

Cognitive Load

Lessons

Cognitive Load



Linguistics

Lessons

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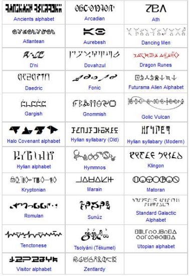

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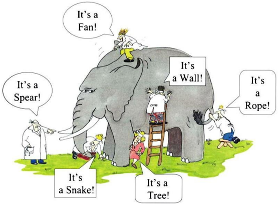

Context

Lessons

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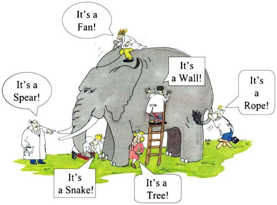

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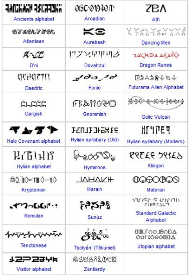


Linguistics

New meta-mathematics

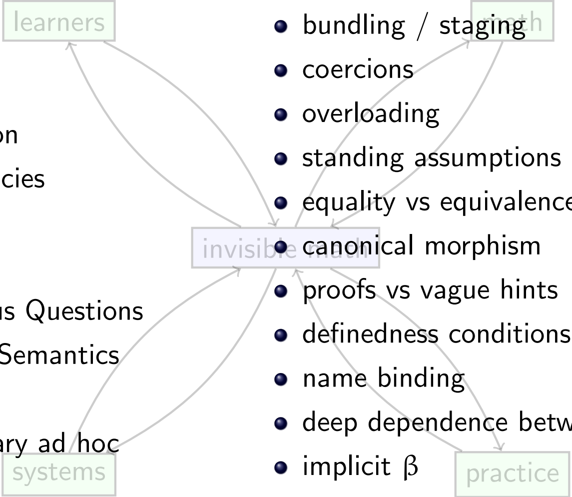
Correct

is

Boring

Enriching

Shades of Invisible Math



- Elaboration
- Dependencies
- Scope
- Context
- Ambiguous Questions
- Syntax / Semantics
- Notation
- Unnecessary ad hoc

- bundling / staging
- coercions
- overloading
- standing assumptions
- equality vs equivalence
- canonical morphism
- proofs vs vague hints
- definedness conditions
- name binding
- deep dependence between def'ns
- implicit β
- interop of equiv. signatures
- underlying use of universal algebra