CAS 750 & CES 750 Model-Based Image Reconstruction - Winter 2009

News

First class Tues, Jan 6, 10:30, ITB 225. Let me know if you can't make it. The choice of material for both lectures and student presentations will depend on the student project areas. Students without other interests will be assigned a problem area from Magnetic Resonance Image Reconstruction.

Outline

Calendar Description

An overview of three themes in advanced image processing: functional analysis (e.g., Fourier, Wavelet and SVD methods), PDEs (e.g., anisotropic diffusion), optimization of statistical models (e.g. Total Variation regularization). And, a detailed look at specific methods and techniques for applying these methods in new areas: medical imaging, visual process control. Including all phases of application development from mathematical modelling, through complexity analysis.

Course Objective

In this course you will learn (1) what an inverse problem is (2) what the instructor classifies as an inverse imagining problem (3) methods of mathematical modelling which lead to inverse problems, using examples from Magnetic Resonance Imaging (4) methods of solving such inverse probems (5) factors which affect the performance of the solver and (6) factors which affect the stability of the solver.

Recommended Texts

- Deblurring Images: Matrices, Spectra, and Filtering; Per Christian Hansen, James G. Nagy, and Dianne P. OLeary, 2006 / xiv+130 pages / Softcover, ISBN-13: 978-0-898716-18-4 / ISBN-10: 0-89871-618-7 (cheaper if ordered from SIAM using free student membership)
- Computational Methods for Inverse Problems; Curtis R. Vogel, ISBN-13: 978-0-898715-50-7. (cheaper if ordered from SIAM using free student membership)

- Introduction to the Mathematics of Medical Imaging, Charles L. Epstein, 768 pages, 2003, ISBN-13: ISBN 978-0130675484
- Magnetic Resonance Imaging Physical Principles and Sequence Design; Haacke, E. Mark / Brown, Robert W. / Thompson, Michael R. / Venkatesan, Ramesh; 1999, Wiley, 914 Pages, Hardcover ISBN-10: 0-471-35128-8 ISBN-13: 978-0-471-35128-3
- Geometric Level Set Methods In Imaging, Vision, And Graphics, Osher, Stanley; Paragios, Nikos

Instructor

Christopher Anand, ITB-213, x24895. anandc (circled a) (name of university) (country -- no! not California). Make an appointment by email.

Schedule

T 10:30-12:30 ITB 225 (except Jan 27, ITB 101a)
F 11:30-12:30 ITB 222
We may not meet every week. Class time will include both lectures on course topics, and presentations by students. Rough order: (1) Overview of objectives, what is expected in a project, methodolgy, and discussion of possible areas for student projects. (2-3) Introduction to MRI. (4) Introduction to Model Based Image Reconstruction. (5) Example of a previous student project. (6) Student presentations of assigned problem areas. (Longer discussion of application areas other than MRI as required.) (7-8) Advanced topics in modeling MRI. (9) Student presentations of chosen problem. (10-11) Advanced topics in inverse problems. (12-13) Student presentations of final results.

Evaluation

10 percent assignments. 10 percent for class participation. 20 percent for presentation of assigned problem; 20 percent for presentation of selected problem; 20 percent for presentation of results in class; 20 percent for written documentation, including code, and demonstrations.

The number and weighting of presentations may change depending on the number of students registered.

Each student will choose a problem, and one or method of solution, as approved by the instructor, and carry out all steps in the above procedure. Each student will be evaluated primarily based on a final written report including log (see below) (9/12) and a presentation of the results (3/12). Final reports must be submitted by April 15th, via email. Collaboration on implementation of the solver is encouraged, but all collaboration must be documented in a log in a way which makes the nature of the collaboration clear. For this purpose, it is recommended that all students use a version-control system such as subversion, and use it to record all of their work on source code, documentation and their report.

ACADEMIC INTEGRITY

You are expected to exhibit honesty and use ethical behaviour in all aspects of the learning process. Academic credentials you earn are rooted in principles of honesty and academic integrity. Academic dishonesty is to knowingly act or fail to act in a way that results or could result in unearned academic credit or advantage. This behaviour can result in serious consequences, e.g. the grade of zero on an assignment, loss of credit with a notation on the transcript (notation reads: "Grade of F assigned for academic dishonesty"), and/or suspension or expulsion from the university.
It is your responsibility to understand what constitutes academic dishonesty. For information on the various types of academic dishonesty please refer to the Academic Integrity Policy, located at http://www.mcmaster.ca/academicintegrity
The following illustrates only three forms of academic dishonesty: 1. Plagiarism, e.g. the submission of work that is not one's own or for which other credit has been obtained. 2. Improper collaboration in group work. 3. Copying or using unauthorized aids in tests and examinations.

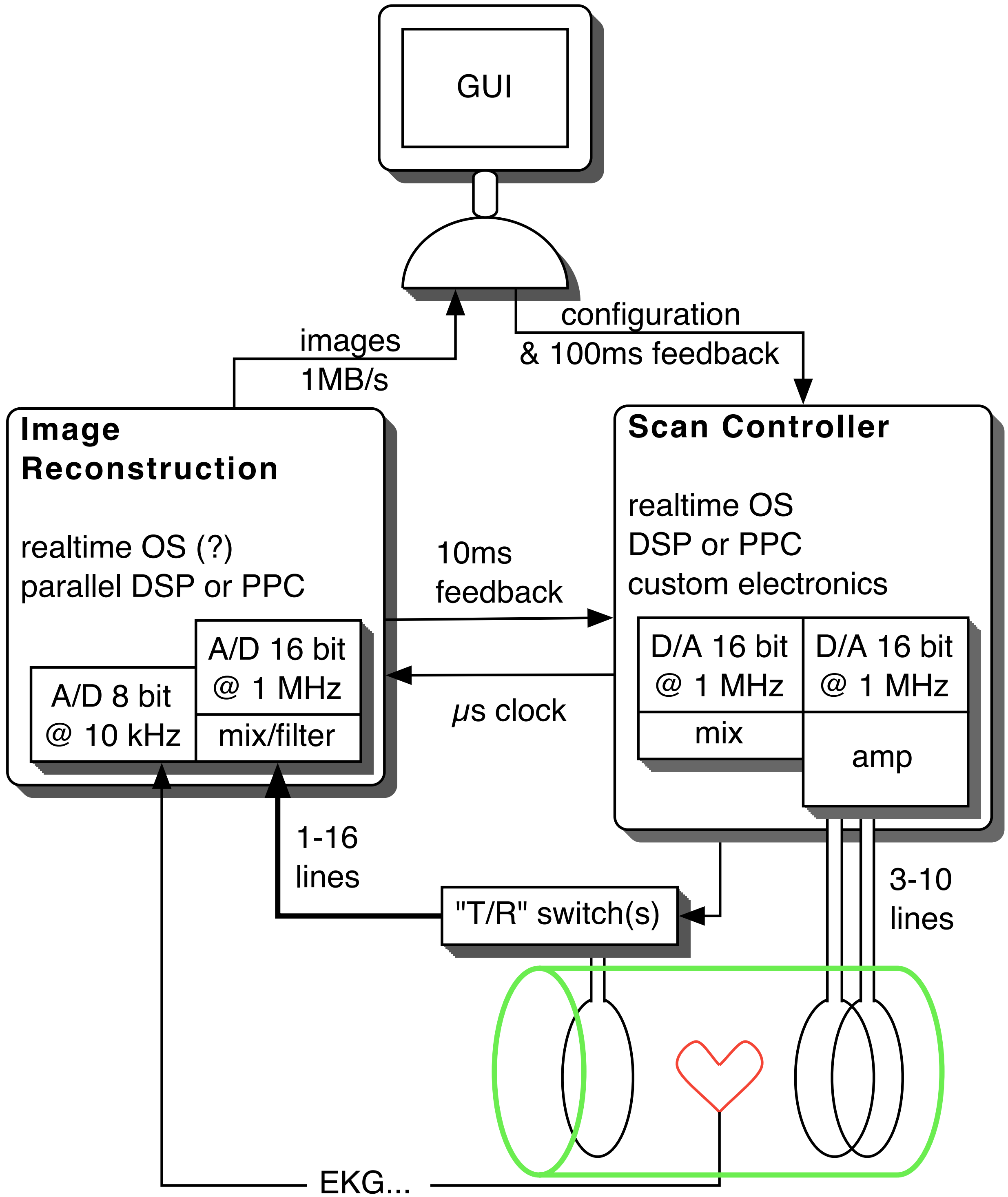
If in doubt, ask the instructor how this applies to your work.

TURNITIN.COM

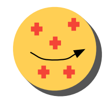
In this course we reserve the right to use a web-based service (Turnitin.com) to reveal plagiarism. Students will be expected to submit their work electronically to Turnitin.com and in hard copy so that it can be checked for academic dishonesty. Students who do not wish to submit their work to Turnitin.com must still submit a copy to the instructor. No penalty will be assigned to a student who does not submit work to Turnitin.com. All submitted work is subject to normal verification that standards of academic integrity have been upheld (e.g., on-line search, etc.). To see the Turnitin.com Policy, please go to www.mcmaster.ca/academicintegrity

Personal Information

In this course we will be using subversion, email and other on-line discussion fora. Students should be aware that, when they access the electronic components of this course, private information such as first and last names, user names for the McMaster e-mail accounts, and program affiliation may become apparent to all other students in the same course. The available information is dependent on the technology used. Continuation in this course will be deemed consent to this disclosure. If you have any questions or concerns about such disclosure please discuss this with the course instructor.


GUI
images
1MB/s
configuration
& 100ms feedback
Image Reconstruction
realtime OS (?)
parallel DSP or PPC
A/D 8 bit
@ 10 kHz
A/D 16 bit
@ 1 MHz
mix/filter
10ms
feedback
μs clock
Scan Controller
realtime OS
DSP or PPC
custom electronics
D/A 16 bit
@ 1 MHz
mix
D/A 16 bit
@ 1 MHz
amp
1-16
lines
"T/R" switch(s)
3-10
lines
EKG...

Our Choice:
Classical Model

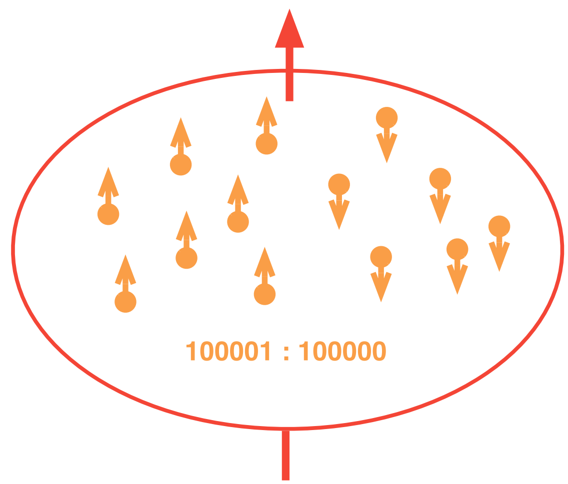
protons • spinning charge  has magnetic moment 

external field • leans on local field

• parallel is lower energy

thermal motion • jostles protons

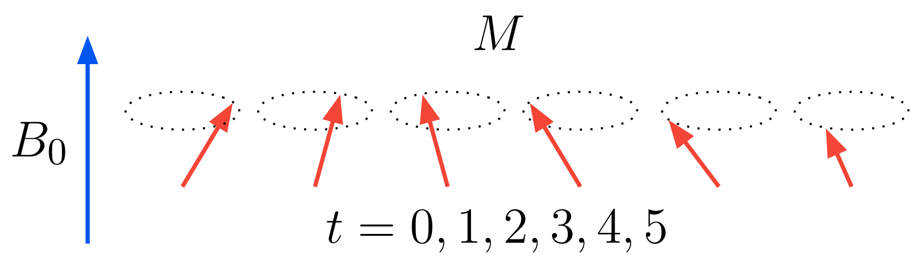
• thermal equilibrium:



net moment () • modelled as continuous function of position

Precession

- main field B_0 acts on moment M
- same as gravity acting on spinning top



$$\frac{dM}{dt} = M \times B_0$$

Exercise 1. *Write equations for perpendicular components (M_x and M_y). Solve. What constant determines the rate of rotation?*

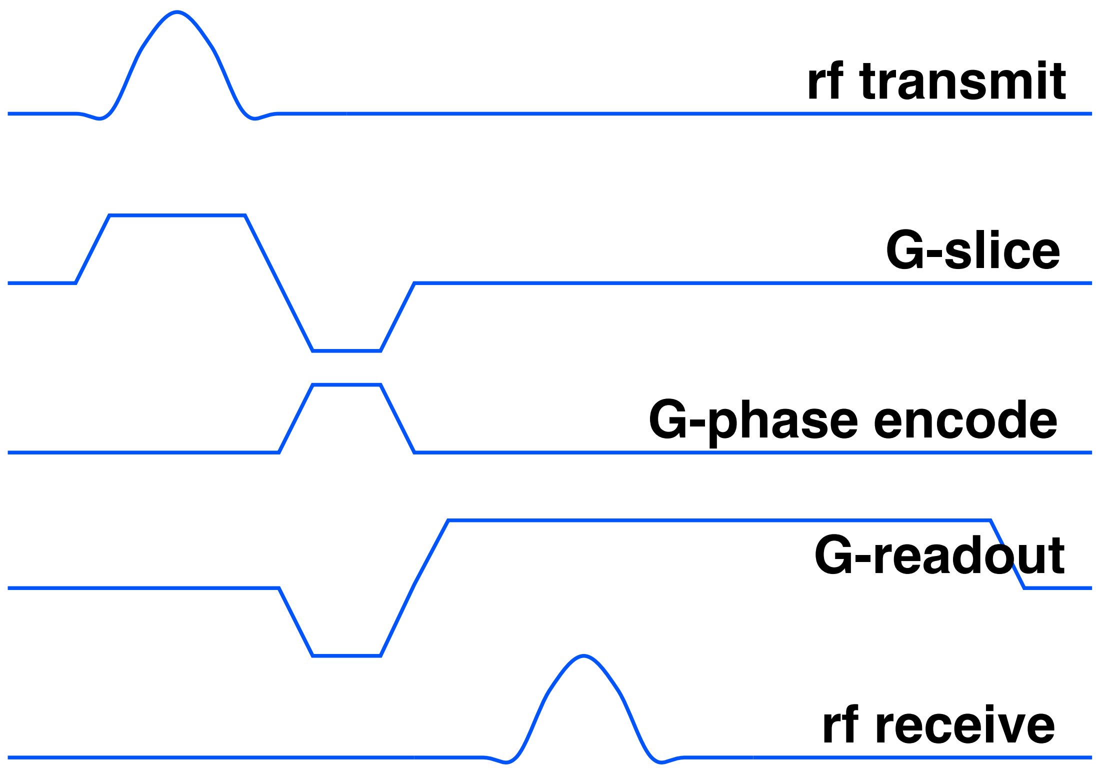
Relaxation: All good things fade away.

• equilibrium is $M \| B_0$ ∘ other positions unstable ∘ thermal collisions bring it back	$\mapsto$ spin-lattice regrowth
• external field B is not uniform ∘ imperfections in magnet ∘ permeability of tissues changes field ∘ protons interact with other particles	$\mapsto$ spin-spin dephasing

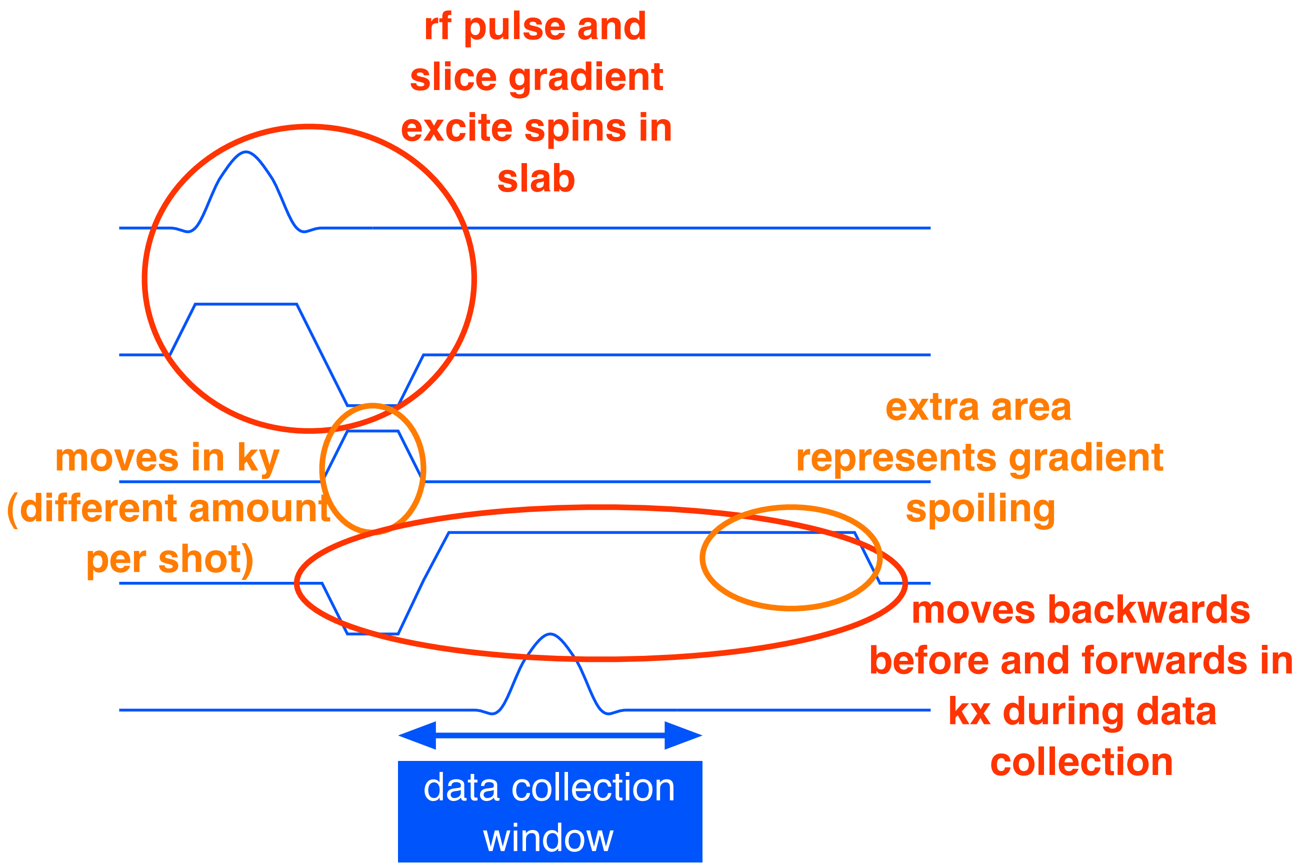
Bloch Equations:

$$\frac{dM}{dt} = M \times B_{\text{external}} + \frac{1}{T_1}(M_0 - M_z) - \frac{1}{T_2}(M - M_z)$$

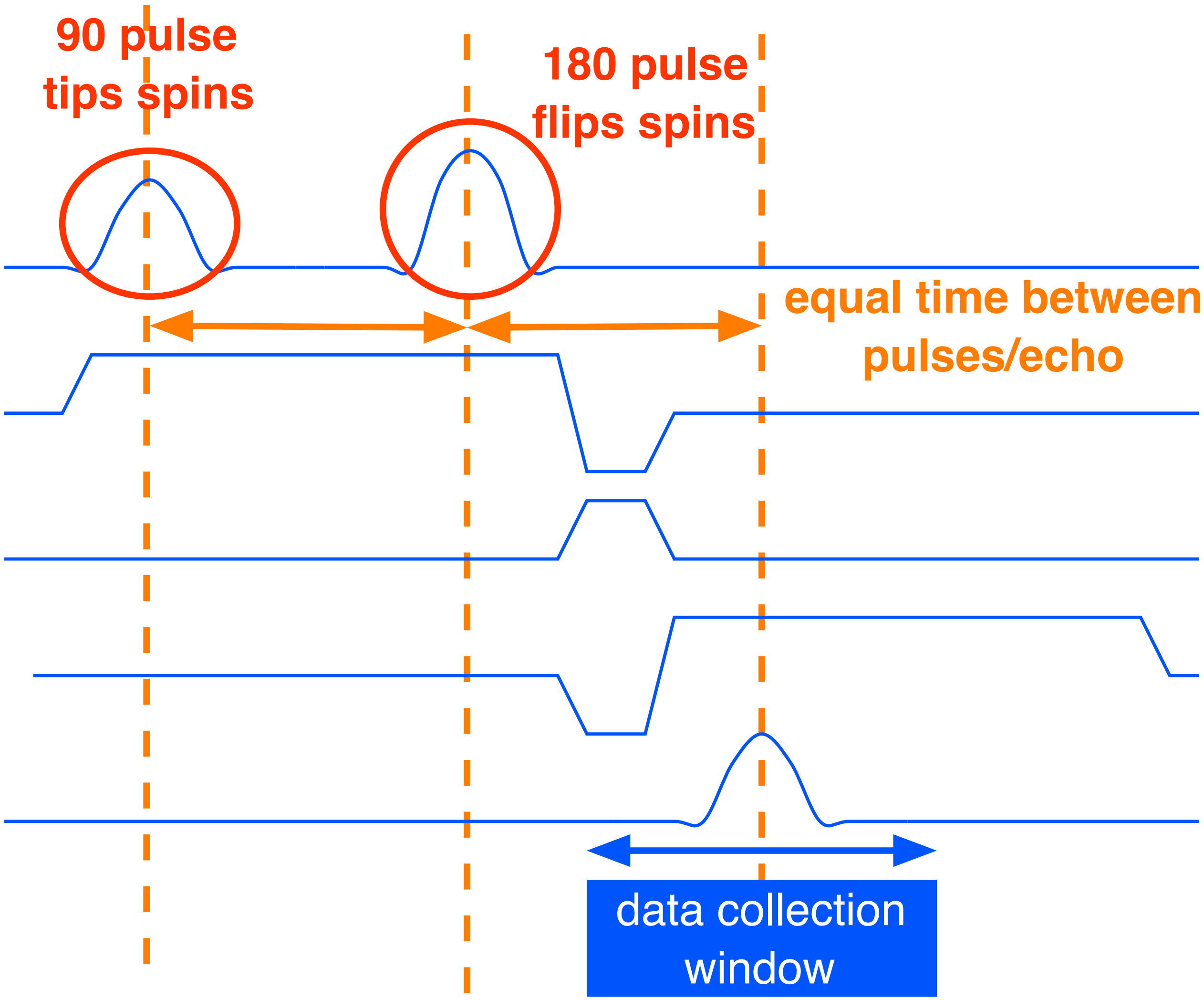
Gradient Echo



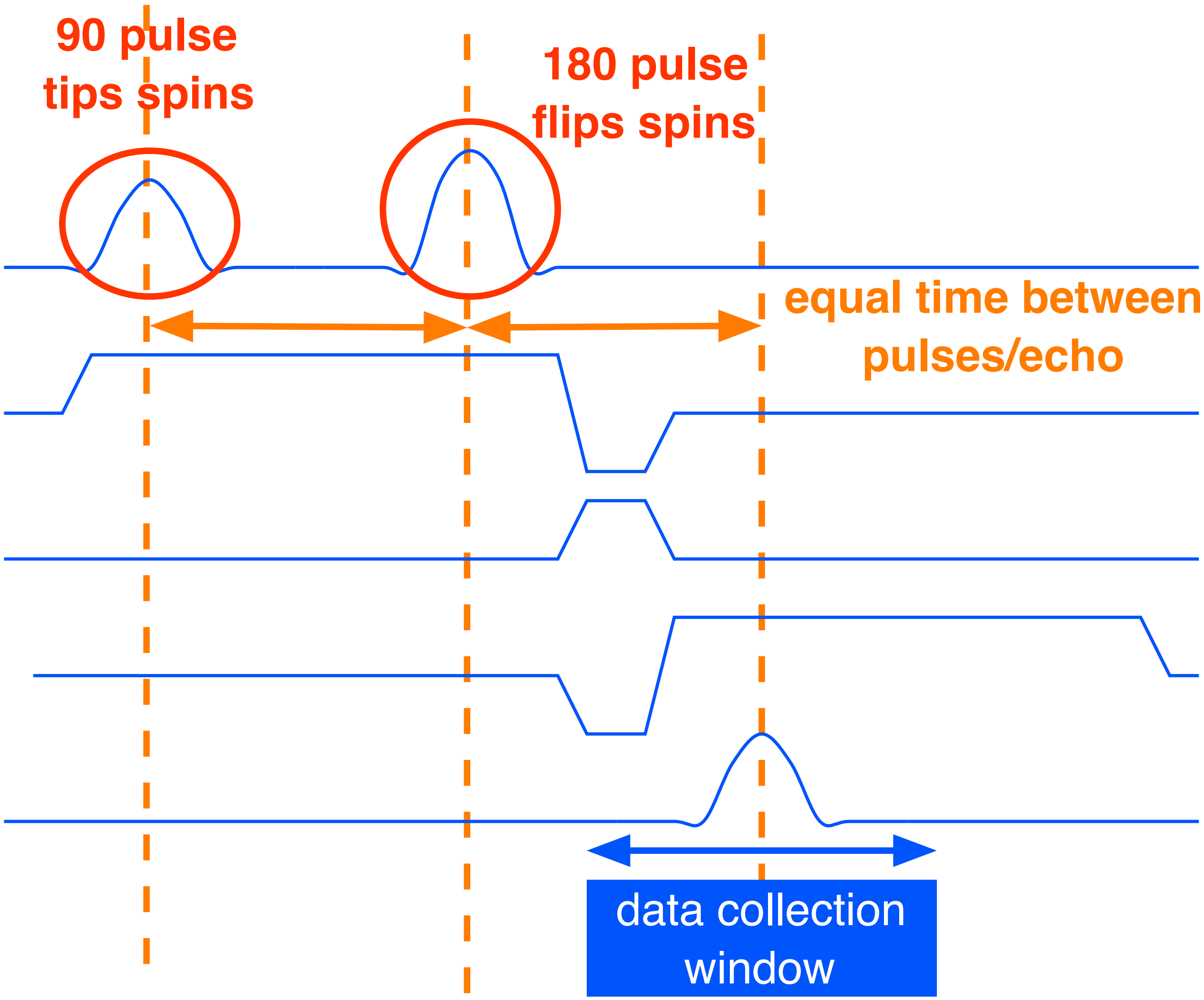
Gradient Echo Explained



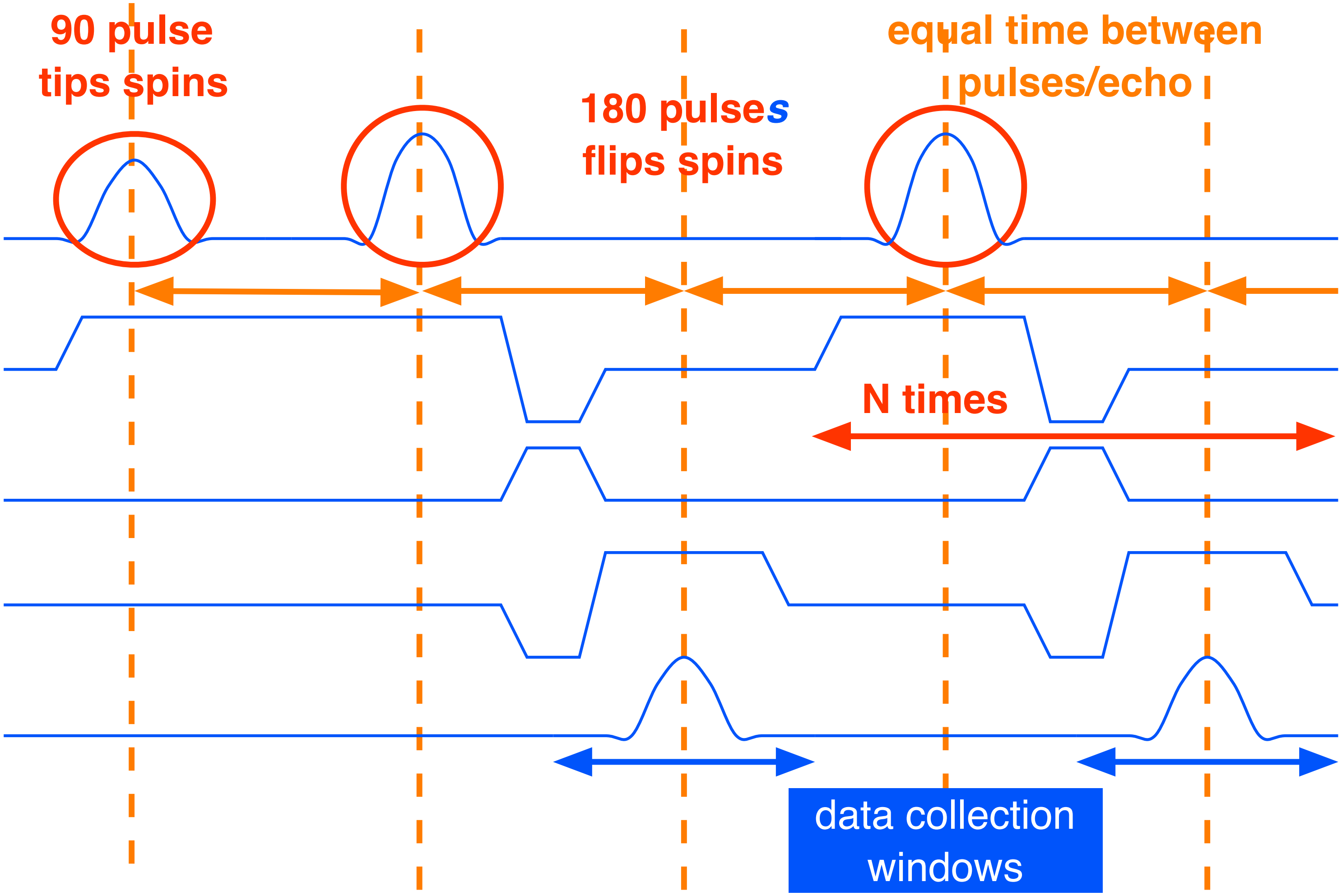
Spin Echo Explained



Spin Echo Explained



Fast Spin Echo Explained



So far...

- Bloch equation terms

$M \times B_{\text{external}}$	M rotates around external field
$-\frac{1}{T_2}(M - M_z)$	M disappears (dephases) (slow)
$+\frac{1}{T_1}(M_0 - M_z)$	M regrows (slower)

- two-part experiment
  - excitation
    - rf-field at resonance
    - M tips over
  - readout
    - measure perpendicular components
    - gradients $\rightarrow$ linear variation in $|B|$
    - $|B| \rightarrow$ phase evolution

k-Space

- basic solution $e^{i|B_0|t}$ in presence of B_0
- basic solution 1 in rotating frame
- $e^{i(xG_x+yG_y+zG_z)t}$ in precense of constant gradient
- $e^{i(xk_x+yk_y+zk_z)t}$; $k = \int G(t)dt$ otherwise
- k is the position in k-space

$$\begin{aligned} s(t) &= \int \rho(x,y,z)e^{i(B_0+xk_x+yk_y+zk_z)t}dxdydz \\ &= e^{iB_0t}\int \rho(x,y,z)e^{i(xk_x+yk_y+zk_z)t}dxdydz \\ &= e^{iB_0t}FT(\rho) \text{ (for the right choice of } G(t)) \end{aligned}$$

Example 1. *For the gradient encoding example, we used*

$$G_x(t) = \begin{cases} G_0 & 0 < t < 1 \\ -G_0 & 1 < t < 3 \end{cases}$$

Readouts

- spin-warp = spin-echo
  - 1-d encoding example with initial G_y
- fast spin echo
  - multiple spin-echoes in sandwich with 180 pulses
- fast gradient echo
  - sandwich of (reversed) spin-echoes with G_y blips
- radial / backprojection
  - read a rotated line through the origin
- echo planar imaging
  - $G_x = \sin(t)$; G_y blips
- spiral imaging
  - k follows spiral
  - most efficient readout

What if...

Exercise 1. *What effect would a delay between the beginning of data-readout and G waveforms have on spin-echo, fast-spin-echo, fast gradient echo, epi, spiral imaging?*

Exercise 2. *What effect would an overall change of phase ($e^{i\phi}$) between odd and even echoes of the fast gradient echo have on the image?*

Exercise 3. *What effect would a miscalibration of G_x have on spin echo, fast gradient echo, and epi, ie if $G_{x,\mathit{actual}}(t) = \epsilon + G_{x,\mathit{desired}}(t)$?*

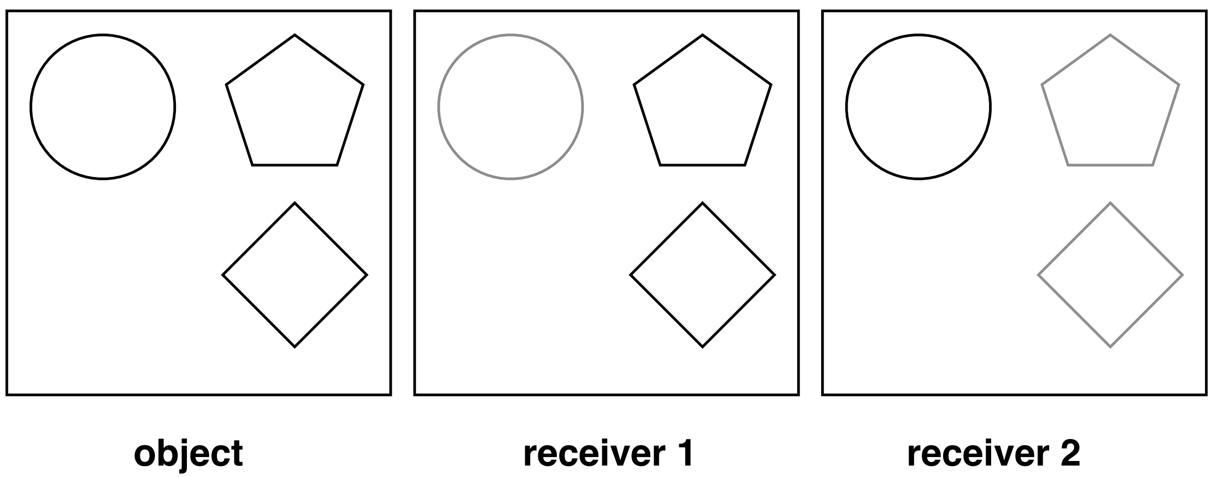
Exercise 4. *What effect would a miscalibration of G_y have on spin echo, fast gradient echo, and epi, ie if $G_{y,\mathit{actual}}(t) = \epsilon + G_{y,\mathit{desired}}(t)$?*

Exercise 5. *Why is there a phase variation in the spiral image? (See code on web site.)*

Multiple receivers

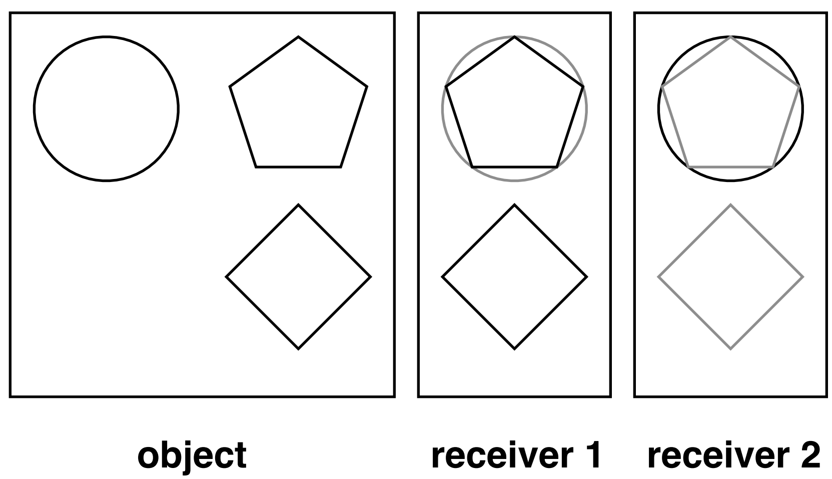
- multiple receivers
  - ◦ improved signal/noise
    - ⋄ averaging noise
    - ⋄ smaller coverage/coil means less noise/coil
  - ◦ conventional reconstruction:

$$I(x,y) = (\sum ||I_{\text{receiver}}(x,y)||)^{1/2}$$

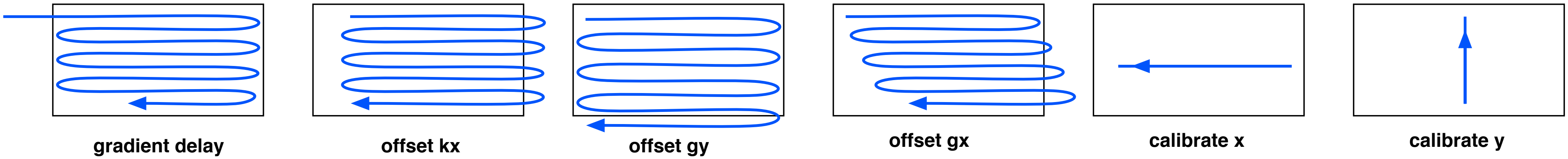


SENSE

- SENSE
  - sensitivity map, $S_{\text{pixel,receiver}}$
  - reduced FOV images, I_{receiver}
  - object, O_{pixel}
  - equation, $I_{\text{receiver}} = \sum S_{\text{pixel,receiver}} O_{\text{pixel}}$
  - SENSE reconstruction:
    - find S, I
    - $O = S^{-1} I$

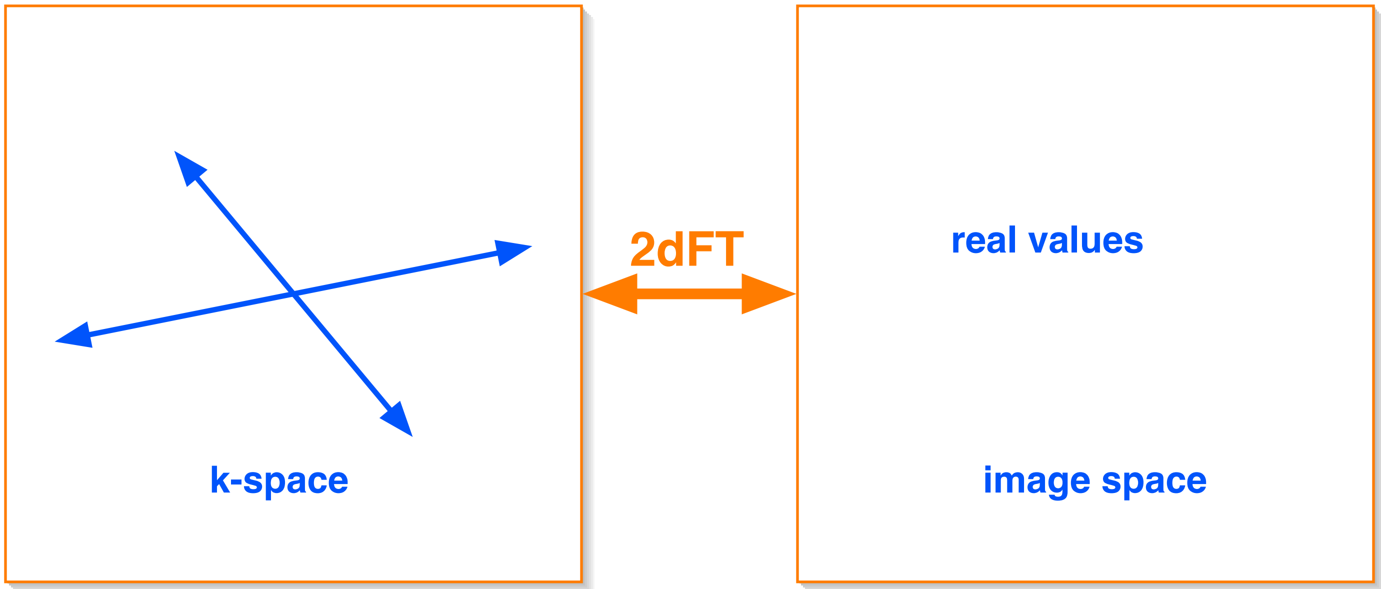


Corrections



- different sources
- one landmark: k_0
- one solution: revisit k_0
  - chose line through k_0
  - pass through multiple times
  - FT of one line = projection onto line
  - relative phase wraps between lines measures shift of k_0
    - ◇ constant odd/even shift = delay
    - ◇ progressive shift = offset (in readout direction)
    - ◇ signal loss = offset in perpendicular (calibrate this first, or bootstrap)
    - ◇ phase difference = phase difference
  - correct in waveforms or in reconstruction

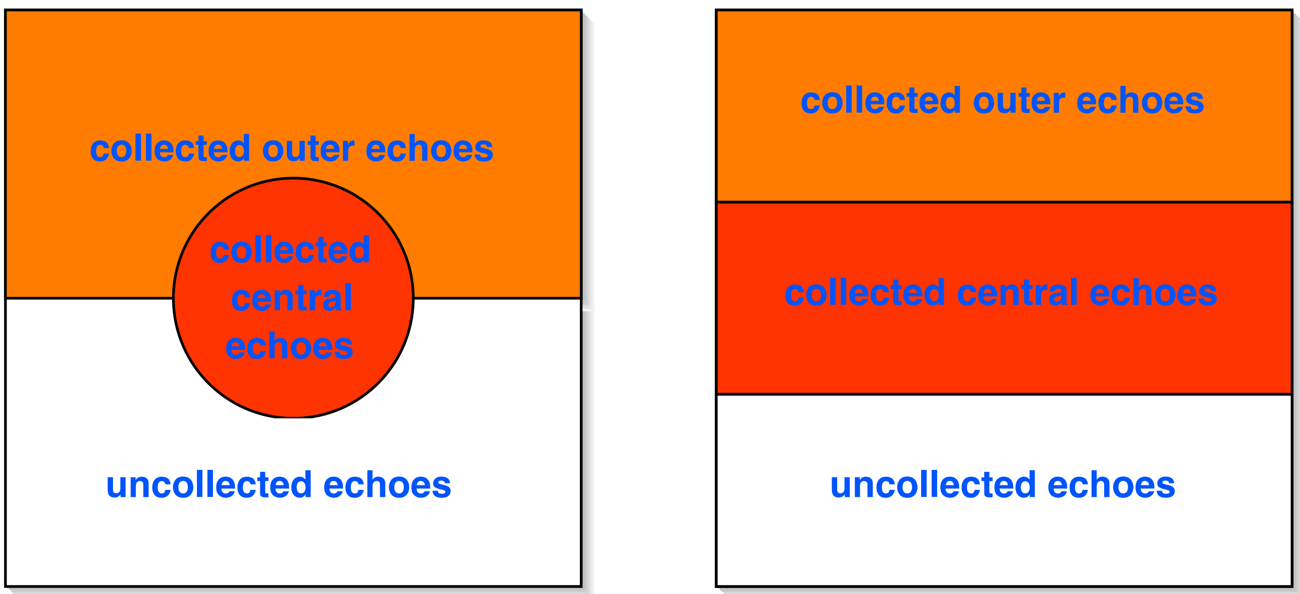
Conjugate Symmetry



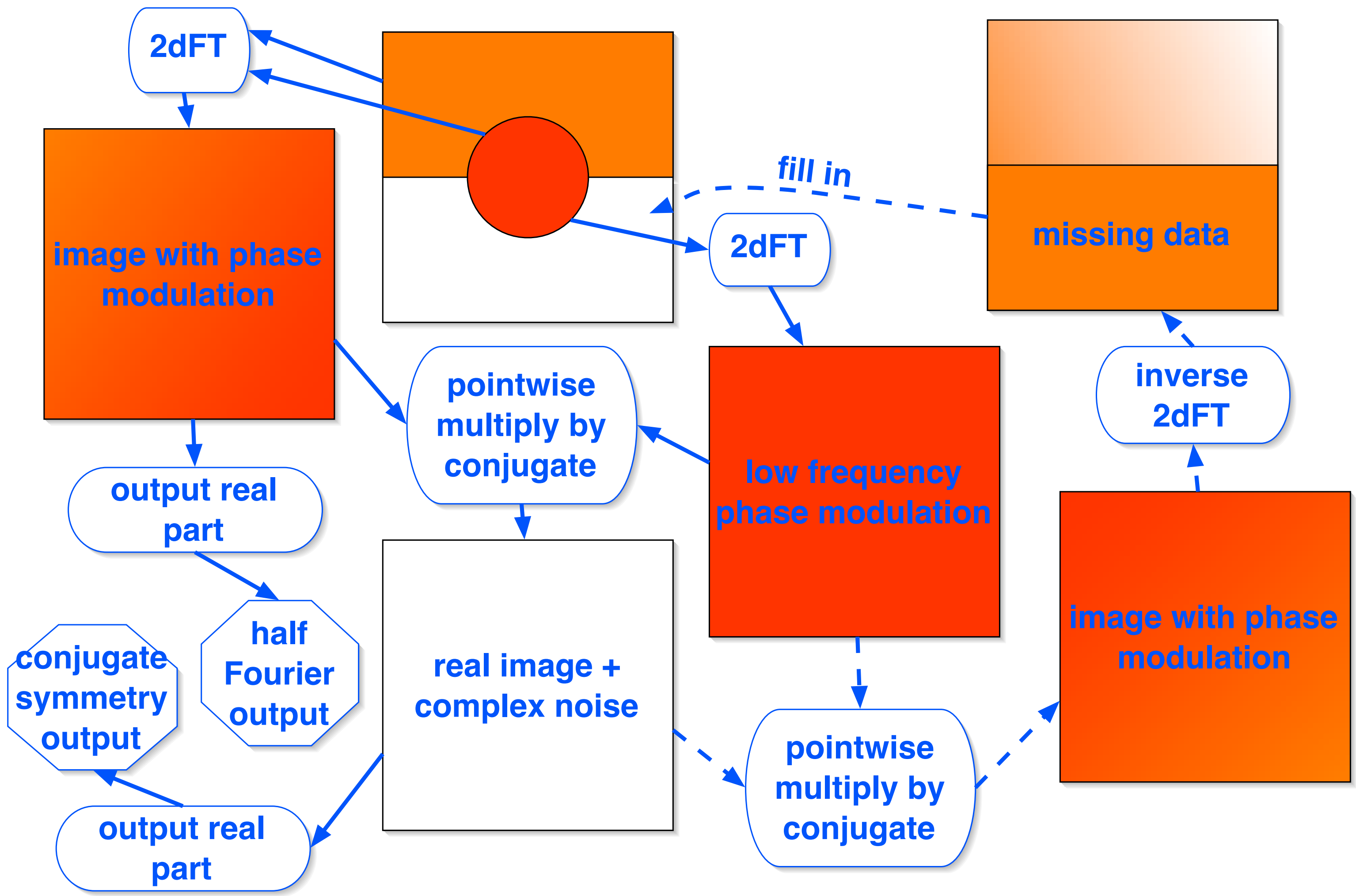
- if we expect a real image
- we should expect a symmetric k-space
- true in any dimension

Skip the review

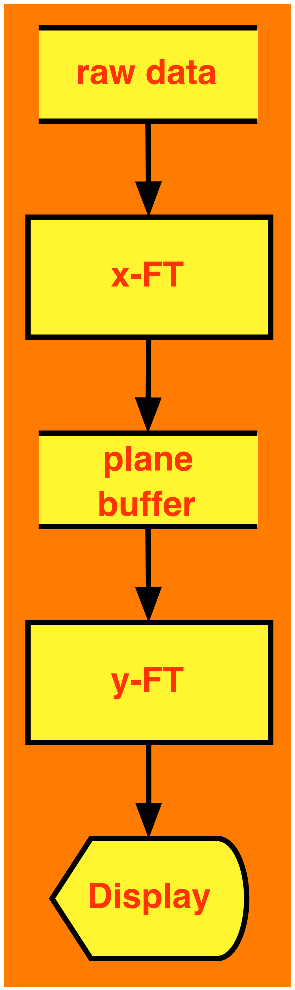
- so collect only half the data...
- or a little bit more to correct for phase modulation



Conjugate Symmetry Flow

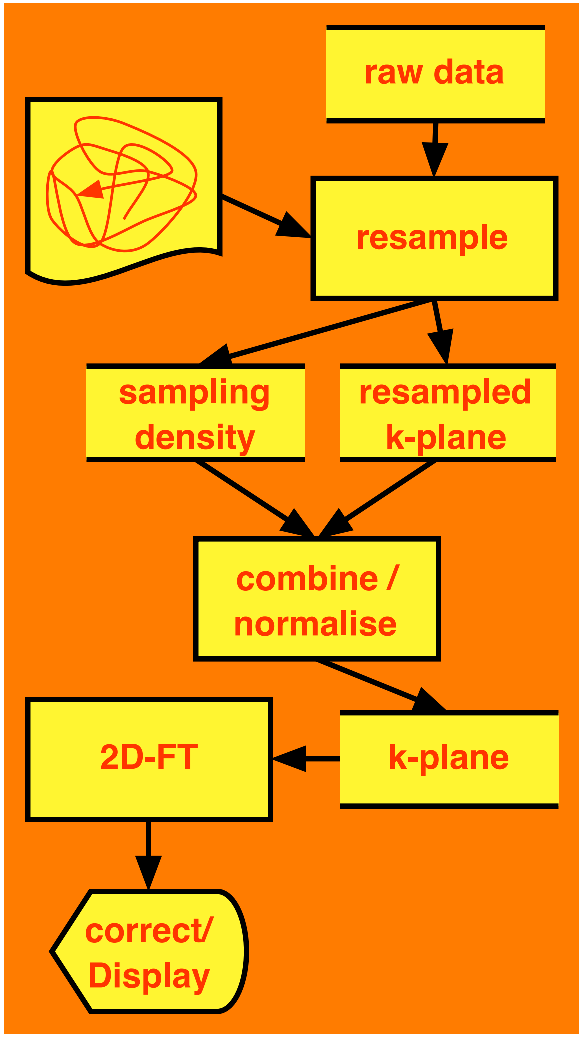


2D Spin-Echo Data Flow



- k_x lines are acquired together
- FT complete lines for low latency
- FT columns on image completion
- convert to magnitude & display

Nonraster 2D Data Flow

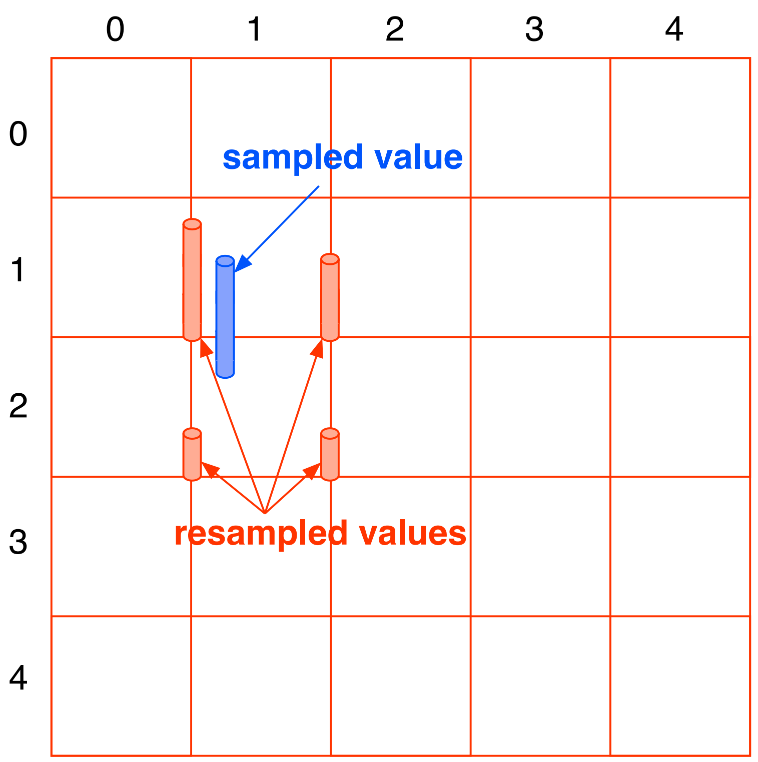


- samples are not at integral points
  - ◦ resample before FT
- sampling density is not uniform
  - ◦ density would filter image
  - ◦ calculate density and normalize

◊ more computations

◊ higher latency

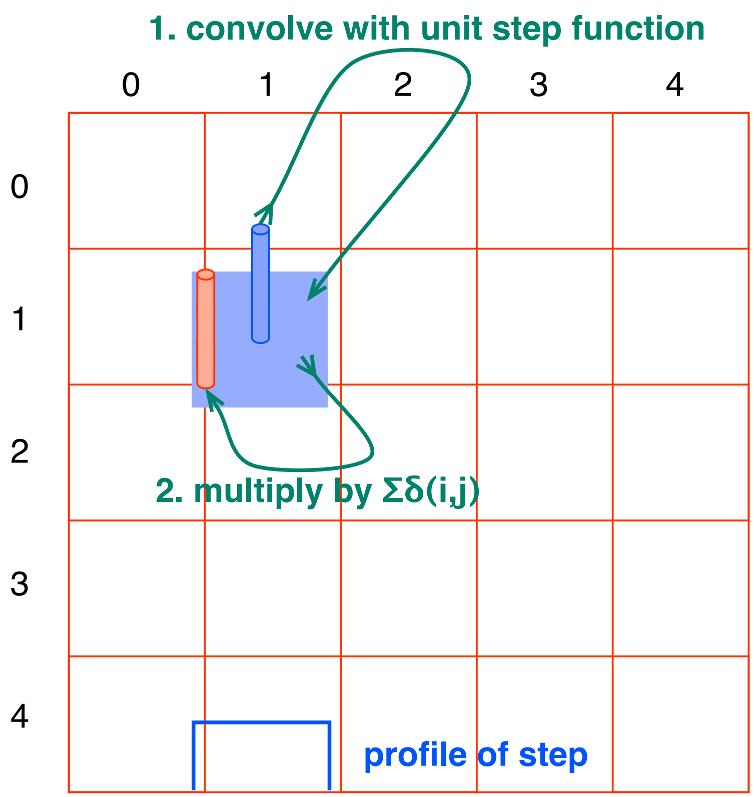
Resampling



- input: (k_x, k_y, s) sample
  - ∘ delta function $\{(k_x, k_y)\} \longrightarrow \mathbb{C}$
- output: $\tilde{s} : \mathbb{Z}^2 \longrightarrow \mathbb{C}$
- method 1: nearest neighbour
  - ∘ simplest, fastest
  - ∘ round k_x, k_y to nearest integer
  - ∘ $\tilde{s} = s \cdot \delta_{\lfloor k_x + .5 \rfloor, \lfloor k_y + .5 \rfloor}$

? What effect on image?

Nearest Neighbour = Convolution



$$S_0(k) = \sum_i s(k_i)\delta_{k_i} \text{(sampled data)}$$

$$S_1(k) = g \star \sum_i s_i \delta_{k_i}$$

$$S_2(k) = \sum_{(i,j)\in\mathbb{Z}^2} \delta_{i,j} \cdot S_1$$

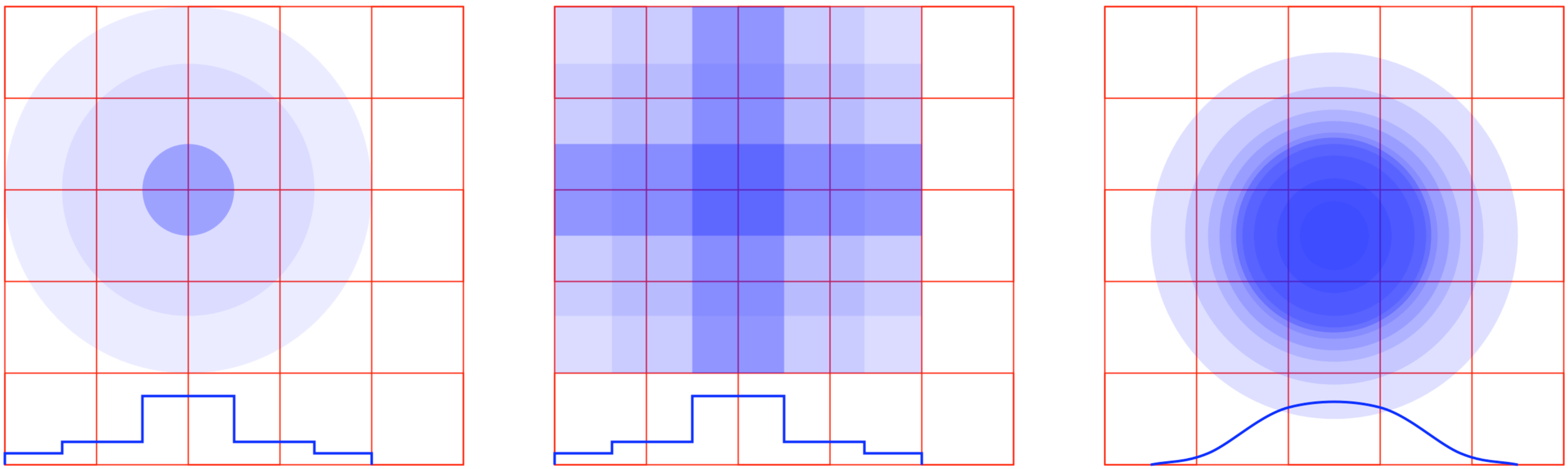
(resampled data)

for nearest neighbour

$$g(k) = \begin{cases} 1 & -\frac{1}{2} < k_x < \frac{1}{2} \ \wedge \ -\frac{1}{2} < k_y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Other Convolution Kernels

Some kernels are...



	symmetric	separable	both
$g =$	$\tilde{g}(k_x^2 + k_y^2)$	$g_1(k_x)\, g_2(k_y)$	e.g. $e^{-(k_x^2 + k_y^2)}$

but, alas, none can be separable, symmetric and compact.

Old Friend

$$
\begin{aligned}
FT(S_2) &= FT(\sum_{(i,j)\in\mathbb{Z}^2} \delta_{i,j}) \star FT(S_1) \\
&= \text{sos} \star (FT(g) \cdot FT(S_0)) \\
&= \text{sos} \star (FT(g) \cdot (FT(s) \star FT(\sum_i \delta_{k_i}))
\end{aligned}
$$

sos $\star$ (son of sinc)

- partial sum of sinc's Fourier series
- result of sampling (always present)
- approaches δ_0 as sampling $\to \infty$

$FT(g)\cdot$ interpolation window

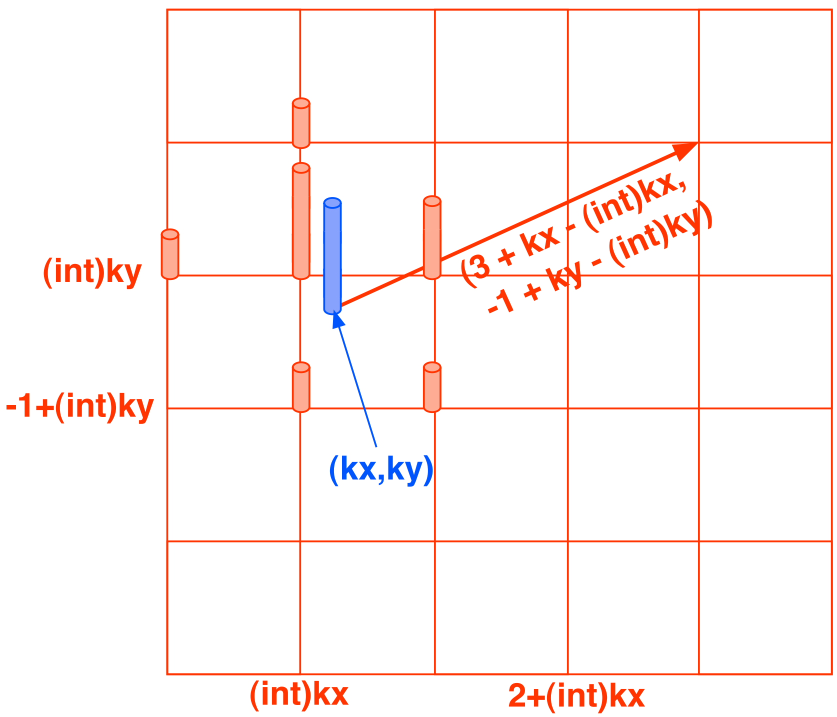
can be undone where $|FT(g)(x,y)| > \epsilon > 0$

$FT(\sum_i \delta_{k_i})\star$ spiral artifact/bluring
caused by uneven sampling
goes away if
∘ no holes bigger than Nyquist
∘ s_i replaced by s_i/local density

fix known nonintersecting trajectory
∘ replace s_i by $(k_i, g_i) \cdot s_i$

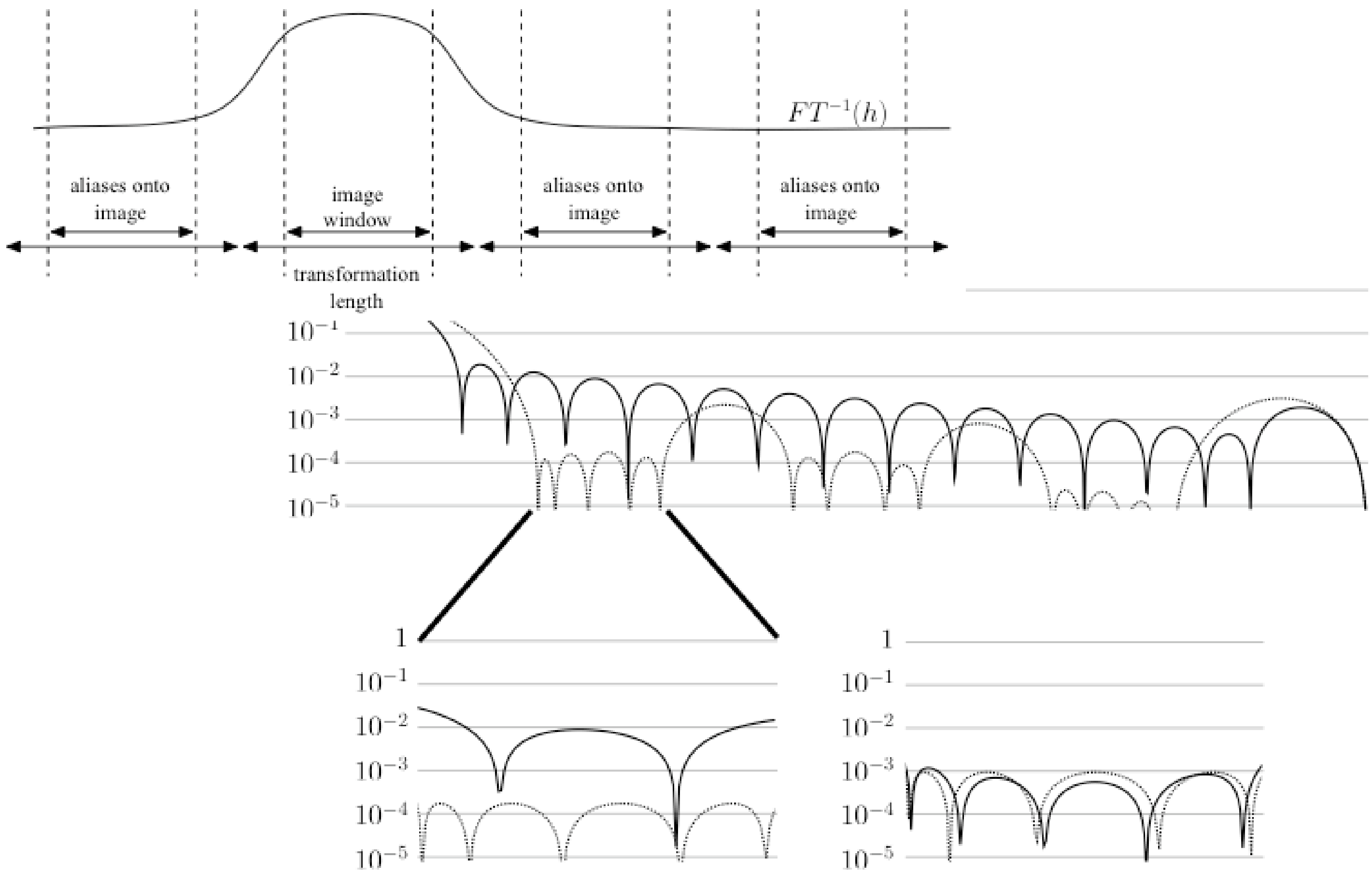
∂fix unknown or intersecting trajectory
∘ replace resampled $\{(s_i, k_i)\}$ by
resampled $\{(s_i, k_i)\}$ / resampled $\{(1, k_i)\}$

Resampling: HowTo




```
void resamplePoint(float kx, float ky, float s) {
 int x,y;
 float xFrac = kx - (int)kx;
 float yFrac = ky - (int)ky;
 for (x = -halfKernel; x <= halfKernel; x++)
  for (y = -halfKernel; y <= halfKernel; y++)
   kPlane[(int)ky + y][(int)kx + x] +=
              s * kernel(x - xFrac, y - yFrac);
}
```

Sequential LP



nonlinear vs KB nonlinear vs linear

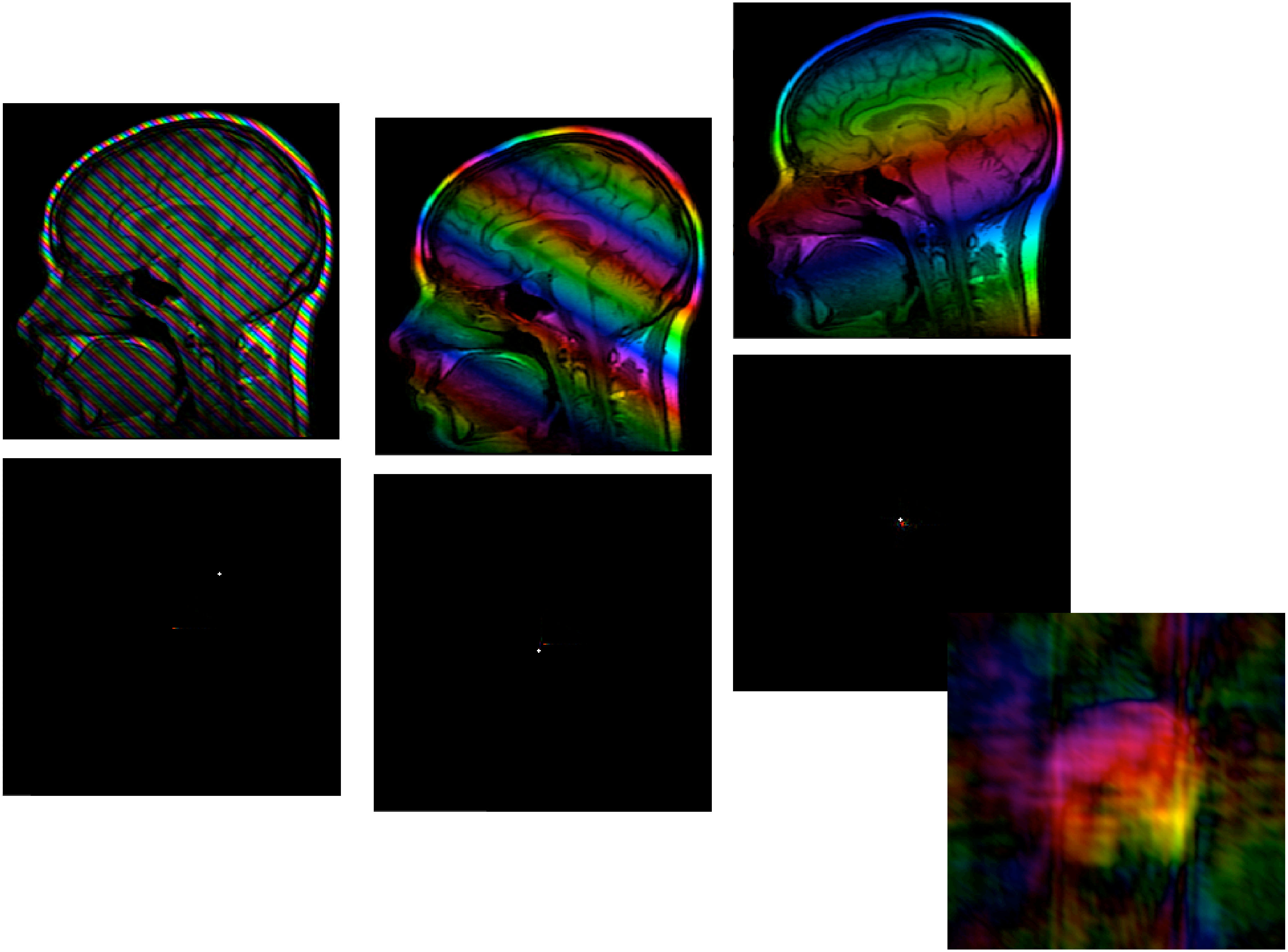
Fourier Transform

one sample at a time

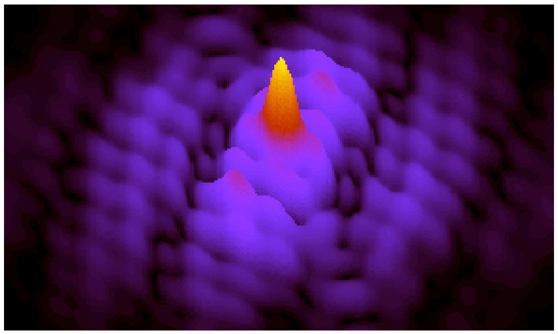
trajectory in dual space

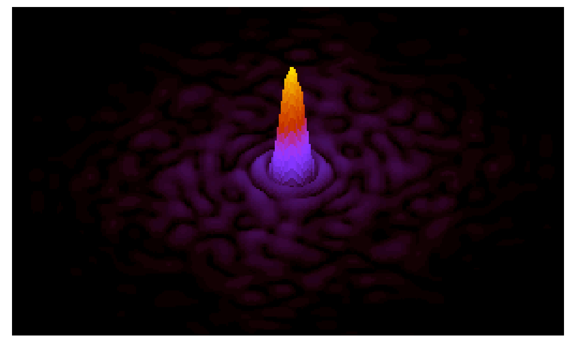
$$s(t) = \int_{\mathbb{R}^3} e^{i\langle x, k(t)\rangle} \rho(x) dx,$$

image



Point-Spread Function





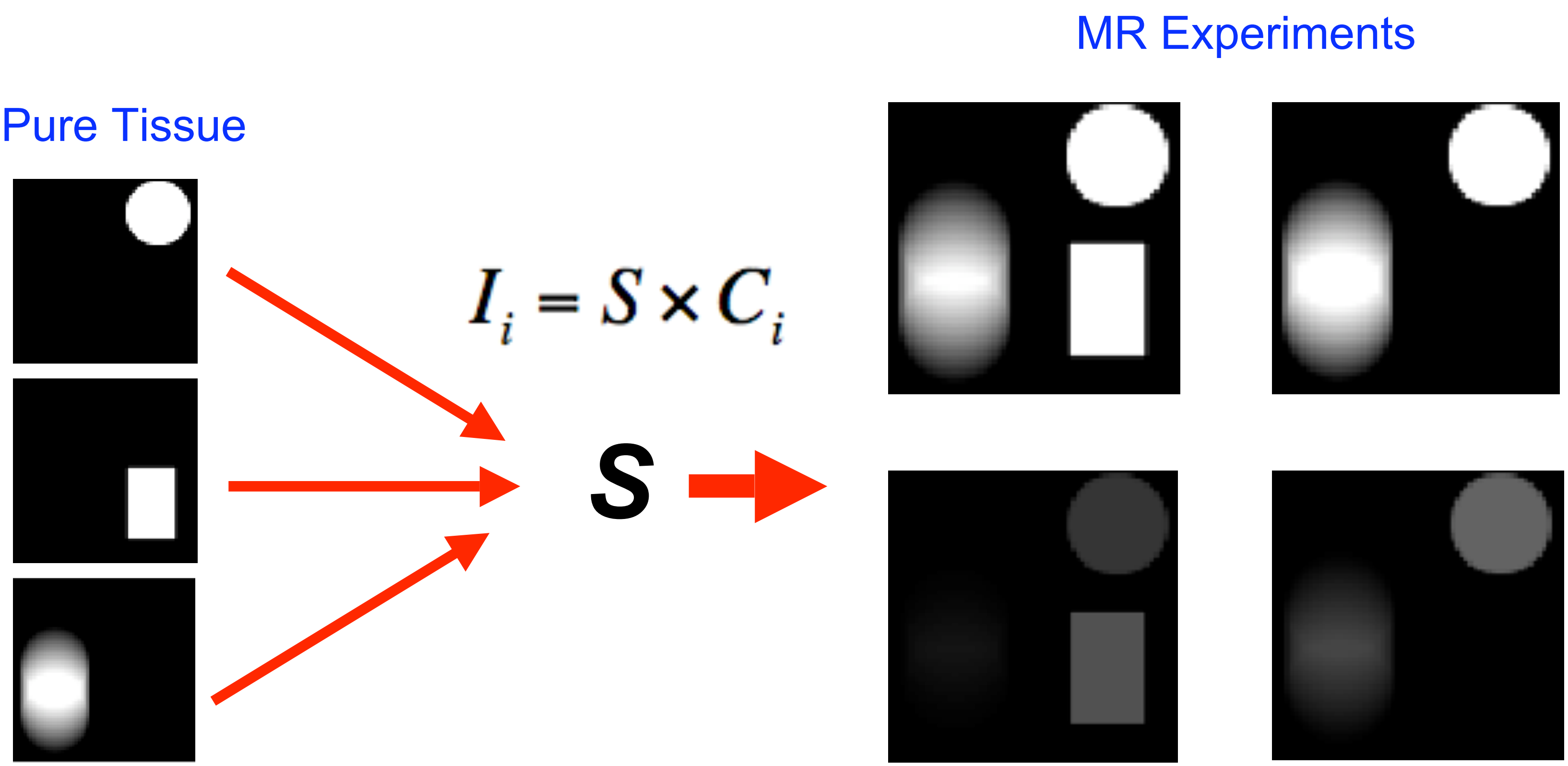
↔
2 voxels

Inverse Segmentation Problem (1)

• ***What is the Forward Problem?***

Having the pure tissues, we are looking for the expected MR images, I_i. In other words, knowing the number of experiments, we are trying to mix all these *t* pure tissue images in order to make MR experiments.

$$I_i = S \times C_i$$



Tissue Quantification (3)

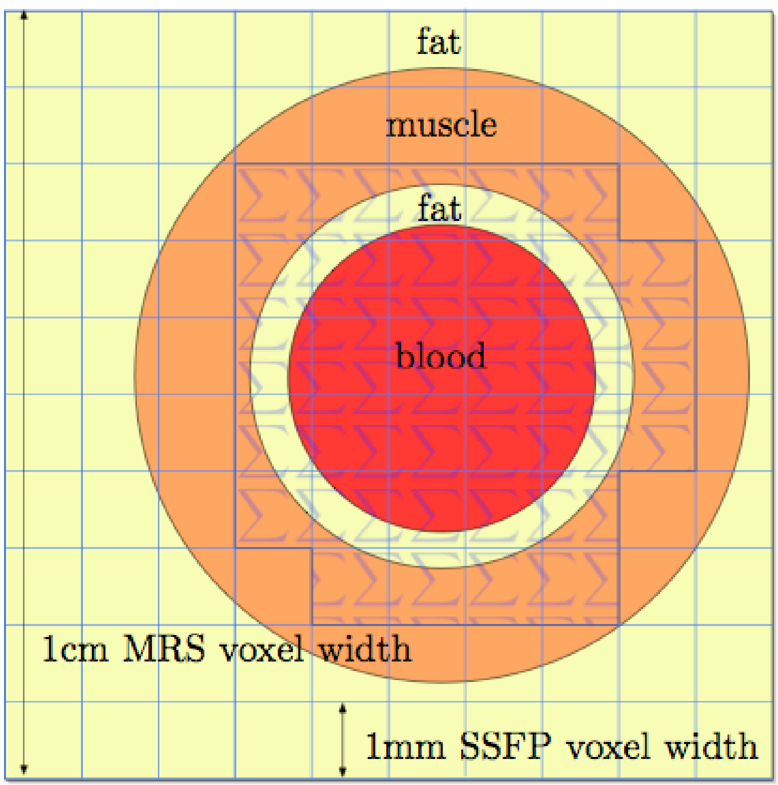
Finding pixel value for 1 Experiment:

$$\left\| \begin{pmatrix} \alpha_1 \\ \beta 1 \end{pmatrix} \rho_1 + \begin{pmatrix} \alpha_2 \\ \beta 2 \end{pmatrix} \rho_2 + \begin{pmatrix} \alpha_3 \\ \beta 3 \end{pmatrix} \rho_3 - \begin{matrix} \text{Real(Pixel Value)} \\ \text{Imag(Pixel Value)} \end{matrix} \right\|_2^2$$

Finding a pixel value for *n* Experiments and *t* Tissues:

$$\left\| \begin{pmatrix} \alpha_{11} \\ \beta_{11} \\ \alpha_{21} \\ \beta_{21} \\ \vdots \\ \alpha_{n1} \\ \beta_{n1} \end{pmatrix} \rho_1 + \begin{pmatrix} \alpha_{12} \\ \beta_{12} \\ \alpha_{22} \\ \beta_{22} \\ \vdots \\ \alpha_{n2} \\ \beta_{n2} \end{pmatrix} \rho_2 + \ldots + \begin{pmatrix} \alpha_{1t} \\ \beta_{1t} \\ \alpha_{2t} \\ \beta_{2t} \\ \vdots \\ \alpha_{nt} \\ \beta_{nt} \end{pmatrix} \rho_t - \begin{pmatrix} \text{Real(Pixel - Value Exp1)} \\ \text{Imag(Pixel - Value Exp}_1\text{)} \\ \text{Real(Pixel - Value Exp1)} \\ \text{Imag(Pixel - Value Exp}_2\text{)} \\ \vdots \\ \text{Real(Pixel - Value Exp}_n\text{)} \\ \text{Imag(Pixel - Value Exp}_n\text{)} \end{pmatrix} \right\|_2^2$$

Optimal Segmentation



$$
\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & S^T S - \lambda I \succeq 0 \\
& A(u_l, t_k) M_{SS}(u_l, t_k) = b(u_l, t_k) \quad \forall l, k
\end{aligned}
$$

$$
S(u_1, \ldots, u_n, t_1, \ldots, t_m) = \begin{pmatrix} M_{\mathrm{SS},x}(u_1, t_1) & \ldots & M_{\mathrm{SS},x}(u_1, t_m) \\ M_{\mathrm{SS},y}(u_1, t_1) & \ldots & M_{\mathrm{SS},y}(u_1, t_m) \\ \vdots & \vdots & \vdots \\ M_{\mathrm{SS},x}(u_n, t_1) & \ldots & M_{\mathrm{SS},x}(u_n, t_m) \\ M_{\mathrm{SS},y}(u_n, t_1) & \ldots & M_{\mathrm{SS},y}(u_n, t_m) \end{pmatrix}
$$

3.2. **Solution Method (we hope).** Now we have a small SDP with non-linear constraints. We don't have a package to solve this, but we can use implicit differentiation to obtain a linear approximation of the non-linear constraints, (or a quadratic approximation to the messy (25) constraint) and solve the SDP in a trust region.

For example, we can calculate $\frac{\partial M_{\mathrm{SS}}}{\partial x}$ (where x is any variable) by solving

$$A \frac{\partial M_{\mathrm{SS}}}{\partial x} = \frac{\partial B}{\partial x} - M_{\mathrm{SS}} \frac{\partial A}{\partial x}.$$

with A evaluated at our current guess for variable values. This linear system has the same LHS as the system for finding M_{SS}, which we

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Regularization (2)

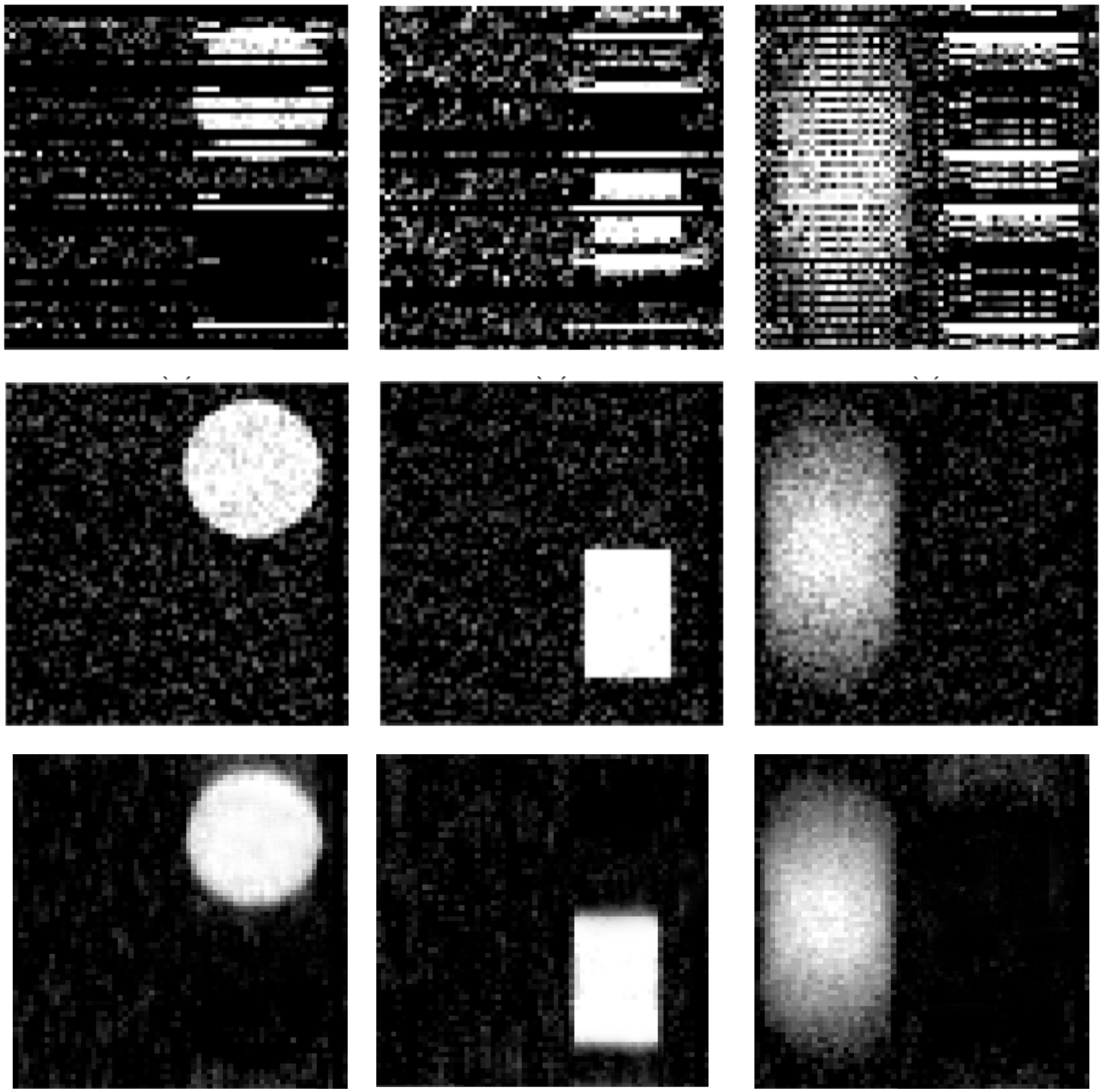
$$R = \begin{bmatrix}
\begin{pmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix} & & & 0 \\
& \begin{pmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix} & & \\
0 & & \ddots & \\
& & & \begin{pmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} & & 0 \\
& \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} & \\
& & \ddots & \ddots \\
0 & & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}
\end{bmatrix}$$

$$\|A.x - b\|^2 + \alpha.\|R.x\|^2$$

$$\left\| \begin{bmatrix} A \\ \alpha R \end{bmatrix} . x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|^2$$

We are using lsqlin() solver of Matlab for the least squares problem.

Numerical Results (5)



Noise +
Partial Data
No Regularization

Noise +
Partial Data+
Regularization $\alpha = 1$

Noise +
Partial Data+
Regularization $\alpha = 3.5$

Numerical Results (6)

Regularization improves the Eigenvalues of the inverse problem