Are Three Squares Impossible? II — Fourteen Subcases!

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Abstract: Series of Talks

These talks describe work done over the last 30 years or so both to understand and to compute repetitions in string — especially since 1999. We will discover that, although much has been learned, much combinatorial insight gained, there remains much more that is unknown about the occurrence of repetitions in strings and the restrictions they are subject to. I present combinatorial results discovered only recently, and I suggest that possibly extensions of these results can be used to compute repetitions in an entirely new way. I hope that members of the audience will be motivated to work on some of the many open problems that remain, to extend combinatorial knowledge even further.

Abstract of Talk II

We describe a major extension of the Three Squares Lemma, the complications that ensue, and the methods developed to handle them. Specifically, we wonder what happens when two squares $\boldsymbol{u}^2$ and $\boldsymbol{v}^2$ occur at the same location, $u < v$, while a third square $\boldsymbol{w}^2$ begins a distance k to the right. In this talk we explore the consequences of supposing that $3u/2 < v < 2u$, $0 \leq k < v-u$, and $v-u < w < v$, $w \neq u$. We uncover evidence to support the strange notion that ...

Three Squares Are Impossible!

Outline

Current Knowledge

The only neighbouring squares problem investigated so far ($\boldsymbol{u}^2$ & $\boldsymbol{v}^2$ at the same position, $\boldsymbol{w}^2$ distance $k \geq 0$ to the right) breaks down into two cases:

(1) $3u/2 < v < 2u,\ 0 \leq k < v-u,\ v-u < w < v, w \neq u$
- $*$ this case breaks down further into 14 (FOURTEEN!) subcases, alas ...

(2) $u < v \leq 3u/2$
- $*$ don't even need $\boldsymbol{k}$ or $\boldsymbol{w}$: we get a very specific breakdown that enables us to count all the runs in $\boldsymbol{v}^2$ of period $\geq v-u$.

In this talk we discuss problem (1); we present the accumulated results of research given in

[FSS05, PST05, S05, FPST06, S07, KS12, FFSS12].

Just Two Squares

$$\overbrace{\underbrace{u_1 \mid u_2 \mid u_1}_{\boldsymbol{u}} \mid u_1 \mid u_2}^{\boldsymbol{v}} \mid u_1 \mid u_2 \mid u_1 \mid u_1 \mid u_2$$

Lemma

Suppose $\boldsymbol{x} = \boldsymbol{v}^2$ *has proper prefix* $\boldsymbol{u}^2$. *Then*

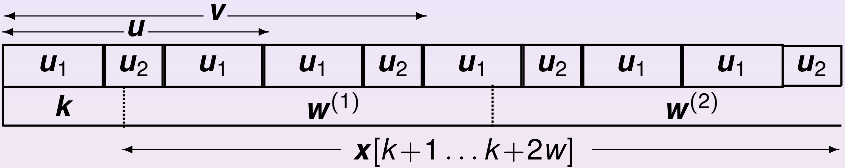
$$\boldsymbol{x} = \boldsymbol{u_1 u_2 u_1 u_1 u_2 u_1 u_2 u_1 u_1 u_2} \tag{1}$$

with $u_1 = 2u - v > 0,\ u_2 = 2v - 3u > 0$ *if and only if* $3u/2 < v < 2u$.

So just two squares provide a lot of structure:

$$\boldsymbol{u} = \boldsymbol{u_1 u_2 u_1},\ \boldsymbol{v} = \boldsymbol{u_1 u_2 u_1 u_1 u_2}.$$

Three Squares Provide Even More Structure

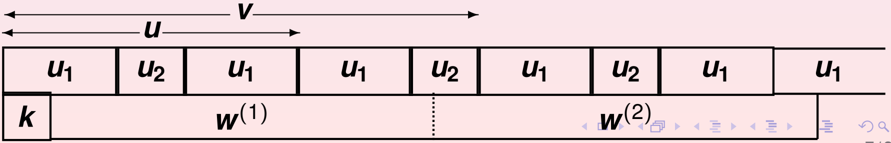


Here we show Subcase 13 in which

$$u_1 < k < u_1 + u_2, \; v < k + w \leq 2u.$$

On the other hand, Subcase 5 is

$$0 \leq k \leq u_1, \; u + u_1 < k + w \leq v :$$



The Dreaded 14 Subcases

Here are the subcases that we identify (based on k and w):

Subcase S	k	$k+w$	$k+2w$	Special Conditions
1	$0 \leq k \leq u_1$	$k+w \leq u$	$k+2w \leq u+u_1$	$k \geq u_2$
2	$0 \leq k \leq u_1$	$k+w \leq u$	$k+2w \leq u+u_1$	$k < u_2$
3	$0 \leq k \leq u_1$	$k+w \leq u$	$k+2w > u+u_1$	—
4	$0 \leq k \leq u_1$	$u < k+w \leq u+u_1$	—	—
5	$0 \leq k \leq u_1$	$u+u_1 < k+w \leq v$	—	—
6	$0 \leq k \leq u_1$	$v < k+w < 2u$	—	—
7	$u_1 < k < u_1+u_2$	$k+w \leq u+u_1$	$k+2w \leq 2u$	—
8	$u_1 < k < u_1+u_2$	$k+w \leq u+u_1$	$k+2w > 2u$	—
9	$u_1 < k < u_1+u_2$	$u+u_1 < k+w \leq v$	—	$w < u$
10	$u_1 < k < u_1+u_2$	$k+w \leq v$	$k+2w \leq u+v$	$w > u$
11	$u_1 < k < u_1+u_2$	$k+w \leq v$	$u+v < k+2w \leq 2v - u_2$	—
12	$u_1 < k < u_1+u_2$	$k+w \leq v$	$2v - u_2 < k+2w$	—
13	$u_1 < k < u_1+u_2$	$v < k+w \leq 2u$	—	—
14	$u_1 < k < u_1+u_2$	$2u < k+w < 2u+u_2-1$	—	—

- **Our First Hope:** We might find a combinatorial breakdown of $\boldsymbol{v}^2$ for each of these cases — **probably true**.
- **Our Second Hope:** The proof for one case might extend to others — **no way!!**

Jenya Kopylova's Algorithm *construct_x* [KS12]

construct_x$(u_1\_max, u_2\_max, k, S;\ \boldsymbol{u_1}, \boldsymbol{u_2}, k, w)$:
$u_1 \in 1..u_1\_max,\ u_2 \in 1..u_2\_max.$

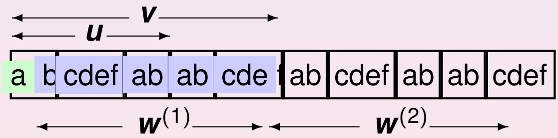
— *For every subcase* (u_1, u_2, k, w) *determined by* u_1_max, u_2_max,
— *compute subcase identifier* S*, maximum alphabet* σ *and* $\boldsymbol{u_1}$, $\boldsymbol{u_2}$.

for $u_1 = 1$ **to** u_1_max **do**
  for $u_2 = 1$ **to** u_2_max **do**
    for $k = 0$ **to** u_1+u_2-1 **do**
      for $w = u_1+u_2+1$ **to** $v-1$ **do**
        if $w \neq 2u_1+u_2$ **then**
          $\sigma = u_1+u_2$
          — *Assign distinct letters to positions in* $\boldsymbol{u_1}$, $\boldsymbol{u_2}$.
          $\boldsymbol{u_1} = 1\,2 \cdots u_1$; $\boldsymbol{u_2} = u_1{+}1\ u_1{+}2 \cdots u_1{+}u_2$
          $(\sigma, \boldsymbol{u_1}, \boldsymbol{u_2}) \leftarrow$ ***force_square***$(\sigma, \boldsymbol{u_1}, \boldsymbol{u_2}, k, w)$
          $S \leftarrow$ ***compute_subcase*** — *from Subcase Table*
          return $(k, w, S, \sigma, \boldsymbol{u_1}, \boldsymbol{u_2})$

Example — Subcase 5

$$\text{Let } X = \{u_1 = 2, u_2 = 4, \text{k} = 1, \text{w} = 12\}$$

$$\Rightarrow 0 \leq k \leq u_1$$
$$\Rightarrow \boldsymbol{w}^{(1)} \text{ ends in } \boldsymbol{u}_2^{(2)}$$



$\sigma = u_1 + u_2 = 6$

Example — Subcase 5

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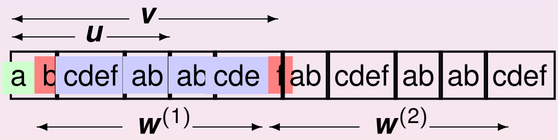


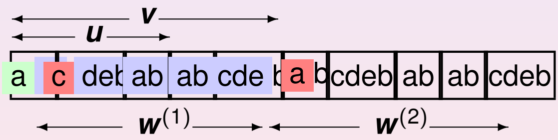
- Begin with $\boldsymbol{x}[k+1] = \boldsymbol{x}[2] =$ b and $\boldsymbol{x}[k+w+1] = \boldsymbol{x}[14] =$ f
- Since $b \neq f$ and $\min(b, f) = b$, then replace all $f's$ with $b's$ in $\boldsymbol{x}$: $\sigma = 5$

Example — Subcase 5

Let $X = \{u_1 = 2, u_2 = 4,$ k $= 1,$ w $= 12\}$

$\Rightarrow 0 \leq k \leq u_1$

$\Rightarrow \boldsymbol{w}^{(1)}$ ends in $\boldsymbol{u}_2^{(2)}$



- Next, $\boldsymbol{x}[k+2] = \boldsymbol{x}[3] =$ c and $\boldsymbol{x}[k+w+2] = \boldsymbol{x}[5] =$ a
- Since $c \neq a$ and $\min(c, a) = a$, then replace all $c's$ with $a's$ in $\boldsymbol{x}$: $\sigma = 4$

Example — Subcase 5

$$\text{Let } X = \{u_1 = 2, u_2 = 4, \text{k} = 1, \text{w} = 12\}$$

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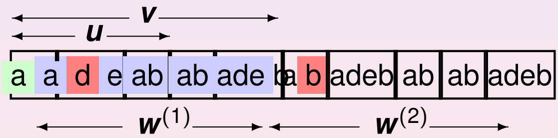

- Continuing with this process until the end of $\boldsymbol{w}$, we find d $=$ b :

$\sigma = 3$

Example — Subcase 5

$$\text{Let } X = \{u_1 = 2, u_2 = 4, \text{k} = 1, \text{w} = 12\}$$

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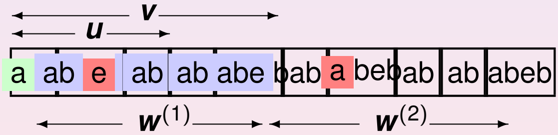


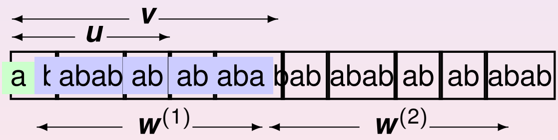
- Then e = a :

  $\sigma = 2$

Example — Subcase 5

$$\text{Let } X = \{u_1 = 2, u_2 = 4, \text{k} = 1, \text{w} = 12\}$$

$$\Rightarrow 0 \leq k \leq u_1$$
$$\Rightarrow \boldsymbol{w}^{(1)} \text{ ends in } \boldsymbol{u}_2^{(2)}$$



- In this example, the string reduces to a repetition of period 2 after 5 comparisons.

Trickier Example — Subcase 3

$$X = \{u_1 = 3, u_2 = 9, \text{k} = 2, \text{w} = 13\}$$

* We find $\sigma = 2 > 1 = d = \gcd(3, 9, 13)$.
* Then $\boldsymbol{x} = \boldsymbol{z}^{\alpha}\boldsymbol{z}[1 \dots u_1 \bmod z]\boldsymbol{z}^{\gamma}\boldsymbol{z}[1 \dots u_1 \bmod z]\boldsymbol{z}^{\varepsilon}$, where $z = \gcd(u - w, w - u_1) = \gcd(2, 10) = 2$, and $\alpha = \lfloor u/z \rfloor = \lfloor 15/2 \rfloor = 7; \gamma = \lfloor v/z \rfloor = \lfloor 27/2 \rfloor = 13; \varepsilon = (u_1 + u_2)/2 = 6$.
* Then $\boldsymbol{x}$ has the breakdown

$$\boldsymbol{x} = \boldsymbol{z}^7\boldsymbol{z}[1]\boldsymbol{z}^{13}\boldsymbol{z}[1]\boldsymbol{z}^6,\ z = 2.$$

* The program-generated string is $\boldsymbol{x} = (ab)^7 a(ab)^{13} a(ab)^6$.

Trickier Example — Subcase 3

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  where $z = \gcd(u - w, w - u_1) = \gcd(2, 10) = 2$,
  and $\alpha = \lfloor u/z \rfloor = \lfloor 15/2 \rfloor = 7; \gamma = \lfloor v/z \rfloor = \lfloor 27/2 \rfloor = 13; \varepsilon = (u_1 + u_2)/2 = 6$.
* Then $\boldsymbol{x}$ has the breakdown

$$\boldsymbol{x} = \boldsymbol{z}^7\boldsymbol{z}[1]\boldsymbol{z}^{13}\boldsymbol{z}[1]\boldsymbol{z}^6, \; z = 2.$$

$\boldsymbol{v}$

$\boldsymbol{u}$

ab	a	babababab	aba	aba	babababab	aba	babababab	aba	aba	babababab

$\boldsymbol{w}^{(1)}$ $\boldsymbol{w}^{(2)}$

* The program-generated string is $\boldsymbol{x} = (ab)^7 a(ab)^{13} a(ab)^6$.

Statistics Chart

1	2	3	4	5	6	7
S	# strings	σ_{max}	$\# \sigma = d$	$\# \sigma > d$	$\# \Sigma = \{1, 2, \ldots, \sigma\}$	# gaps
1	7840	7	7840	0	7840	0
2	8960	10	8960	0	8960	0
3	131100	29	118305	12795	131100	0
4	283620	30	276799	6821	278132	5488
5	227505	30	227505	0	227505	0
6	121800	15	121800	0	121800	0
7	47250	27	44548	2702	44860	2390
8	51640	15	51640	0	51640	0
9	90335	15	90335	0	90335	0
10	64050	10	64050	0	64050	0
11	54000	15	51707	2293	54000	0
12	16800	15	15612	1188	16800	0
13	201405	30	197860	3545	201405	0
14	109620	15	108770	850	108831	789

* Statistics for 1,415,925 strings generated using $u_1\_max = u_2\_max = 30$; $d = \gcd(u_1, u_2, w)$

Generated Conjectures

Subcases S	Conditions	Breakdown of $\boldsymbol{x}/\boldsymbol{v}^2$
1,2,5,6,8–10	$(\forall \boldsymbol{x}, \sigma = d)$	$\boldsymbol{x} = \boldsymbol{d}^{(x/d)}$
3,4,7	$\sigma = d$	$\boldsymbol{x} = \boldsymbol{d}^{(x/d)}$
	$\sigma > d$	$\boldsymbol{x} = \boldsymbol{z}^{\alpha}\boldsymbol{z}[1..s]\boldsymbol{z}^{\gamma}\boldsymbol{z}[1..s]\boldsymbol{z}^{\varepsilon}$
11–14	$\sigma = d$	$\boldsymbol{x} = \boldsymbol{d}^{(x/d)}$
	$\sigma > d$	$\boldsymbol{u} = \boldsymbol{\delta}^{u/\delta}, \boldsymbol{v} = \left(\boldsymbol{t}^{\delta/t}\right)^{v/\delta}$

$$d = \gcd(u_1, u_2, w);\ z = \gcd(u-w, w-u_1);\ s = u_1 \bmod z$$

$$\alpha = \lfloor u/z \rfloor;\ \gamma = \lfloor v/z \rfloor;\ \varepsilon = (u_1+u_2)/z$$

$$\delta = \gcd(u, v-w);\ t = v \bmod \delta$$

Some Conjectures Become Theorems

- In [S07] Subcase 10 and Subcases 11–14 for $\sigma = d$ are proved.
- In [KS12] Subcases 1,2,5,6,8–10 are proved.
- In [FFSS12] Subcase 4 and Subcases 11–14 for $\sigma > d$ are proved.
- Subcases 3 and 7 remain — they are probably challenging ...

The Proofs are HARD ... and All Different!

Since $v-s = k+w$, therefore $k+s = v-w < u$ by hypothesis, and so $k+s+u_1 < u+u_1 < k+w$. Thus the copy of $\boldsymbol{u_1}$ in $\boldsymbol{w}^{(2)}$ must overlap with $\boldsymbol{u_1}^2$ in $\boldsymbol{w}^{(1)}$, telling us that $\boldsymbol{u_1} = R_{u-w}(\boldsymbol{u_1})$, where $u-w < u_1$. From Lemma 5 we conclude that $\boldsymbol{u_1}$ is a repetition of period $u-w$, and further that the prefix $\boldsymbol{z} = \boldsymbol{r}\boldsymbol{u_1}^2$ of $\boldsymbol{w}$ has period $u-w$. Observe that $u-w \mid u_1$, moreover since $w-u_2 = 2u_1-(u-w)$, that $u-w \mid w-u_2$.
Now consider the occurrence of $\boldsymbol{u}^* = \boldsymbol{u_2}\boldsymbol{u_1}$ that has as its prefix a suffix of $\boldsymbol{w}^{(2)}$. Considering the corresponding suffix of $\boldsymbol{w}^{(1)}$, we see that

$$\boldsymbol{u}^*[1..u_1-(s-r)] = \boldsymbol{u_1}[s-r+1..u_1], \tag{2}$$

a suffix of $\boldsymbol{u_1}$ of length $u_1-(s-r) = u_1-(u-w) \geq u-w$, since $u_1 = q(u-w)$ for some $q > 1$. Therefore $\boldsymbol{u}^*[1..u_1-(s-r)]$ of length $(q-1)(u-w)$ has period $u-w$. Continuing the match, we find that

$$\boldsymbol{u}^*[u_1-(s-r)+1..w-(u_1+s)] = \boldsymbol{u}^*[1..u_2-s], \tag{3}$$

that by virtue of (2) continues the period $u-w$. Thus from (2) and (3) we conclude that $\boldsymbol{w}$ has a suffix of length $w-(u_1+s)$ of period $u-w$, and as we saw earlier, it also has a prefix $\boldsymbol{z}$ of length $r+2u_1$ of period $u-w$. The periodic suffix and prefix of $\boldsymbol{w}$ have an overlap of

$$w-u_1-s+r+2u_1-w = u_1-(u-w),$$

as we have seen divisible by $u-w$. We conclude therefore that $\boldsymbol{w}$ has period $u-w$. Similarly, since (2)–(3) also tell us that $\boldsymbol{u_2}$ has a prefix of length $\min(u_2, w-(u_1+s))$ of period $u-w$, and since as we have seen the suffix of $\boldsymbol{u_2}$ of length s has the same period, we find the overlap

$$\begin{aligned}\min(u_2, w-(u_1+s))+s-u_2 &= \min(s, w-(u_1+u_2))\\ &= \min(s, u-(k+s))\\ &= \min(s, u_1-(s-t)),\end{aligned}$$

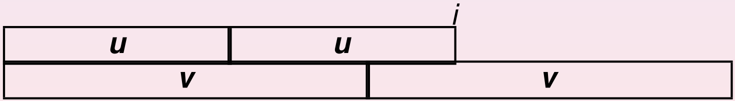
a quantity greater than $u-w$. We conclude therefore that $\boldsymbol{u_2}$ also has period $u-w$. But the periodicity of $\boldsymbol{w}$, taken

The Old Lemma

Lemma (“Periodicity Lemma” [FW65])

Let u and v be two periods of $\boldsymbol{x}$, and let $d = \gcd(u, v)$. If $u+v \leq x+d$, then d is also a period of $\boldsymbol{x}$.

This famous lemma is used in many contexts, and has been used frequently in our proofs:



The string $\boldsymbol{x}[1..i]$ has periods u and v; if $u+v \leq i+\gcd(u, v)$, it also has period $\gcd(u, v)$.

The New Lemma

But a new lemma is particularly relevant to our circumstances, apparently never considered before:

Lemma (Two Rotations, Same Period)

Suppose both $\boldsymbol{x}$ and $R_v(\boldsymbol{x})$, $0 < v < x$, have period u, where $\ell = x \bmod u > 0$ and $t = \lfloor x/u \rfloor$. Let $\boldsymbol{x}_v$ denote $R_v(\boldsymbol{x})$, and let $d = \gcd(u, \ell)$. Then

(a) *if $t = 1$ and $v > \ell$, $\boldsymbol{x}_{v-\ell}[1..2\ell]$ is a square of period ℓ;*

(b) *if $t = 1$ and $v \le \ell$, or if $t > 1$ and $v < u$, $\boldsymbol{x}[1..v+\ell]$ has period ℓ;*

(c) *if $t > 1$ and $u \le v \le x-u$, $\boldsymbol{x}[1..u+\ell]$, hence $\boldsymbol{x}$, is a repetition of period d;*

(d) *if $t > 1$ and $x-u < v$, where $v' = v-(x-u)$, $\boldsymbol{x}[v'+1..u+\ell]$ has period d for $v' = 1$, otherwise period ℓ.*

Two Rotations, Same Period

Recall Subcase 13:

$$u_1 < k < u_1 + u_2, \; v < k + w \leq 2u.$$

$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_1}$	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_1}$	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$
$\boldsymbol{k}$			$\boldsymbol{w}^{(1)}$					$\boldsymbol{w}^{(2)}$	

Here the prefix of $\boldsymbol{w}$ of length $u = 2u_1 + u_2$ matches two distinct rotations of $\boldsymbol{u} = \boldsymbol{u_1 u_2 u_1}$, which therefore have the same period. The new lemma provide precise information: an addition to the stringologist's toolkit!

Case (a) — $t = 1,\ v > \ell$

Suppose $x = 7$, $u = 4$: $\boldsymbol{x}[1..7]$ has period 4. Then $t = \lfloor 7/4 \rfloor = 1$, $\ell = 7 \bmod 4 = 3$, and $\boldsymbol{x} = abcdabc$ for some a, b, c, d.

Suppose $v = 6$, so that $R_6(\boldsymbol{x}) = c\ abcdab$ also has period u. Then $R_{v-\ell}(\boldsymbol{x}) = \boldsymbol{x}_{v-\ell} = \boldsymbol{x}_3 = dabcabc$ has a square prefix $dab\ cab$ of period 3: therefore $d = c$ and $\boldsymbol{x} = abccabc$.

But if $v = 4$, $R_4(\boldsymbol{x}) = abc\ abcd$ also has period u. In this case $\boldsymbol{x}_{v-\ell} = \boldsymbol{x}_1$ has a square prefix $bcdabc$ of period 3, so that $a = b, b = c, c = d$ and $\boldsymbol{x} = a^7$.

Case (b) — $t = 1,\ v \le \ell$

$x = 7, u = 4 \implies t = 1, \ell = 3, \boldsymbol{x} = abcdabc.$

Suppose $v = 3$: $R_3(\boldsymbol{x})$ has period 4; that is, $\boldsymbol{x}[1..v+\ell] = abcdab$ has period 3: $\boldsymbol{x} = a^7$.

Suppose $v = 1$: $R_1(\boldsymbol{x})$ has period 4; that is, $\boldsymbol{x}[1..v+\ell] = abcd$ has period 3: $d = a$, $\boldsymbol{x} = abcaabc$.

Case (c) — $t > 1,\ u \leq v \leq x - u$

Suppose $x = 11$, $u = 4$: $\boldsymbol{x}[1..11]$ has period 4. Then $t = \lfloor 11/4 \rfloor = 2$, $\ell = 11 \bmod 4 = 3$, and $\boldsymbol{x} = (abcd)^2abc$.

For $v \in u..x-u$ — that is, $4 \leq v \leq 7$ — $\boldsymbol{x}$ is a repetition of period $d = \gcd(u, \ell) = \gcd(4, 3) = 1$. For every v in the range, $\boldsymbol{x} = a^7$.

Open Questions

- What is the upper bound on the alphabet size σ? It seems that $\sigma \leq (u_1 + u_2)/2$.
- Apparently $\sigma = \gcd(u_1, u_2, w)$ implies that $\boldsymbol{x}$ is a repetition of period d — why? Can such a fact be used to simplify proofs?
- If $\boldsymbol{x}$ includes a maximum letter greater than σ, does it follow that $\sigma > \gcd(u_1, u_2, w)$? Why are such cases so rare?
- Can we prove Subcases 3 & 7?
- What happens if three squares occur within some distance k of each other, but without the restriction that two of them (our $\boldsymbol{u}$ and $\boldsymbol{v}$) occur together, and $\boldsymbol{w}$ possibly preceding one or both of $\boldsymbol{u}$ and $\boldsymbol{v}$. *No-one has thought through these cases to understand which values of u, v, w, k are possible and/or interesting — that is the next lecture ...*

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