

A New Approach to the Periodicity Lemma on Strings with Holes ^{☆,☆☆}

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Abstract

We first give an elementary proof of the periodicity lemma for strings containing one hole (variously called a “wild card” or a “don’t-care” or an “indeterminate letter” in the literature). The proof is modelled on Euclid’s algorithm for the greatest common divisor and is simpler than the original proof given in [BB99]. We then study the two hole case, where our result agrees with the one given in [BSH02] but is more easily proved and enables us to identify a maximum-length prefix or suffix of the string to which the periodicity lemma does apply. Finally we extend our result to three or more holes using elementary methods and state a version of the periodicity lemma that applies to all strings with or without holes. We describe an algorithm that, given the locations of the holes in a string, computes maximum length substrings to which the periodicity lemma applies, in time proportional to the number of holes. Our approach is quite different from the one in [BSH02, BS04] and also simpler.

Key words: periodicity, periodicity lemma, indeterminate string, hole

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1. Introduction

Over the last few years researchers have shown interest [BB99, IMM⁺03, BSH02] in strings that may contain *don't-care* letters; that is, letters $*$ that match every letter in a given alphabet Σ . More generally, several papers [HS03, HSW06, HSW08] have studied “indeterminate” strings that may contain “indeterminate” letters — those that match various subsets of Σ . In this article we study the more general model.

Let Σ be an alphabet and let λ_i , $|\lambda_i| \geq 2$, $1 \leq i \leq m$, be pairwise distinct subsets of Σ . We form a new alphabet $\Sigma' = \Sigma \cup \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and define a new relation *match* (\approx) on Σ' as follows:

- for every $\mu_1, \mu_2 \in \Sigma$, $\mu_1 \approx \mu_2$ if and only if $\mu_1 = \mu_2$;
- for every $\mu \in \Sigma$ and every $\lambda \in \Sigma' - \Sigma$, $\mu \approx \lambda$ and $\lambda \approx \mu$ if and only if $\mu \in \lambda$;
- for every $\lambda_i, \lambda_j \in \Sigma' - \Sigma$, $\lambda_i \approx \lambda_j$ if and only if $\lambda_i \cap \lambda_j \neq \emptyset$.

This idea seems to have first been mentioned in [FP74].

We observe that *match* is reflexive and symmetric but not necessarily transitive; for example, if $\lambda = \{a, b\}$, then $a \approx \lambda$ and $b \approx \lambda$ does not imply $a \approx b$. In this paper $\mathbf{x} = \mathbf{x}[1..n]$ is always a nonempty string on Σ' that may therefore contain some $\lambda \in \Sigma' - \Sigma$ at some position $h \in 1..n$; that is, $\mathbf{x}[h] = \lambda$. We refer to an occurrence of λ in \mathbf{x} as a *hole*, generalizing the usage in [BB99, BSH02, BS04], where always $\Sigma' = \Sigma \cup \{\Sigma\}$. Here a hole is equivalent to an *indeterminate letter* as defined in [HS03]. We also sometimes refer to the position h itself as a hole.

A string \mathbf{x} has *period* (*strong period*) p if and only if for every $i, j \in 1..n$ such that $i \equiv j \pmod{p}$, $\mathbf{x}[i] \approx \mathbf{x}[j]$; \mathbf{x} has *weak period* p if and only if for every $i, j \in 1..n$ such that $j = i+p$, $\mathbf{x}[i] \approx \mathbf{x}[j]$. For example, in the following table \mathbf{x} has a weak period but not a strong period of length 2.

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{x} = & a & b & a & * & a & c \end{array}$$

On strings without holes, periodicity and weak periodicity are equivalent.

2. Strings With One Hole

We first consider strings with exactly one hole. In [BB99] a variant of the periodicity lemma [FW65] for such strings was stated, proved, and shown to be sharp:

Lemma 1. *If \mathbf{x} with one hole has weak periods p and $q > p$, and $n \geq p+q$, then \mathbf{x} has strong period $d = \gcd(p, q)$.*

We prove this lemma here based on the Euclidean algorithm, extending the proof given in [Smy03] for the original periodicity lemma. As observed in [BB99], it suffices to establish the case $n = p+q$, since therefore for larger n , the lemma holds for every factor of length $p+q$, hence for \mathbf{x} itself. We first prove a preliminary result:

Lemma 2. *Suppose $\mathbf{x} = \mathbf{x}[1..p+q]$ has weak periods p and $q > p$ with a single hole $\mathbf{x}[h] = \lambda$.*

(a) $h \in 1..q \Rightarrow \mathbf{x}[1..q]$ has weak periods p and $q-p$;

(b) $h \in p+1..p+q \Rightarrow \mathbf{x}[p+1..p+q]$ has weak periods p and $q-p$.

Proof. We prove (a); the proof of (b) is analogous. Since \mathbf{x} has weak periods p and $q > p$, therefore $\mathbf{x}[1..q]$ has weak period p . Since for $i > p$, $i+(q-p) > q$, we need consider only $i \in 1..p$. For these values of i , it follows from weak q periodicity that $\mathbf{x}[i] \approx \mathbf{x}[i+q]$ and from weak p periodicity that $\mathbf{x}[i+q] \approx \mathbf{x}[i+q-p]$. Since $h \leq q$, we know that $\mathbf{x}[i+q] \neq \lambda$, hence that $\mathbf{x}[i] \approx \mathbf{x}[i+q-p]$. Therefore $\mathbf{x}[1..q]$ also has weak period $q-p$, as required. \square

Since h satisfies the hypothesis of either Lemma 2(a) or Lemma 2(b) (or both), we can always reduce \mathbf{x} with a single hole, whose length $p+q$ is the sum of its distinct weak periods p and q , to a substring \mathbf{y} with a single hole whose length q is the sum of its (not necessarily distinct) weak periods p and $q-p$: \mathbf{y} is either a prefix $\mathbf{x}[1..q]$ or a suffix $\mathbf{x}[p+1..p+q]$ of \mathbf{x} . If $p = q-p$, we have computed $p = \gcd(p, q) = q/2$; if not, we can perform another reduction. Let us write $\mathbf{x}^{(0)} = \mathbf{x}$ and for $r \geq 0$, let $\mathbf{x}^{(r+1)}$ be the reduction (hence a substring) of $\mathbf{x}^{(r)}$. By the correctness of the Euclidean algorithm, a finite number $k \geq 1$ of reductions yields a string $\mathbf{x}^{(k)} = \mathbf{x}^{(k)}[1..2d]$ that contains one hole and has weak period $d = \gcd(p, q)$. But then, since $\mathbf{x}^{(k)}$ takes the form $\mathbf{u}\mathbf{u}$, where $\mathbf{u} = \mathbf{x}[1..d]$, it actually has strong period d . We illustrate this reduction process with an example in Tables 1–4. Starting with a string $\mathbf{x}^{(0)}$ that has weak periods $q^{(0)} = 8$ and $p^{(0)} = 6$, we recursively reduce it to $\mathbf{x}^{(3)}$ that has a strong period 2.

$\mathbf{x}^{(0)} =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	a	b	a	b	a	b	*	b	(a	b	a	b	a	b)

Table 1: $|\mathbf{x}^{(0)}| = 14, q^{(0)} = 8, p^{(0)} = 6, q^{(0)} - p^{(0)} = 2$

Lemma 3. *If for some $r \in 1..k$, $\mathbf{x}^{(r)}$ has strong period d , then $\mathbf{x}^{(r-1)}$ also has strong period d .*

Proof. According to the nature of a reduction, $\mathbf{x}^{(r-1)}$ has weak periods p and $q > p$ that are divisible by $d = q-p$, and $|\mathbf{x}^{(r-1)}| = p+q$. We want to prove that for every $i, j \in 1..p+q$ such that $i \equiv j \pmod{d}$, $\mathbf{x}^{(r-1)}[i] \approx \mathbf{x}^{(r-1)}[j]$. We consider three cases:

$$\mathbf{x}^{(1)} = \begin{array}{cccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ (a & b) & a & b & a & b & * & b & & & & & & & & \end{array}$$

Table 2: $|\mathbf{x}^{(1)}| = 8, q^{(1)} = 6, p^{(1)} = 2, q^{(1)} - p^{(1)} = 4$

$$\mathbf{x}^{(2)} = \begin{array}{cccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ & & & (a & b) & a & b & * & b & & & & & & & \end{array}$$

Table 3: $|\mathbf{x}^{(2)}| = 6, q^{(2)} = 4, p^{(2)} = 2, q^{(2)} - p^{(2)} = 2$

1. both i and j lie in $\mathbf{x}^{(r)}$;
2. one position (say i) lies in $\mathbf{x}^{(r)}$, but not j ;
3. neither i nor j lies in $\mathbf{x}^{(r)}$.

Case (1) is straightforward since $\mathbf{x}^{(r)}$ is strongly d periodic.

In case (2), assume without loss of generality that $\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)}[1..q]$ — the proof for suffix $\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)}[p+1..p+q]$ is analogous. By the weak periodicity of $\mathbf{x}^{(r-1)}$, $\mathbf{x}^{(r-1)}[j-q] \approx \mathbf{x}^{(r-1)}[j]$ and $\mathbf{x}^{(r-1)}[j-p] \approx \mathbf{x}^{(r-1)}[j]$, where $j-q < j-p \leq q$, so that both $j-q$ and $j-p$ are positions in $\mathbf{x}^{(r)}$. Since there is exactly one hole in $\mathbf{x}^{(r)}$, we may denote by j^* any one of $j-q, j-p$ that is *not* a hole. Since $i \equiv j \pmod{d}$ and d divides both p and q , $i \equiv j^* \pmod{d}$. Then by the strong d periodicity of $\mathbf{x}^{(r)}$,

$$\mathbf{x}^{(r-1)}[i] \approx \mathbf{x}^{(r-1)}[j^*] \approx \mathbf{x}^{(r-1)}[j].$$

Since j^* is not a hole, $\mathbf{x}^{(r-1)}[i] \approx \mathbf{x}^{(r-1)}[j]$, as required.

In case (3) we again need only consider prefix $\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)}[1..q]$. Using the same argument as in case (2), we can find $j^* < q$, not a hole, such that $\mathbf{x}^{(r-1)}[j^*] \approx \mathbf{x}^{(r-1)}[j]$. But now the same construction applies also to $i > q$, allowing us to find $i^* < q$, not a hole, such that $\mathbf{x}^{(r-1)}[i^*] \approx \mathbf{x}^{(r-1)}[i]$. Since $i \equiv j \pmod{d}$, it follows that $i^* \equiv j^* \pmod{d}$, so that by the strong d periodicity of $\mathbf{x}^{(r)}$, $\mathbf{x}^{(r-1)}[i^*] \approx \mathbf{x}^{(r-1)}[j^*]$. Thus $\mathbf{x}^{(r-1)}[i] \approx \mathbf{x}^{(r-1)}[j]$. (In fact, in this case, $\mathbf{x}^{(r-1)}[i] = \mathbf{x}^{(r-1)}[j]$.) \square

Lemma 3 allows us to reconstruct \mathbf{x} by reversing the reduction, and shows that every intermediate substring $\mathbf{x}^{(r)}$ has the same strong period. Using again the example in Tables 4–1, we see that starting with $\mathbf{x}^{(3)}$ of strong period 2, every intermediate substring $\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)}$, and eventually $\mathbf{x}^{(0)}$ will have the same strong period 2.

Therefore, Lemmas 2–3 imply Lemma 1, the periodicity lemma for strings with one hole.

$\mathbf{x}^{(3)} =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
					a	b	$*$	b						

Table 4: $|x^{(3)}| = 4, q^{(3)} = 2, p^{(3)} = 2, q^{(3)} - p^{(3)} = 0$

3. Strings With Two Holes

Let $\mathbf{x} = \mathbf{x}[1..n]$ be a string with two holes that is weakly p, q periodic with $q > p$, where $n \geq 2(p+q) - d$, $d = \gcd(p, q)$. Let $L_0 = p+q-d, L_1 = p+q$, and observe that $L_1 > L_0 \geq q$. Consider the prefix $\mathbf{x}_1 = \mathbf{x}[1..L_0]$ of length L_0 and the suffix $\mathbf{x}_2 = \mathbf{x}[n-L_1+1..n]$ of length L_1 . Since there are only two holes, no matter where they lie at least one of \mathbf{x}_1 and \mathbf{x}_2 must, by the periodicity lemmas for no-hole and one-hole strings, be d periodic. Of course the same statement holds for $\mathbf{x}_1 = \mathbf{x}[1..L_1]$ and $\mathbf{x}_2 = \mathbf{x}[n-L_0+1..n]$.

Since part of \mathbf{x} is strongly d periodic, we are encouraged to investigate whether there is a way to extend the d periodic portion(s), perhaps to all of \mathbf{x} . The following definition provides one basis for such an extension:

Definition 4. Suppose that $\mathbf{x} = \mathbf{x}[1..n]$ is a string with at most two holes that is weakly p, q periodic, $q > p$. For $i \in L_0+1..n$, we say that $\mathbf{x}[1..i-1]$ is *right-extendible* (RE) if at least one of the following conditions holds:

1. $\mathbf{x}[i-p] \in \Sigma$;
2. $\mathbf{x}[i-q] \in \Sigma$;
3. $i+p \leq n$ and $\mathbf{x}[i+p-q] \in \Sigma$;

For example, in Table 5, x has weak periods $q = 6$ and $p = 4$. Since $d = \gcd(6, 4) = 2$, $L_0 = 6+4-2 = 8$ and $L_1 = 6+4 = 10$. There is no hole in $x[1..L_0]$, therefore according to the original periodicity lemma, $x[1..L_0]$ is (strongly) d periodic. Furthermore, according to Definition 4, for all $i \in 9..13$, $x[1..i]$ is right-extendible, while $x[1..14]$ is not right-extendible.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$x = a$	b	a	b	a	b	a	b	$*$	b	$*$	b	a	b	c	b	a	b

Table 5: Example: Right extendibility of a string with two holes

We remark that if neither condition (1) nor (2) in Definition 4 is satisfied, then both $i-p$ and $i-q$ are holes; since \mathbf{x} contains at most two holes, therefore for $i+p \leq n$, $\mathbf{x}[i+p] \in \Sigma$, and so condition (3) can fail to hold only in the case that $q = 2p$ — thus $i+p-q = i-p$. This is the “special” case described in [BSH02].

We shall see in the next section that for strings with an arbitrary number of holes, a weaker (and more general) definition of RE suffices. Based on the

RE property, the following lemma allows us to extend a d periodic prefix to the right:

Lemma 5. *Suppose that a string \mathbf{x} on Σ' with at most two holes is weakly p, q periodic, $q > p$, and let $d = \gcd(p, q)$. If $\mathbf{x}[1..i-1]$ is d periodic and RE, then $\mathbf{x}[1..i]$ is d periodic.*

Proof. We need only prove that for every $j \in 1..i$ such that $j \equiv i \pmod{d}$, $\mathbf{x}[j] \approx \mathbf{x}[i]$.

Suppose condition (1) of Definition 4 holds. By d periodicity, for every $j \in 1..i-1$ such that $j \equiv (i-p) \pmod{d}$, $\mathbf{x}[j] \approx \mathbf{x}[i-p]$. By weak p periodicity we know that $\mathbf{x}[i] \approx \mathbf{x}[i-p]$. Because $\mathbf{x}[i-p]$ is not a hole, it follows that for every $j \in 1..i$ such that $j \equiv i \equiv (i-p) \pmod{d}$, $\mathbf{x}[j] \approx \mathbf{x}[i]$, so that $\mathbf{x}[1..i]$ is d periodic.

The proof for condition (2) is analogous.

Suppose that neither condition (1) or condition (2) holds, but that (3) is true. By d periodicity, for every $j \in 1..i-1$ such that $j \equiv (i+p-q) \pmod{d}$, $\mathbf{x}[j] \approx \mathbf{x}[i+p-q]$. Since there are at most two holes, $\mathbf{x}[i+p] \in \Sigma$ and so $\mathbf{x}[i] = \mathbf{x}[i+p]$; by weak q periodicity, $\mathbf{x}[i+p] \approx \mathbf{x}[i+p-q]$; since moreover $\mathbf{x}[i+p-q] \in \Sigma$, in fact $\mathbf{x}[i] = \mathbf{x}[i+p-q]$. It follows that for every $j \in 1..i$ such that $j \equiv i \equiv (i+p-q) \pmod{d}$, $\mathbf{x}[j] \approx \mathbf{x}[i]$, so that again $\mathbf{x}[1..i]$ is d periodic. \square

A symmetrical definition and lemma enable us to extend a d periodic suffix to the left:

Definition 6. Suppose that $\mathbf{x} = \mathbf{x}[1..n]$ is a string with zero or more holes that is weakly p, q periodic, $q > p$. For $i \in 1..n-L_0$, we say that $\mathbf{x}[i+1..n]$ is **left-extendible** (LE) if at least one of the following conditions holds:

1. $\mathbf{x}[i+p] \in \Sigma$;
2. $\mathbf{x}[i+q] \in \Sigma$;
3. $i > p$ and $\mathbf{x}[i-p+q] \in \Sigma$;

Lemma 7. *Suppose that a string \mathbf{x} on Σ' with at most two holes is weakly p, q periodic, $q > p$, and let $d = \gcd(p, q)$. If $\mathbf{x}[i+1..n]$ is d periodic and LE, then $\mathbf{x}[i..n]$ is d periodic.* \square

We see that under specified conditions, we can extend a strongly d periodic prefix/suffix of \mathbf{x} by one to the right/left, respectively. If this process can be iterated to cover all of \mathbf{x} , then \mathbf{x} is d periodic. We summarize our results as

Lemma 8. *Suppose that $\mathbf{x} = \mathbf{x}[1..n]$ is a string with two holes and weak periods p and $q > p$, where $n \geq L_0+L_1$, $d = \gcd(p, q)$. Then:*

- (a) *At least one of $\mathbf{x}[1..L_0]$ and $\mathbf{x}[n-L_1+1..n]$ is d periodic.*
- (b) *If $\mathbf{x}[1..L_0]$ is d periodic and for every $i \in L_0+1..n$, $\mathbf{x}[1..i-1]$ is RE, then \mathbf{x} is d periodic.*

- (c) If $\mathbf{x}[n-L_1+1..n]$ is d periodic and for every $i \in 1..n-L_1$, $\mathbf{x}[i+1..n]$ is LE, then \mathbf{x} is d periodic. \square

As suggested earlier, this result can also be stated in terms of $\mathbf{x}[1..L_1]$ and $\mathbf{x}[n-L_0+1..n]$; note also that it applies to strings with any form of hole, not only don't-cares. Lemma 8 basically agrees with the result given in [BSH02], where d periodicity of \mathbf{x} is shown to depend on \mathbf{x} being “not $(2, p, q)$ -special”. However, the iterative approach given here is simpler and leads directly to a straightforward $\Theta(n)$ -time algorithm to compute the maximum-length d periodic suffix/prefix of $\mathbf{x}[1..n]$ with two holes.

To understand this better, again we consider the weakly 4, 6 periodic two-hole string of Table 5. By Lemma 5 the 2 periodic prefix $\mathbf{x}[1..8]$ can be iteratively extended to the right, yielding the conclusion that $\mathbf{x}[1..14]$ is 2 periodic. Since none of the conditions (1)-(4) of Definition 4 is satisfied in position 15, no further extension is possible. This makes sense since $\mathbf{x}[15] = c$, so that $\mathbf{x}[1..15]$ is not 2 periodic. Observe however that even if we transform \mathbf{x} into \mathbf{x}' by changing position 15 from c to a , $\mathbf{x}'[1..14]$ can still not be right-extended, because of the definition. Nevertheless \mathbf{x}' is in fact 2 periodic.

In order to resolve such situations, we state a more precise version of Lemma 8, as follows:

Corollary 9. *Suppose that $\mathbf{x} = \mathbf{x}[1..n]$ is a string with two holes h_1 and $h_2 > h_1$ and weak periods p and $q > p$, where $n \geq L_0 + L_1$, $d = \gcd(p, q)$.*

- (a) *If $h_2 - h_1 \neq q - p$, then \mathbf{x} is d -periodic.*
- (b) *If $h_2 - h_1 = q - p$, then*
 - (i) *$h_2 + p > n$ or $h_1 \leq p \Rightarrow \mathbf{x}$ is d periodic;*
 - (ii) *otherwise, $\mathbf{x}[h_2 + p] = \mathbf{x}[h_1 - p] \Leftrightarrow \mathbf{x}$ is d periodic.*

Proof.

- (a) If the gap between the holes is never $q - p$, then either condition (1) or condition (2) of both Definitions 4 and 6 will hold for every i . Thus one of Lemmas 5 and 7 can be used to extend the d periodic segment of \mathbf{x} to the full range $1..n$.
- (b) Suppose then that the gap between holes is exactly $q - p$. Even so, if $h_2 + p > n$ (respectively, $h_1 \leq p$), there can exist no i such that conditions (1)-(3) of Definition 4 (respectively, 6) all fail to hold. Again, the d periodic segment can be extended, either right or left, to the full range.

Suppose then that $h_2 + p \leq n$ and $h_1 > p$. Since $n \geq L_0 + L_1$, either $\mathbf{x}[1..h_2 + p - 1]$ or $\mathbf{x}[h_1 - p + 1..n]$ is d periodic. In both cases, to establish whether the d periodic range can be extended (to $\mathbf{x}[1..h_2 + p]$ or to $\mathbf{x}[h_1 - p..n]$), it suffices to perform the single comparison

$$\mathbf{x}[h_2 + p] : \mathbf{x}[h_1 - p],$$

where, since two holes are accounted for, both must be regular letters in Σ . If unequal, then the d periodic range cannot be extended; if equal, then since the remainder of the string contains no holes, the entire string is d periodic.

□

This result yields the following simple constant-time algorithm:

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function  $d$ -range( $\mathbf{x}, n, p, q, h_1, h_2$ )
if  $h_2 - h_1 \neq q - p$  or  $h_2 + p > n$  or  $h_1 \leq p$  then
    return  $1, n$ 
elseif  $\mathbf{x}[h_2 + p] = \mathbf{x}[h_1 - p]$  then
    return  $1, n$ 
elseif  $h_1 + h_2 > n$  then
    return  $1, h_2 + p - 1$ 
else
    return  $h_1 - p + 1, n$ 

```

Figure 1: For weakly p, q periodic $\mathbf{x}[1..n]$, $q > p$, $n \geq L_0 + L_1$, identify the maximum d periodic range that contains holes h_1 and $h_2 > h_1$.

Our methodology extends easily and naturally to three or more holes, as discussed in the next section.

4. Strings With Zero or More Holes

For a string \mathbf{x} with three holes and length $n \geq 2L_1$, again we consider a prefix $\mathbf{x}_1 = \mathbf{x}[1..L_1]$ and a suffix $\mathbf{x}_2 = \mathbf{x}[n - L_1 + 1..n]$: now both of them have length L_1 . Note that since there are only three holes, at least one of these substrings has no more than one hole. If at least two holes lie in \mathbf{x}_1 , so that at most one hole lies in \mathbf{x}_2 , then by Lemma 1 we know that \mathbf{x}_2 is d periodic; otherwise \mathbf{x}_1 is d periodic. In either case, at least a substring (prefix or suffix) of \mathbf{x} is d periodic. Figure 2 shows possible positions of these three holes, where in this case \mathbf{x}_1 is d periodic.

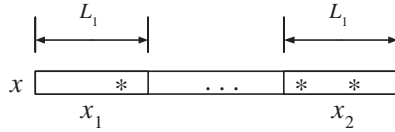


Figure 2: Possible positions of three holes

We can extend this result to any number of holes. For $d = \gcd(p, q)$, in addition to $L_0 = p + q - d$, $L_1 = p + q$, for $k \geq 2$ define $L_k = L_{k-2} + L_1$. Thus for odd k , $L_k = \lceil (k+1)/2 \rceil (p+q)$, while for even k , $L_k = L_{k+1} - d$. We claim that the following lemma holds:

Lemma 10. For a string \mathbf{x} with $k \geq 0$ holes, if \mathbf{x} is weakly p, q periodic and $|\mathbf{x}| \geq L_k$, then a substring of \mathbf{x} of length at least L_0 is d periodic, where $d = \gcd(p, q)$.

Proof. We prove this result by induction. For $k = 0$ and $k = 1$, the lemma holds by the periodicity lemmas for zero hole and one hole. If it holds for $k - 2$, then for a string \mathbf{x} with $|\mathbf{x}| \geq L_k$, we consider its prefix $\mathbf{x}_1 = x[1..L_{i-2}]$ and its suffix $\mathbf{x}_2 = x[n - L_1 + 1..n]$ of length L_1 . If the number of holes in \mathbf{x}_1 is less than or equal to $k - 2$, then by the inductive assumption \mathbf{x}_1 has a d periodic substring of length L_0 . Otherwise the number of holes in \mathbf{x}_1 is greater than $k - 2$, so that the number of holes in \mathbf{x}_2 is at most 1, implying by Lemma 1 that \mathbf{x}_2 is d periodic. \square

Note that unlike the 2-hole and 3-hole cases, in a string \mathbf{x} with more than three holes the substring of \mathbf{x} (let's call it \mathbf{x}_d) that may initially be d periodic is not necessarily a prefix or a suffix of \mathbf{x} . Therefore if \mathbf{x}_d can be extended both to the left and to the right until all of \mathbf{x} is covered, we may still claim that all of \mathbf{x} is d periodic. Observe that \mathbf{x}_d must itself contain a substring of length d without holes:

- * in the case that $|\mathbf{x}_d| = L_0$, \mathbf{x}_d contains no holes and $L_0 \geq 2d$;
- * if $|\mathbf{x}_d| = L_1$, \mathbf{x}_d contains at most one hole and $L_1 \geq 3d$.

Figure 3 demonstrates a possible position of \mathbf{x}_d and a substring of \mathbf{x}_d without holes.

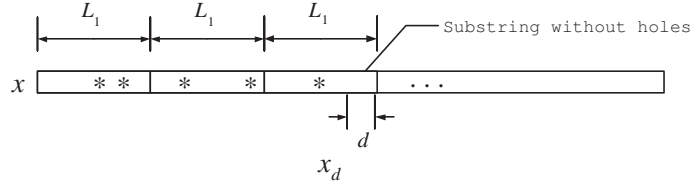


Figure 3: Possible position of \mathbf{x}_d

To accommodate three or more holes, we give a more general definition of RE and LE as follows:

Definition 11. Suppose a string \mathbf{x} with zero or more holes is weakly p, q periodic, $q > p$, with a substring $\mathbf{x}_d = \mathbf{x}[i..j]$, $j - i \geq p - 1$, that is d periodic, $d = \gcd(p, q)$.

- (a) \mathbf{x}_d is said to be RE iff $\mathbf{x}[j + 1] = \{\Sigma\}$ (hole) or there exists an integer sequence s_1, s_2, \dots, s_t , $t \geq 2$, such that
- * $s_1 = j + 1 \leq n$ and $s_t \in i..j$;
 - * for every $\ell \in 2..t$, $\mathbf{x}[s_\ell] \in \Sigma$ and $|s_\ell - s_{\ell-1}| = p$ or q .

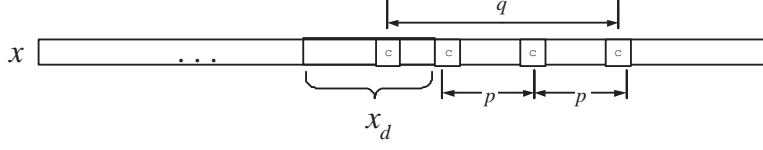


Figure 4: Example of RE and a path

(b) Symmetrically, \mathbf{x}_d is LE iff $\mathbf{x}[i-1] = \{\Sigma\}$ or there exists an integer sequence s_1, s_2, \dots, s_t , $t \geq 2$, such that

- * $s_1 = i - 1 \geq 1$ and $s_t \in i..j$;
- * for every $\ell \in 2..t$, $\mathbf{x}[s_\ell] \in \Sigma$ and $|s_\ell - s_{\ell-1}| = p$ or q .

Intuitively, this definition means that if we can find a path starting from $\mathbf{x}[j+1]$ that at each step identifies a next position p or q positions away and not a hole, terminating at a position that lies between i and j — then $\mathbf{x}[i..j]$ is RE (similarly for LE). Figure 4 illustrates an example of RE and such a path.

Note that Definitions 4 and 6 given in the previous section are special cases of this general definition.

Lemma 12. *Suppose that a string \mathbf{x} with zero or more holes is weakly p, q periodic, $q > p$, with $d = \gcd(p, q)$. If there exist i and $j \geq i + p - 1$ such that $\mathbf{x}[i..j]$ is d periodic and RE (respectively, LE), then $\mathbf{x}[i..j+1]$ (respectively, $\mathbf{x}[i-1..j]$) is d periodic.*

Proof. We prove the RE case only. If $\mathbf{x}[j+1] = \{\Sigma\}$ then certainly for every $\ell \in i..j$ such that $\ell \equiv (j+1) \pmod{d}$, $\mathbf{x}[\ell] \approx \mathbf{x}[j+1]$. Otherwise there exists a sequence s_1, s_2, \dots, s_t as described in Definition 11(a). We see that

$$\mathbf{x}[j+1] \approx \mathbf{x}[s_2] \approx \mathbf{x}[s_3] \approx \dots \approx \mathbf{x}[s_t],$$

and since every $\mathbf{x}[s_\ell] \in \Sigma$, $2 \leq \ell \leq t$, it follows that $\mathbf{x}[j+1] \approx \mathbf{x}[s_t]$. Since moreover $j+1 \equiv s_\ell \pmod{d}$ for every $\ell \in 2..t$, we conclude in particular that $j+1 \equiv s_t \pmod{d}$. Since $s_t \in i..j$ and $\mathbf{x}[i..j]$ is d periodic, therefore $\mathbf{x}[j+1] \approx \mathbf{x}[r]$ for every $r \in i..j$ such that $r \equiv (j+1) \pmod{d}$. Thus $\mathbf{x}[i..j+1]$ is d periodic, as required. \square

We now define functions **Right-Extend** and **Left-Extend** as follows:

Definition 13. Suppose that \mathbf{x} is weakly p, q periodic, $q > p$, with a d periodic substring $\mathbf{x}[i..j]$, where $d = \gcd(p, q)$ and $j - i \geq p - 1$. The function **Right-Extend** maps the pair (i, j) to $(i, j+1)$ if $\mathbf{x}[i..j]$ is RE and to (i, j) otherwise. The function **Left-Extend** maps the pair (i, j) to $(i-1, j)$ if $\mathbf{x}[i..j]$ is LE and to (i, j) otherwise.

Using these functions, we can state a general characterization of the left and right extensions that guarantee that \mathbf{x} is d periodic.

Lemma 14. *If \mathbf{x} with $k \geq 0$ holes has weak periods p and $q > p$, and $|\mathbf{x}| \geq L_k$, then at least a substring $\mathbf{x}[i..j]$ of length L_0 is d periodic, where $d = \gcd(p, q)$. If there exists a concatenation of functions $E = E_1 \circ E_2 \circ \dots \circ E_t$ where for every $\ell \in 1..t$, $E_\ell \in \{\mathbf{Right-Extend}, \mathbf{Left-Extend}\}$, and such that $E(i, j) = (1..n)$, then \mathbf{x} is d periodic. \square*

This is a statement of the periodicity lemma that applies to all strings with or without holes. However, as in the two-hole case (Corollary 9), we can be more precise: we now describe a straightforward algorithm that identifies a maximum-length d periodic substring of \mathbf{x} that contains a substring initially known to be d periodic. The algorithm uses a list of the k holes in \mathbf{x} and executes in $O(k)$ time.

Consider $\mathbf{x} = \mathbf{x}[1..n]$, $n \geq L_k$, with $k \geq 0$ holes. Suppose an array $H[1..k]$ gives the locations of all the holes in \mathbf{x} in ascending order. We add $H[0] = 0$ and $H[k+1] = n+1$. By Lemma 10 we may suppose that a $\Theta(k)$ scan of H has yielded a range $i..j$ in \mathbf{x} such that $\mathbf{x}[i..j]$ is d periodic, as well as a position s in H such that $H[s] < j$, $H[s+1] > j$, where in addition one of the following holds:

- * $j-i > L_0$ and $H[s] < i$;
- * $j-i > L_1$ and $H[s-1] < i$, $H[s] \in i..j$.

In either of these cases $\mathbf{x}[i..j]$ contains a substring $\mathbf{x}[\ell..\ell+d-1]$ such that for every $i' \in \ell..\ell+d-1$, $\mathbf{x}[i'] \in \Sigma$ (i' not a hole).

In addition to H , it is convenient also to compute a Boolean array $N[1..k]$ defined as follows: for every $s \in 1..k$, $N[s] = \mathbf{TRUE}$ if $\mathbf{x}[H[s]+q-p]$ is a hole, $N[s] = \mathbf{FALSE}$ otherwise. Figure 5 describes the preprocessing that computes N in $\Theta(k)$ time.

```

N[k] ← FALSE; r ← 2; δ ← q-p
for s ← 1 to k-1 do
  START ← H[s]
  while r ≤ k and H[r]-START < δ do
    r ← r+1
  if r > k or H[r]-START ≠ δ or H[r]+p > n then
    N[s] ← FALSE
  else
    N[s] ← TRUE; r ← r+1

```

Figure 5: Preprocessing: compute $N = N[1..k]$ in $\Theta(k)$ time from the array H of holes.

We are now in a position to describe an algorithm that extends a d periodic range $i..j$ in \mathbf{x} to the right by processing H and N from left to right, with minimal access to \mathbf{x} itself. The function **right-extend** shown in Figure 6 uses a current hole s to extend the current range: it returns $s+1$ and an extended right boundary j if further extension to the right may be possible; otherwise, it returns $s = k+1$ and the absolute rightmost boundary j of the

d periodic substring. It executes in constant time for each position s in H . (Note that here we assume the mathematical `mod` operation can be performed in constant time, since $(a \bmod b = a - \lfloor a/b \rfloor \cdot b)$; thus the complexity of `mod` is equivalent to that of division and multiplication.) A corresponding algorithm `left-extend` deals with left extension of range $i..j$. Overall, repeated execution of `right-extend` and `left-extend` will yield a maximum-length d periodic substring that contains the original d periodic range $i..j$, thus generalizing the algorithm described in Figure 1 for the two-hole case.

```

function right-extend( $H, N, s, k, \mathbf{x}, i, j, \ell, n, p, q, d$ )
if  $j - H[s] \geq q$  or not  $N[s]$  then
     $s \leftarrow s + 1$ ;  $j \leftarrow \max\{j, \min\{H[s] + q - 1, n\}\}$ 
else
     $j \leftarrow H[s] + q$ 
    if  $\mathbf{x}[j] \approx \mathbf{x}[\ell + (j - i) \bmod d]$  then
         $s \leftarrow s + 1$ 
    else
         $j \leftarrow j - 1$ ;  $s \leftarrow k + 1$ 
return  $j, s$ 

```

Figure 6: This function uses a single hole $H[s]$ to extend the d periodic range $i..j$ to the right.

We remark that a little further preprocessing may be done to form an array $\mathbf{z}[1..d] = \mathbf{x}[\ell..\ell + d - 1]$. Apart from H , N and \mathbf{z} , at most one reference to $\mathbf{x}[j]$ is then required in order to right-extend range $i..j$.

For a string \mathbf{x} with multiple holes and with weak periods $p = 4$ and $q = 6$, we illustrate the right extend process in Tables 6–8. Starting in Table 6, we first identify a periodic substring $\mathbf{x}[1..10]$ of length $p + q = 10$ with strong period $d = \gcd(4, 6) = 2$. As we already know, the existence of such a substring is guaranteed by Lemma 10. Let $[\ell..\ell + d - 1]$ be $\mathbf{x}[1..2]$. Since the position of the first hole $H[s] = 9$, we immediately know that $\mathbf{x}[1..9 + q - 1]$ is d periodic. Because every position in $\mathbf{x}[9 + 1..9 + q - 1]$ is RE according to Definition 11, Lemma 12 tells us that $\mathbf{x}[1..9 + q - 1]$ is d periodic. Since $N[s] = \text{TRUE}$ indicates that both $\mathbf{x}[15 - p]$ and $\mathbf{x}[15 - q]$ are holes, we have to compare $\mathbf{x}[15]$ with $\mathbf{x}[2]$. Since they match, we right-extend j from position 10 to 15.

Next we consider $H[s] = 11$ in Table 7. Since $N[s] = \text{FALSE}$, without any comparison we know that $\mathbf{x}[1..11 + q]$ is d periodic and therefore right-extend j to position 17.

Finally we consider $H[s] = 12$ in Table 8. Because $N[s] = \text{TRUE}$ and $\mathbf{x}[12 + q]$ does not match $\mathbf{x}[2]$, the algorithm correctly returns the maximum d periodic range $1..17$.

5. Summary and Future Work

The periodicity lemma is perhaps the fundamental result of stringology. In this paper we extend this result to strings with holes, an increasingly important

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathbf{x} =$	a	b	a	b	a	b	a	b	*	b	*	*	a	*	a	b	a	c	a	b
	i									j										

Table 6: $H[s] = 9, N[s] = TRUE, x[15] \approx x[1], j \leftarrow 15$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathbf{x} =$	a	b	a	b	a	b	a	b	*	b	*	*	a	*	a	b	a	c	a	b
	i														j					

Table 7: $H[s] = 11, N[s] = FALSE, j \leftarrow 17$

algorithmic topic. Throughout this paper we have used elementary and simple methods independent of number theory. In the case that the number of holes is arbitrary, we have taken a quite different approach than the graph-theoretical one of [BS04]. Our Lemma 14 is very general, covering indeterminate strings whose holes are not necessarily don't-cares; it leads to the algorithm that identifies maximum-length d periodic substrings of \mathbf{x} . We would like to extend other important results in stringology to strings with holes (indeterminate strings).

References

- [BB99] J. Berstel and L. Boasson. Partial words and a theorem of Fine and Wilf. *Theoret. Comput. Sci.*, 218:135–141, 1999.
- [BS04] F. Blanchet-Sadri. Periodicity on partial words. *Computers and Mathematics with Applications*, 47:71–82, 2004.
- [BSH02] F. Blanchet-Sadri and Robert A. Hegstrom. Partial words and a theorem of fine and wilf revisited. *Theor. Comput. Sci.*, 270(1-2):401–419, 2002.
- [FP74] M. J. Fischer and M. S. Paterson. String matching and other products. In R.M.Karp, editor, *SIAM-AMS Proceedings*, number 7 in Complexity of Computation, pages 113–125, 1974.
- [FW65] N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. In *Proc. Amer. Math. Soc.*, volume 16, pages 109–114, 1965.
- [HS03] Jan Holub and W. F. Smyth. Algorithms on indeterminate strings. In Mirka Miller and Kunsoo Park, editors, *Proc. 14th Australasian Workshop on Combinatorial Algorithms*, pages 36–45, 2003.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathbf{x} =$	a	b	a	b	a	b	a	b	$*$	b	$*$	$*$	a	$*$	a	b	a	c	a	b
	i																			j

Table 8: $H[s] = 12$ $N[s] = TRUE$, $not(x[18] \approx x[2])$, **return**

- [HSW06] Jan Holub, W. F. Smyth, and Shu Wang. Hybrid pattern-matching algorithms on indeterminate strings. *London Algorithmics and Stringology 2006*, J. Daykin, M. Mohamed and K. Steinhofel (eds.), King's College London Series Texts in Algorithmics, pages 115–133, 2006.
- [HSW08] Jan Holub, W. F. Smyth, and Shu Wang. Fast pattern-matching on indeterminate strings. *J. Discrete Algorithms*, 6(1):37–50, 2008.
- [IMM⁺03] C.Š. Iliopoulos, Manal Mohamed, Laurent Mouchard, Katerina G. Perdikuri, W.F. Smyth, and Athanasios K. Tsakalidis. String regularities with don't cares. *Nordic J. Computing*, 10(1):40–51, 2003.
- [Smy03] Bill Smyth. *Computing Patterns in Strings*. Pearson Addison Wesley, 2003.