GRAPHS WITH SMALL GENERALIZED CHROMATIC NUMBER

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ABSTRACT

Let $G = (V, E)$ denote a finite simple undirected connected graph of order $n = |V|$ and diameter D. For any integer $k \in [1, D]$, a *proper k-colouring* of G is a labelling of the vertices V such that no two distinct vertices at distance k or less have the same label. We let $\gamma_k(G)$, the *k-chromatic number* of G, denote the least number of labels required to achieve a proper k-colouring of G. In this paper we show that there exists an infinite class $\mathcal{G}^*$ of graphs of order n and diameter $D \geq 3$ such that, over all graphs $G \in \mathcal{G}^*$, $\gamma_{D-1}(G) \in \Theta\left(\sqrt{Dn}\right)$. Constructions are specified for graphs in the class $\mathcal{G}^*$.

1 INTRODUCTION

Let $G = (V, E)$ denote a finite simple undirected connected graph of order $n = |V|$ and diameter $D > 0$. For any integer $k \in [1, D]$, a *proper k-colouring* of G is a labelling of the vertices V such that no two distinct vertices at distance k or less have the same label. For given G, we let $\gamma_k(G)$, the *k-chromatic number* of G, denote the least number of labels required to achieve a proper k-colouring of G.

Problems associated with the estimation of the k-chromatic number were surveyed some years ago by Gionfriddo [G87]. A common approach has been to define the *k-density* $\rho_k(G)$ to be the maximum order over all subgraphs of G with diameter k. (For example, if $k = 1$ and G is triangle-free, then $\rho_1(G) = 2$.) The nonnegative quantity

$$\gamma'_k(G) = \gamma_k(G) - \rho_k(G)$$

is then considered; in particular, attention focusses on small values of k and small values of $\gamma'_k(G)$ — that is, cases in which the k-density is close to the k-chromatic number. Now let $v_k(h)$ denote the least integer such that there exists a graph G of order $v_k(h)$ for which $\gamma'_k(G) = h$. (For example, if $k = 1$ and $h = 0$, then $v_1(0) = 2$ corresponding to $G = P_2$, the path of length 1.) Some progress has been made establishing bounds on $v_k(h)$ in more general cases [GV85], but exact calculation seems to be very difficult, even for $k = 2$ and small values of h.

In this paper, a different approach is adopted: bounds for $\gamma_k(G)$ are related to the diameter of G. It turns out that, for $k = D - 1$, it is possible to determine quite sharp bounds which correspond to interesting constructions.

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Thus, for each choice $D = 1, 2, \dots.$ and for every integer $n > D$, let $\mathcal{G}_{D,n}$ denote the class of all graphs G of diameter D and order n. Then, over all graphs $G \in \mathcal{G}_{D,n}$, let $\alpha_{D,n,k}$ (respectively, $\beta_{D,n,k}$) denote the minimum (respectively, maximum) value attained by $\gamma_k(G)$. As Theorem 1.1 shows, the upper bound is easy to determine:

Theorem 1.1 $\beta_{D,n,k} = n - D + k$.

Proof We look for the greatest number of labels that can possibly be used in a minimum proper k-colouring of G. Observe first that there exists a shortest path P_{D+1} in G of length D. Then a minimum proper k-colouring of P_{D+1} requires $k+1$ distinct labels. If the remaining $n - D - 1$ vertices of G must all be given labels distinct from the $k+1$ labels used in P_{D+1}, then a total of $n - D + k$ labels will be used to colour G. Thus this total is the maximum possible value that $\gamma_k(G)$ could take. But this value is actually attained for the graph G formed by joining one end vertex of a path P_D of length $D-1$ to every vertex of a complete graph K_{n-D}: the resulting graph G then has n vertices, diameter D, and k-chromatic number $n - D + k$. $\square$

As a special case of this result, we see that $\beta_{D,n,D} = n$; since also $\alpha_{D,n,D} = n$, it follows that

$$\gamma_D(G) = n$$

for any given graph G. At the other extreme, for $k = 1$, Theorem 1.1 tells us that $\beta_{D,n,1} = n - D + 1$. To determine $\alpha_{D,n,1}$, consider the graph G formed by joining $n - D$ isolated vertices $\overline{K}_{n-D}$ to one end of a path P_D of length $D-1$. The resulting graph G has n vertices, diameter D, and 1-chromatic number 2. (Indeed, G may be any tree on n vertices with diameter D.) Since for any $k \geq 1$, $\alpha_{D,n,k} > 1$, it follows that $\alpha_{D,n,1} = 2$, so that, for any given graph G,

$$2 \leq \gamma_1(G) \leq n - D + 1.$$

For $1 < k < D$, the value of $\alpha_{D,n,k}$ is more difficult to determine. In the remainder of this article, we consider the case $k = D-1$; to simplify notation, we write $\alpha_{D-1} \equiv \alpha_{D,n,D-1}$.

For the estimation of α_{D-1}, the first interesting case that arises is $D = 3$, which we now begin to consider. It follows from Theorem 5 of [BRZ68] that if $D \geq 3$ for a graph G, then the diameter of the complement graph $\overline{G}$ is $\overline{D} \leq 3$. (This result was later rediscovered in [HR85].) Recall now the result of Bloom, Kennedy and Quintas [BKQ87] that G has diameter 2 if and only if $\overline{G}$ is not empty and $\overline{G}$ is not spanned by a double star. (A *spanning double star* of G is a spanning tree of G which consists of two stars with centres u and v joined by the edge uv.) Thus for $D = 3$ there are exactly two possibilities:

(1) $\overline{D} = 3$, in which case G is spanned by a double star;
(2) $\overline{D} = 2$, in which case G is NOT spanned by a double star.

In this section we deal with the first and more straightforward of these possibilities:

Theorem 1.2 Let $\mathcal{G}'_{3,n}$ denote the set of all graphs $G \in \mathcal{G}_{3,n}$ with complements $\overline{G} \in \mathcal{G}_{3,n}$. Denote by α'_2 the restriction of α_2 to $\mathcal{G}'_{3,n}$. Then

$$\alpha'_2 = \lceil n/2 \rceil + 1.$$

Proof Since G is spanned by a double star, we may divide the vertices V into three non-empty sets U, S and T: U consists of the centres of the two spanning stars, S consists of the radial vertices of one star, T the radial vertices of the other. Without loss of generality, suppose that $|S| \leq |T|$. Now consider a proper 2-colouring of G. We see that all labels in $S \cup U$ must be distinct, as so also must all labels in $T \cup U$. Then, for given n, a minimum number of labels will be used if and only if the following two conditions are satisfied:

(1) every label used in S is also used in T;

(2) $|T| - |S| \leq 1$.

When n is even, so that $|T| = |S|$, we have $n = 2|S| + 2$ and $\gamma_2(G) = n/2 + 1$. For odd n, we have similarly $|T| = |S| + 1$, $n = 2|S| + 3$, and $\gamma_2(G) = (n+1)/2 + 1$. Thus a double star which satisfies condition (2) can be labelled to yield a minimum 2-colouring using $\lceil n/2 \rceil + 1$ labels. From this fact the result follows. □

As we shall see below, the lower bound of Theorem 1.2 is exceptional. It appears that for other classes of graphs, $\alpha_{D-1} \in O(\sqrt{Dn})$. We begin our consideration of these graphs in Section 2, by investigating the graphs of diameter 3 whose complements have diameter 2. In Section 3 we look in detail at constructions for graphs of even diameter $D \geq 4$, and then in Section 4 we indicate how these constructions may be extended to graphs of odd diameter $D \geq 5$.

2 DIAMETER $D = 3$

In this section we study graphs $G \in \mathcal{G}''_{3,n}$, where $\mathcal{G}''_{3,n}$ denotes the set of all graphs of $\mathcal{G}_{3,n}$ whose complements are in $\mathcal{G}_{2,n}$. In particular, we display an infinite subclass $\mathcal{G}^*_{3,n} \subset \mathcal{G}''_{3,n}$ of graphs G whose 2-chromatic number

$$\gamma_2(G) = \sqrt{2n+4} - 1.$$

The graphs in this subclass are constructed based on an integer parameter $\nu = 3, 4, \ldots,$, and so we denote them G_ν. We shall see that G_ν has order $n = 2(\nu^2 - 1)$, size $m = \nu(\nu^2 - 1)$, and of course diameter $D = 3$.

To construct the G_ν, we begin with two copies of the complete graph $K_{\nu-1}$, which we call $K^{(1)}_{\nu-1}$ and $K^{(2)}_{\nu-1}$. Corresponding to each $i = 1, 2$, we introduce $\nu - 1$ disjoint sets of ν isolated vertices $\overline{K}_\nu$, which we call $\overline{K}^{(i,1)}_\nu, \overline{K}^{(i,2)}_\nu, \ldots, \overline{K}^{(i,\nu-1)}_\nu$. Let $U^{(i)} = \{u^{(i)}_1, u^{(i)}_2, \ldots, u^{(i)}_{\nu-1}\}$ denote the vertices of $K^{(i)}_{\nu-1}$, and for each $j = 1, 2, \ldots, \nu-1$, let $V^{(i,j)} = \{v^{(i,j)}_1, v^{(i,j)}_2, \ldots, v^{(i,j)}_\nu\}$ denote the vertices of $\overline{K}^{(i,j)}_\nu$. We now introduce two sets of edges (beyond those already found in the $K^{(i)}_{\nu-1}$):

(1) For every $i = 1, 2$; $j = 1, 2, \ldots, \nu - 1$; $h = 1, 2, \ldots, \nu$: $u^{(i)}_j v^{(i,j)}_h$ is an edge. (This joins each vertex of $K^{(i)}_{\nu-1}$ to each of ν vertices in one of the sets $V^{(i,j)}$.)

(2) For every $j = 1, 2, \ldots, \nu - 1$; $j' = 1, 2, \ldots, \nu - 1$; $h = 1, 2, \ldots, \nu$: $v^{(1,j)}_h v^{(2,j')}_{h'}$ is an edge if and only if h' is computed by the following algorithm:

$$\begin{aligned}
&h' \longleftarrow j + j' + h - 1;\\
&\textbf{if } h' > \nu \textbf{ then}\\
&\quad \{h' \longleftarrow h' - \nu;\\
&\quad \textbf{if } j + j' - 1 = \nu \textbf{ then}\\
&\qquad h' \longleftarrow h' + 1\}.
\end{aligned}$$

(This joins each vertex of $V^{(1,j)}$ to a single vertex in each of $V^{(2,j')}$, $j' = 1, 2, \ldots, \nu - 1$, in such a way that $v_h^{(1,j)} v_h^{(2,j')}$ is never an edge for any choice of j, j' and h.)

To the vertices of G_ν we now assign labels L as follows: for every $i = 1, 2$ and for every $j = 1, 2, \ldots, \nu - 1$ set

$$L(u_j^{(i)}) \longleftarrow j; \text{ and}$$

$$\forall h = 1, 2, \ldots, \nu : \quad L(v_h^{(i,j)}) \longleftarrow \nu + h - 1.$$

Thus a total of $2\nu - 1$ distinct labels are assigned; we claim that $\gamma_2(G_\nu) = 2\nu - 1$.

To prove this claim, we first show that in fact G_ν has diameter $D = 3$. Consider first the edges $v_h^{(1,j)} v_{h'}^{(2,j')}$ and observe that, for fixed j and j', as h assumes the values $1, 2, \ldots, \nu$, h' assumes distinct values in the cyclic permutation of $1, 2, \ldots, \nu$ which begins at one of $\{j + j', j + j' - \nu, j + j' - \nu + 1\}$; thus every vertex of $K_\nu^{(1,j)}$ is adjacent to a single distinct vertex of $K_\nu^{(2,j')}$. Clearly this statement is true also when the superscripts 1 and 2 are interchanged. It follows then that every vertex of $K_{\nu-1}^{(i)}$ is distance exactly 2 from every vertex of $\overline{K}_\nu^{(3-i,j)}$ for every $j = 1, 2, \ldots, \nu - 1$, and hence distance exactly 3 from every vertex of $K_{\nu-1}^{(3-i)}$. Furthermore, we see that whenever $v_h^{(i,j)} v_{h'}^{(3-i,j')}$ is an edge, then $v_h^{(i,j)}$ is distance exactly 3 from every vertex $v_{h''}^{(3-i,j')}$, where $h'' \in [1, \nu]$ and $h'' \neq h'$. Finally, observe that $v_h^{(i,j)}$ is distance exactly 2 from every vertex $v_{h'}^{(i,j)}$, $h' \neq h$, and distance exactly 3 from every vertex $v_{h'}^{(i,j')}$, $j' \neq j$. We conclude that $D = 3$ and note that, moreover, every vertex of G_ν is peripheral. Note also that G_ν is not spanned by a double star.

To show that the labelling specified yields a proper 2-colouring, it is necessary to show that all vertices with the same label are distance 3 from each other. This is clearly true for the vertices of $K_{\nu-1}^{(i)}$. To prove this result for the labels $\nu, \nu + 1, \ldots, 2\nu - 1$, it suffices to show that

* $v_h^{(i,j)} v_h^{(3-i,j')}$ is never an edge;
* $v_h^{(i,j)}$ is always adjacent to $\nu - 1$ vertices with distinct labels.

The first of these propositions is a direct consequence of the method of calculation of h' specified in (2) above: if $h' = h$ then either $j + j' - 1 = 0$, an impossibility, or $j + j' - 1 = \nu$, in which case the value of h' is incremented by one, so as to be no longer equal to h. To prove the second proposition, consider first the edges joining a vertex $v_h^{(1,j)}$ to $V^{(2,1)}$, for every $h = 1, 2, \ldots, \nu$. These edges will be $v_h^{(1,j)} v_{h_1}^{(2,1)}$, where h_1 is one element of the set $H_0 = \{j + h, j + h - \nu, j + h - \nu + 1\}$. More generally, the vertices $v_{h_1}^{(2,1)}, v_{h_2}^{(2,2)}, \ldots, v_{h_{\nu-1}}^{(2,\nu-1)}$ of $V^{(2,1)}, V^{(2,2)}, \ldots, V^{(2,\nu-1)}$, respectively, which are adjacent to $v_h^{(1,j)}$ are identified by the cyclic permutation $(h_1, h_2, \ldots, h_{\nu-1})$ of $1, 2, \ldots, \nu$ which begins at the element of H_0 specified by (2) and omits h. These vertices will have $\nu - 1$ distinct labels $\nu + h_1 - 1, \nu + h_2 - 1, \ldots, \nu + h_{\nu-1} - 1$. A similar argument establishes also that vertices $v_{h_1}^{(1,1)}, v_{h_2}^{(1,2)}, \ldots, v_{h_{\nu-1}}^{(1,\nu-1)}$ adjacent to a given vertex $v_h^{(2,j)}$ all have distinct labels. We conclude that $\gamma_2(G_\nu) = 2\nu - 1$, as required. We state this result formally as follows:

Theorem 2.1 For every integer $\nu \geq 3$, the graphs G_ν of order $n = 2(\nu^2 - 1)$ and diameter 3 have 2-chromatic number $\sqrt{2n+4} - 1$. Thus

$$\alpha_2 \in O(\sqrt{n}). \quad \square$$

Finally, we remark that for $\nu = 3$ a slight improvement can be made to the above construction, yielding $n = 2\nu^2 = 18$ and $\gamma_2(G_3) = \sqrt{2n} - 1 = 5$.

3 EVEN DIAMETER $D \geq 4$

In this section we first present a construction for graphs $G \in \mathcal{G}_{4,n}$ for which the (D-1)-chromatic number

$$\gamma_{D-1}(G) = \left\lceil 2\sqrt{n-1} \right\rceil.$$

We then show how to generalize this construction to graphs $G \in \mathcal{G}_{2d,n}$ of arbitrary even diameter $D = 2d$, where $d = 2, 3, \ldots$.

We note in passing the result of Bosák, Rosa and Znám [BRZ68] (later rediscovered in [S86]) that for graphs G of diameter $D \geq 4$, the complement graph $\overline{G}$ must have diameter $\overline{D} = 2$. Thus the special case dealt with in Theorem 1.2 does not arise for $D \geq 4$.

Consider a graph $G = (V, E) \in \mathcal{G}_{4,n}$. There exists a shortest path $s_1 u_1 u u_2 s_2$ in G joining peripheral vertices s_1 and s_2. Let $U = \{u, u_1, u_2\}$, and for $j = 1, 2$ let S_j denote the set of all vertices $v^{(j)} \neq u$ such that $u_j v^{(j)} \in E$. Then, in particular, $s_j \in S_j$. Observe that a proper $(D-1)$-colouring of G requires that all the labels assigned to $U \cup S_j$ be distinct; that is,

$$\gamma_3(G) \geq \max_{j=1,2} |S_j| + 3.$$

The lower bound can of course be attained by a graph G whose *every* shortest path from S_1 to S_2 is of length D.

Suppose now more generally that V contains $p \geq 2$ distinct subsets $S_1, S_2, \ldots, S_p$ such that every shortest path from one subset to another

* is of length D;

* passes through u.

Then the graph G of least order n satisfying these conditions has vertex set $V = U \cup \left(\cup_{j=1}^{p} S_j\right)$, where $U = \{u, u_1, u_2, \ldots, u_p\}$ and

$$\gamma_3(G) = \max_{1 \leq j \leq p} |S_j| + (p+1).$$

Let s^* denote the average order of the S_j; that is, $s^* = (|S_1| + |S_2| + \cdots + |S_p|)/p$. Then

$$n = ps^* + p + 1. \qquad \ldots(3.1)$$

Observe now that γ_3 can be minimized by imposing the further condition on G that

$$\lfloor s^* \rfloor \le |S_j| \le \lfloor s^* \rfloor + 1, \qquad \ldots(3.2)$$

for every $j = 1, 2, \ldots, p$. Then, using (3.1),

$$\begin{aligned} \gamma_3(G) &= \lceil s^* \rceil + p + 1 \\ &= \lceil (n-1)/p \rceil + p, \end{aligned}$$

an expression which achieves its minimum value for

$$p = \left\lceil \sqrt{n-1} \right\rceil. \qquad \ldots(3.3)$$

Thus, if the subsets of V are chosen to satisfy (3.2) and (3.3), then

$$\gamma_3(G) = \left\lceil (n-1) \Big/ \left\lceil \sqrt{n-1} \right\rceil \right\rceil + \left\lceil \sqrt{n-1} \right\rceil. \qquad \ldots(3.4)$$

We now state three noteworthy identities (left as exercises for the reader). For any positive real number x, let

$$C(x) \equiv \left\lceil x \Big/ \left\lceil \sqrt{x} \right\rceil \right\rceil + \left\lceil \sqrt{x} \right\rceil.$$

For $x \ge 1$, let

$$F(x) \equiv \left\lceil x \Big/ \left\lfloor \sqrt{x} \right\rfloor \right\rceil + \left\lfloor \sqrt{x} \right\rfloor.$$

Then

$$C(x) = \left\lceil 2\sqrt{\lceil x \rceil} \right\rceil; \qquad \ldots(3.5a)$$

$$F(x) = \left\lceil 2\sqrt{\lceil x \rceil} \right\rceil, \qquad \ldots(3.5b)$$

if there exists no integer N such that $N^2 - 1 < x < N^2$; and

$$F(x) = \left\lceil 2\sqrt{\lceil x \rceil} \right\rceil + 1, \qquad \ldots(3.5c)$$

otherwise. Thus, in view of (3.5a), (3.4) becomes

$$\gamma_3(G) = \left\lceil 2\sqrt{n-1} \right\rceil. \qquad \ldots(3.6)$$

A graph G which satisfies (3.6) may be characterized as a graph of diameter $D = 4$, radius $d = 2$, a single centre u, and $n - \left\lceil \sqrt{n-1} \right\rceil - 1$ peripheral nodes divided as equally as possible into $\left\lceil\sqrt{n-1}\right\rceil$ mutually peripheral subsets. An example of such a graph is formed by $\left\lceil\sqrt{n-1}\right\rceil$ complete graphs K_s, where $\lceil s^*\rceil+1 \leq s \leq \lceil s^*\rceil+2$, each with a single vertex adjacent to u.

Consider now a graph $G \in \mathcal{G}_{D,n}$, where $D = 2d$ for some integer $d \geq 2$, and suppose as before that the vertex set V contains $p \geq 2$ distinct subsets $S_1, S_2, \ldots, S_p$ as defined above. Defining s^* as before, we find

$$n = ps^* + p(d-1) + 1, \qquad \ldots(3.7)$$

analogous to (3.1). Then, applying (3.2) and (3.7), we find that

$$\begin{aligned}\gamma_{D-1}(G) &= \lceil s^*\rceil + p(d-1) + 1 \\ &= \lceil (n-1)/p\rceil + (d-1)(p-1) + 1.\end{aligned} \qquad \ldots(3.8)$$

Now consider the function $\gamma^*(p) = (n-1)/p + (d-1)(p-1) + 1$, differentiable in any interval not containing $p = 0$. This function attains its minimum for

$$\frac{d\gamma^*}{dp} = -(n-1)/p^2 + (d-1) = 0;$$

that is, for $p = \sqrt{\frac{n-1}{d-1}}$. Then (3.8) is minimized by choosing either $p = p_1 \equiv \left\lceil\sqrt{\frac{n-1}{d-1}}\right\rceil$ or $p = p_2 \equiv \left\lfloor\sqrt{\frac{n-1}{d-1}}\right\rfloor$. To estimate the minimum value of (3.8), consider first

$$\gamma^*(p_1) = (d-1)\left\{\left(\frac{n-1}{d-1}\right)\Big/\left\lceil\sqrt{\frac{n-1}{d-1}}\right\rceil + \left\lceil\sqrt{\frac{n-1}{d-1}}\right\rceil - 1\right\} + 1.$$

We see that

$$(d-1)\left(C\left(\frac{n-1}{d-1}\right) - 2\right) \leq \gamma^*(p_1) \leq (d-1)\left(C\left(\frac{n-1}{d-1}\right) - 1\right),$$

where $C\left(\frac{n-1}{d-1}\right)$ is given by (3.5a). From (3.5b) and (3.5c), it follows then that

$$\gamma^*(p_1) \leq \gamma^*(p_2) \leq \gamma^*(p_1) + 1;$$

and since, for every positive value of p,

$$0 \leq \gamma_{D-1}(G) - \gamma^*(p) \leq 1,$$

we have the main result of this section:

Theorem 3.1 For every even integer $D = 2d$, $d \geq 2$, and for every integer $n > D$, there exists a graph $G \in \mathcal{G}_{D,n}$ such that

$$(d-1)\left(\left\lceil 2\sqrt{\left\lceil\frac{n-1}{d-1}\right\rceil}\right\rceil - 2\right) \leq \gamma_{D-1}(G) - 1 \leq (d-1)\left(\left\lceil 2\sqrt{\left\lceil\frac{n-1}{d-1}\right\rceil}\right\rceil - 1\right).$$

Hence $\alpha_{D-1} \in O(\sqrt{Dn})$. $\square$

As in the special case $d = 2$, a graph G which satisfies Theorem 3.1 is characterized by radius d, a single centre u, and $n - (d-1)p_1 - 1$ peripheral nodes divided as equally as possible into p_1 mutually peripheral subsets, where $p_1 = \left\lceil \sqrt{\frac{n-1}{d-1}} \right\rceil$.

4 ODD DIAMETER $D \geq 5$

Here we consider graphs $G \in \mathcal{G}_{2d+1,n}$ of odd diameter $D = 2d+1$, $d \geq 2$. The construction is similar to the construction for even diameter $2d$; the main difference is that the centre u is replaced by a complete subgraph K_p, which thus increases the diameter by one. The results apply also to the case $D = 3$ ($d = 1$).

As before, we suppose that V contains $p \geq 2$ mutually peripheral subsets $S_1, S_2, \ldots, S_p$ such that every shortest path from one subset to another passes through at least two vertices of the "central" complete subgraph K_p. If as before s^* denotes the average size of $|S_j|$, $j = 1, 2, \ldots, p$, then, analogous to (3.1),

$$n = p(s^* + d). \qquad \ldots(4.1)$$

By choosing the orders of the S_j to be as nearly equal as possible, we can make

$$\begin{aligned} \gamma_{D-1}(G) &= s^* + pd \\ &= n/p + (p-1)d, \end{aligned} \qquad \ldots(4.2)$$

from (4.1). We find then that $\gamma_{D-1}(G)$ is minimized for $p = \left\lfloor \sqrt{n/d} \right\rfloor$ or $\left\lceil \sqrt{n/d} \right\rceil$. Substituting these values into (4.2) and applying (3.5a)-(3.5c), we find

Theorem 4.1 For every odd integer $D = 2d+1$, $d \geq 1$, and for every integer $n > D$, there exists a graph $G \in \mathcal{G}_{D,n}$ such that

$$d\Big(\Big\lceil 2\sqrt{\lceil n/d \rceil}\Big\rceil - 2\Big) \leq \gamma_{D-1}(G) \leq d\Big(\Big\lceil 2\sqrt{\lceil n/d \rceil}\Big\rceil - 1\Big).$$

Hence $\alpha_{D-1} \in O(\sqrt{Dn})$. $\square$

Note that in the special case $D = 3$, the construction of Section 2 yields a lower $(D-1)$-chromatic number than the construction given here. More generally, it is not known whether the constructions given in this paper are best possible, in the sense of yielding, for given D and n, the least possible value of γ_{D-1}.

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