The Sum Number of the Cocktail Party Graph

Mirka MILLER
Department of Computer Science
The University of Newcastle, NSW 2308, Australia

e-mail: mirka@cs.newcastle.edu.au

Joseph F. RYAN
Department of Mathematics
The University of Newcastle, NSW 2308, Australia

e-mail: joe@frey.newcastle.edu.au

W. F. SMYTH
Department of Computer Science and Systems
McMaster University, Hamilton, Ontario, Canada

School of Computing
Curtin University, Perth WA 6001, Australia

e-mail: smyth@mcmaster.ca

Abstract

A graph G is called a *sum graph* if there exists a labelling of the vertices of G by distinct positive integers such that the vertices labelled u and v are adjacent if and only if there exists a vertex labelled $u+v$. If G is not a sum graph, adding a finite number of isolated vertices to it will always yield a sum graph, and the *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices that will achieve this result. A labelling that realizes $G+\overline{K}_{\sigma(G)}$ as a sum graph is said to be *optimal.* In this paper we consider $G=H_{m,n}$, the complete n-partite graph on $n\geq 2$ sets of $m\geq 2$ nonadjacent vertices. We give an optimal labelling to show that $\sigma(H_{2,n})=4n-5$, and in the general case we give constructive proofs that $\sigma(H_{m,n})\in\Omega(mn)$ and $\sigma(H_{m,n})\in O(mn^2)$. We conjecture that $\sigma(H_{m,n})$ is asymptotically greater than mn, the cardinality of the vertex set; if so, then $H_{m,n}$ is the first known graph with this property. We also provide for the first time an optimal labelling of the complete bipartite graph $K_{m,n}$ whose smallest label is 1.

1 Introduction

Since the introduction of sum graphs by Harary [4], a number of easily-stated but tantalizingly difficult problems have emerged:

1. Do there exist graphs $G = (V, E)$ such that $\sigma(G) \in \Theta(|V|^2)$? This question was answered in the affirmative by Gould & Rödl [3], but their methods provide no means of constructing such graphs. In fact, the only known class of graphs G that even achieves as much as $\sigma(G) \in \Theta(|E|)$ is the class of wheels W_n with n spokes: as shown in [6] and [7],

$$\begin{aligned} \sigma(W_n) &= n/2 + 2 \text{ for } n \text{ even;} \\ &= n \text{ for } n \text{ odd.} \end{aligned}$$

   With the possible exception of the graphs considered in this paper, no graphs are known whose sum number exceeds $|V|$ in an asymptotic sense.

2. Efforts to find graphs of large sum number have of course led to a consideration of graphs with many edges; for example, complete graphs K_n and complete bipartite graphs $K_{m,n}$. But for these graphs it turns out that $\sigma(G) \in \Theta(|V|)$: Bergstrand *et al.* showed [1] that for $n \geq 4$, $\sigma(K_n) = 2n - 3$, while Hartsfield and Smyth showed [5] that for $n \geq m \geq 2$, $\sigma(K_{m,n}) = \lceil (3m + n - 3)/2 \rceil$.

3. Attention has also been directed toward graphs of small sum number, in particular toward *unit graphs*: graphs G such that $\sigma(G) = 1$. Ellingham showed [2] that for every forest F, $\sigma(F) = 1$, and Smyth showed [9] that for all integers $n \geq 2$ and $m \in n-1..\lfloor n^2/4 \rfloor$, there exists a unit graph with n vertices and m edges. Further efforts to characterize unit graphs have so far been unsuccessful.

4. In a labelling of a sum graph, vertices whose label corresponds to an edge (u, v) are said to be *working* vertices. It has been realized that certain graphs can only be labelled in such a way that all the working vertices are also isolates; such graphs are called *exclusive*. Exclusive graphs are of interest for two reasons: they may be easier to label optimally, and they may be more likely to have a large sum number. It turns out that K_n and W_n are exclusive, while F and $K_{m,n}$ are not. We show in this paper that $H_{m,n}$ also is exclusive, but the characterization of exclusive graphs remains an open problem.

5. Optimal labellings that include 1 as the least label are called *minimal* labellings. It is not known whether or not every graph G has at least one minimal labelling. The labellings presented in this paper, for $H_{m,n}$ and $K_{m,n}$, are either minimal or have equivalent minimal labellings.

6. The number of different edges (u, v) to which a vertex label corresponds is called the *multiplicity* [9] of the vertex. Thus nonworking vertices have zero multiplicity. For certain graphs — notably K_n and $K_{m,n}$ — the multiplicity pattern of the vertices is the same for every optimal labelling. Thus finding a means of specifying the multiplicity pattern for these kinds of graphs would eliminate the need to exhibit a specific labelling. No general results for identifying multiplicity patterns have so far been discovered, though the concept is useful in this paper, as in others, for establishing other properties of sum graphs.

In his original paper [4], Harary mentions the curious case of the cycle C_n, $n \geq 3$: for $n \neq 4$, $\sigma(C_n) = 2$, while $\sigma(C_4) = 3$. In a recent paper [8], Sharary finds C_4 also to be an anomaly for "integral" sum graphs. In this paper, we describe a minimal labelling for $H_{2,n}$ (sometimes called the "cocktail party graph") that achieves $\sigma(H_{2,n}) = 4n-5$, $n \geq 2$. Since $H_{2,2}$ is isomorphic to C_4 and $\sigma(H_{2,2}) = 3$, it appears that as a component of a sum graph, C_4 is more naturally regarded as a cocktail party graph than as a cycle. More generally, we give a lower bound for $H_{m,n}$ and show also that $\sigma(H_{m,n}) \in O(mn^2)$. Finally, we extend our discussion of bipartite graphs by exhibiting for the first time a minimal labelling for $K_{m,n}$.

2 The Working Vertices

Throughout this paper we refer to vertices by their labels, usually of the form v_i. If two vertices are not adjacent, we say that they are *independent*. For $H_{m,n}$ we use the symbol $\{i\}$ to denote the *independent set* $\{{v_i}^1, {v_i}^2, \ldots, {v_i}^m\}$; that is, the set of all vertices that are independent of a specified vertex v_i. Note that this set includes v_i itself. The vertex $v_i + v_j$ is said to *correspond* to the edge (v_i, v_j), and we write $v_i + \{j\}$ to denote the set of distinct vertices corresponding to the edges joining v_i to $\{j\}$.

Theorem 2.1 *$H_{m,n}$ is exclusive.*

Proof Let $H_{m,n} = (V, E)$. Assume that $H_{m,n}$ is not exclusive and let $v_k = v_i + v_j$ denote the largest working vertex in V. Without loss of generality, suppose that $v_i < v_j < v_k$. This leads to two cases:

1. v_k is adjacent to both v_i and v_j.

2. v_k is adjacent to exactly one of v_i,v_j (assumed without loss of generality to be v_i).

Case 1: $v_k \notin \{i\} \cup \{j\}$.

In this case the vertices $v_k + v_j^1, \ldots, v_k + v_j^m$ all exist. Then since for every $h \in 1..m$,

$$v_k + v_j{}^h = (v_i + v_j{}^h) + v_j,$$

it follows that all the vertices $v_i + \{j\} \in V$. Similarly, $v_j + \{i\} \in V$. Setting $V' = (v_j + \{i\}) \cup (v_i + \{j\})$, we see that $m \leq |V'| \leq 2m - 1$.

If $|V'| > m$, there is at least one vertex in V' not contained in $\{k\}$, which means that at least one of the vertices in V' must be adjacent to v_k. Then either there exists $p \in 1..m$ such that $(v_i + v_j{}^p, v_k) \in E$, or there exists $q \in 1..m$ such that $(v_j + v_i{}^q, v_k) \in E$. This implies that either $v_i + v_k \in V$ or $v_j + v_k \in V$, contradicting the original assumption that v_k is the largest working nonisolate. We conclude that $|V'| = m$, hence that $V' = \{k\}$.

Let $v_i{}'$, $v_j{}'$ and $v_k{}' \neq v_k$ denote the smallest vertices in $\{i\}$, $\{j\}$ and $\{k\}$, respectively. Then

$$v_k{}' = v_i + v_j{}' = v_i{}' + v_j,$$

and it is clear that $v_i{}' \neq v_i$ and $v_j{}' \neq v_j$. Now consider the edge $(v_k{}', v_j{}')$ to which the vertex

$$v_k{}' + v_j{}' = (v_i{}' + v_j{}') + v_j$$

corresponds. This tells us that $v_i{}' + v_j{}'$ is a working vertex in V that is however not in $\{k\}$. Hence $v_k + v_i{}' + v_j{}'$ is a working vertex, so that $(v_k + v_i{}', v_j{}') \in E$ and $v_k + v_i{}' \in V$. Since $v_k + v_i{}' > v_k$, this contradicts the assumption that v_k is the largest working vertex in V, and so we conclude that Case 1 is impossible.

Case 2: $v_k \in \{j\}$.

Suppose there exists $q \in 1..m$ such that $v_i{}^q + v_j \notin \{j\}$. But since $v_i{}^q + v_k = (v_i{}^q + v_j) + v_i$ must exist, it follows that $v_i{}^q + v_j \in V$. Hence $(v_i{}^q + v_j, v_k)$ is an edge, so that $(v_i{}^q + v_k) + v_j$ is a vertex, revealing $v_i{}^q + v_k$ as a working nonisolate greater than v_k, a contradiction.

We conclude then that $v_i{}^q + v_j \in \{j\}$ for all $q \in 1..m$. That is, each element of $v_j + \{i\}$ is contained in $\{j\}$. But since there are m elements in each set, it therefore follows that $v_j + \{i\} = \{j\}$, an impossibility since $v_j \notin v_j + \{i\}$. Thus Case 2 is also impossible, completing the proof. $\square$

For $n = 2$ note that $H_{m,2} = K_{m,m}$. The complete bipartite graph is not generally exclusive [5] but the symmetric complete bipartite graph $K_{m,m}$ is exclusive, suggesting that for the purposes of sum graphs it more naturally resides in $H_{m,n}$ than in $K_{m,n}$.

3 The Sum Number of $H_{m,n}$

3.1 Lower & Upper Bounds for $\sigma(H_{m,n})$

Theorem 3.1 $\sigma(H_{m,n}) \geq 2mn - 2m - 1$.

Proof Let v' and v^* denote the smallest and largest labels in V, respectively, under some labelling of $H_{m,n}$. Consider the sets of vertices

$$\begin{aligned} V' &= \{v' + v_i \ : \ (v', v_i) \in E\}; \\ V^* &= \{v_i + v^* \ : \ (v_i, v^*) \in E\}. \end{aligned}$$

Observe that $|V'| = |V^*| = mn - m$. Observe also that if $(v', v^*) \in E$, then $v' + v^*$ is both the greatest element in V' and the least element in V^*; otherwise, $V' \cap V^* = \emptyset$. Thus

$$2mn - 2m \geq |V' \cup V^*| \geq 2mn - 2m - 1,$$

from which the result follows. $\square$

Theorem 3.2

$$\begin{aligned} \sigma(H_{m,n}) &\leq ((m - \frac{1}{4})n^2 - \frac{7n}{2} + 8)/2, \ n \text{ even}; \\ &\leq ((m - \frac{1}{4})(n-1)^2 + (3m - \frac{9}{2})(n-1) + 4)/2, \ n \text{ odd}. \end{aligned}$$

Proof We exhibit labellings of $H_{m,n}$ that yield the sum numbers specified.

For even $n = 2l$, number the vertices of pairs H^i of independent sets as follows:

$$H^i = \{k + \sigma_i + \{1, 3, \dots, 2m - 1\}\} \cup \{k + \sigma_i + \{2, 4, \dots, 2m\}\}, \quad 0 \le i \le l - 1,$$

where $\sigma_i = (4m - 3)(2^i - 1)$ and k is chosen sufficiently large to avoid any accidental additional edges (say $k = \sigma_l$).

For the isolates, let I_i be the set of isolates arising from edges between the two independent sets in H^i and let I_{ij} be the set of isolates arising from edges between H^i and H^j $(i \ne j)$. Then

$$I_i = \{2k + 2\sigma_i + \{3, 4, \dots, 4m - 1\}\} \text{ and } I_{ij} = \{2k + (\sigma_i + \sigma_j) + \{2, 3, \dots, 4m\}\}.$$

Simple counting yields $|I_i| = 2m - 1$ and $|I_{ij}| = 4m - 1$. If we further let

$$I1 = \bigcup_i I_i \quad \text{and} \quad I2 = \bigcup_{i<j} I_{ij}$$

and denote the set of isolates by I, then

$$\begin{aligned} I &= I1 \cup I2 \\ \text{and } |I| &= |I1| + |I2| - |I1 \cap I2| \\ &= l(2m - 1) + \binom{l}{2}(4m - 1) - |I1 \cap I2|. \end{aligned}$$

It remains to calculate $|I1 \cap I2|$.

By virtue of the numbering, we have $maxI_i = minI_{0(i+1)}$, $\;i = 0, \dots, l - 1$ along with $minI_1 = maxI_{01}$ accounting for l repetitions. Further repetitions stem from noting that the two greatest terms in I_{0i} are duplicated, as are the two least terms in I_{1i}, accounting for two repetitions from each of $l - 2$ sets. Totalling gives $|I1 \cap I2| = 3l - 4$, so we have

$$\begin{aligned} |I| &= l(2m - 1) + \binom{l}{2}(4m - 1) - (3l - 4) \\ &= 2l^2m - \frac{1}{2}(l^2 + 7l - 8). \end{aligned}$$

For $n = 2l + 1$ label the first independent set J as

$$k + \{1, 2, \dots, m\}$$

and the remaining $2l$ sets in a way similar to the even case:

$$H^i = \{k + \tau_i + \{1, 3, \ldots, 2m-1\}\} \cup \{k + \tau_i + \{2, 4, \ldots, 2m\}\}, \quad 0 \le i \le l-1,$$

where $\tau_i = 2^i(3m-3) + (2^i - 1)(4m-3)$ and $k = \tau_l$.

This time the set of isolates are considered as a union of three sets:

I_i	accounting for	edges between vertices in H^i;
I_{Ji}	accounting for	edges between J and H^i;
I_{ij}	accounting for	edges between H^i and H^j, $i \neq j$;

where

$$\begin{aligned} I_i &= \{2k + 2\tau_i + \{3, 5, \ldots, 4m-1\}\}; \\ I_{Ji} &= \{2k + \tau_i + \{2, 3, \ldots, 3m\}\}; \\ I_{ij} &= \{2k + (\tau_i + \tau_j) + \{2, 3, \ldots, 4m\}\}. \end{aligned}$$

Let

$$I1 = \bigcup_i I_i, \quad I2 = \bigcup_i I_{Ji}, \quad I3 = \bigcup_{i<j} I_{ij},$$

and, similar to the case of even n, we have $I = I1 \cup I2 \cup I3$. In this case, $|I_i| = 2m-1$, $|I_{Ji}| = 3m-1$ and $|I_{ij}| = 4m-1$ and as before

$$\begin{aligned} |I| &= |I1| + |I2| + |I3| - |I1 \cap I2| - |I1 \cap I3| - |I2 \cap I3| + |I1 \cap I2 \cap I3| \\ &= |I1| + |I2| + |I3| - |\text{ duplicates }| \\ &= l(2m-1) + l(3m-1) + \binom{l}{2}(4m-1) - |\text{ duplicates }|, \end{aligned}$$

where as above it remains to calculate the duplicates.

In a similar manner to the even case we have $maxI_i = minI_{J(i+1)}, \quad i = 0, \ldots, l-1$ along with $minI_1 = maxI_{J1}$ accounting for l repetitions (ie $|I1 \cap I2| = l$). Again the two greatest terms in I_{Ji} are duplicated, as are the two least terms in I_{1i}, accounting for two repetitions in each of $l-1$ sets: $|I2 \cap I3| = 2(l-1)$. Since $|I1 \cap I3| = |I1 \cap I2 \cap I3| = 0$, totalling gives the number of repetitions as $3l-2$, so we have

$$\begin{aligned} |I| &= l(2m-1) + l(3m-1) + \binom{l}{2}(4m-1) - (3l-2) \\ &= 2l^2m + 3ml - \frac{1}{2}(l^2 + 9l - 4). \end{aligned}$$

Expressing l in terms of n and doing some manipulation yields the forms given in the statement of the theorem. $\square$

We conjecture that in fact $\sigma(H_{m,n}) \in \Theta(mn^2)$ in general, even though, as the next section shows, a better result holds for $m = 2$.

3.2 The Sum Number of $H_{2,n}$

Substituting $m = 2$ into Theorem 3.1 yields $\sigma(H_{2,n}) \geq 4n - 5$. We now present a labelling of the vertices of $H_{2,n}$ that achieves $\sigma(H_{2,n}) = 4n - 5$.

This labelling is also based on the labelling for the complete graph K_{2n} as presented in [1]. Note that

$$\begin{aligned} \sigma(H_{2,n}) &\geq 4n - 5 \\ &= 4n - 3 - 2 \\ &= \sigma(K_{2n}) - 2. \end{aligned}$$

We construct $H_{2,n}$ by considering K_{2n} and removing n independent edges with just two isolates. By labelling K_{2n} according to

$$v_i = 1 + 4(i - 1),\ i = 1, \ldots, 2n,$$

we induce a labelling on the isolates according to

$$v_j = 6 + 4(i - 1),\ i = 1, \ldots, 2n - 3.$$

To apply this labelling to $H_{2,n}$:

- When n is even, remove the smallest and largest isolates with multiplicity $n/2$ and the corresponding edges from the graph.
- For n odd, remove the smallest isolate with multiplicity $(n - 1)/2$ and the largest isolate with multiplicity $(n + 1)/2$. Remove the corresponding edges.

In both cases we have removed 2 isolates from the sum graph for K_{2n} and n corresponding nonadjacent edges.

4 A Minimal Labelling for $K_{m,n}$

As stated in the Introduction, a minimal labelling is an optimal sum graph labelling in which the smallest label is 1. In [5] Hartsfield and Smyth gave a labelling for $K_{m,n}$ which, in most cases, was not minimal. We present a minimal labelling for all complete bipartite graphs.

Throughout we assume $m \leq n$, and label the partite sets V_m and V_n of cardinality m and n respectively. In keeping with the manner of [5], we first consider the case where $m+n$ is odd and denote V_n as the non intersecting union of two sets of vertices Q_1 and Q_2 with cardinalities respectively q_1 and $q_2 (= n - q_1)$. R refers to the set of isolates. The labelling is as follows;

Number V_m	$1 + ix$	$i = 0, 1, \ldots, m-1$	$x > \max(m, q_1, q_2)$
Number Q_1	$y + jx$	$j = 0, 1, \ldots, q_1 - 1$	$q_1 = \frac{n-m+1}{2}$
Number Q_2	$y + kx + 1$	$k = 0, 1, \ldots, n - q_1 - 1$	$x + 1 < y < 2x - 1$
Number R	$y + lx + 2$	$l = 0, 1, \ldots, \sigma(K_{m,n}) - 1$	

Selecting y in the interval $(x+1, 2x-1)$ removes the possibility of vertices in V_m and V_n having the same label.

The following lemma shows that we have accounted for all edges.

Lemma 4.1 *Suppose that $a \in V_m$. Then*

(i) for $b \in Q_1$, $\quad a + b \in Q_2$*;*
(ii) for $b \in Q_2$, $\quad a + b \in R$*.*

Proof (i)

$$\begin{array}{ll} a = 1 + ix & i = 0, 1, \ldots, m-1 \\ b = y + jx & j = 0, 1, \ldots, q_1 - 1 \\ a + b = y + (i+j)x + 1 & i + j = 0, 1, \ldots, m - 1 + q_1 - 1 \end{array}$$

Note that $m - 1 + q_1 - 1 = m + \frac{n-m+1}{2} - 2 = \frac{n+m-3}{2} = n - q_1 - 1$. So

$$\{a + b : a \in V_m, b \in Q_1\} = Q_2.$$

(ii)

$$
\begin{array}{ll}
a = 1 + ix & i = 0, 1, \ldots, m-1 \\
b = y + kx + 1 & k = 0, 1, \ldots, n - q_1 - 1 \\
a + b = y + (k+i)x + 1 & k + i = 0, 1, \ldots, m - 1 + n - q_1 - 1
\end{array}
$$

where $m - 1 + n - q_1 - 1 = m - 1 + n - \frac{n-m+1}{2} - 1 = \frac{3m+n-3}{2} - 1 = \sigma(K_{m,n}) - 1$. So

$$\{a + b : a \in V_m, b \in Q_2\} = R.$$

$\square$

As in [5] the case $m + n$ even follows by labelling $K_{m,n+1}$ as above and removing one vertex (and incident edges) from Q_1 leaving a labelling for $K_{m,n}$.

Acknowledgements

The work of the third author was supported in part by Grant No. A8180 of the Natural Sciences & Engineering Research Council of Canada and by Grant No. GO-12778 of the Medical Research Council of Canada.

References

[1] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski & J. Wiener, **The sum number of a complete graph**, *Bull. Malaysian Math. Soc. 12* (1989) 25-28.

[2] Mark N. Ellingham, **Sum graphs from trees**, *Ars Combinatoria 35* (1993) 335-349.

[3] R. Gould & V. Rödl, **Bounds on the number of isolated vertices in sum graphs**, *Graph Theory, Combinatorics and Applications* (edited by Y. Alevi, G. Chartrand, O. R. Oellermann & A. J. Schwenk), John Wiley & Sons (1991) 553-562.

[4] F. Harary, **Sum graphs and difference graphs**, *Congressus Numerantium 72* (1990) 101-108.

[5] Nora Hartsfield & W. F. Smyth, **The sum number of complete bipartite graphs**, *Graphs and Matrices* (edited by Rolf Rees), Marcel Dekker (1992) 205-211.

[6] Nora Hartsfield & W. F. Smyth, **A family of sparse graphs of large sum number**, *Discrete Math. 141* (1995) 163-171.

[7] Mirka Miller, Joseph F. Ryan, Slamin & W. F. Smyth, **Labelling wheels for minimum sum number**, submitted for publication (1996).

[8] Ahmad Sharary, **Integral sum graphs from complete graphs, cycles and wheels**, *Arab Gulf J. Sci. Res. 14-1* (1996) 1-14.

[9] W. F. Smyth, **Sum graphs of small sum number**, *Colloq. Math. Soc. János Bolyai 60* (1991) 669-678.