The Sum Number of a Disjoint Union of Graphs

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Abstract

In this paper we consider the disjoint union of graphs as sum graphs. We provide an upper bound on the sum number of a disjoint union of graphs and provide an application for the exclusive sum number of a graph. We conclude with some open problems.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. A graph G is called a *sum graph* if there exists a labelling of the vertices of G by distinct positive integers such that the vertices labelled u and v are adjacent if and only if there exists a vertex labelled $u+v$. If G is not a sum graph, adding a finite number of isolated vertices to G will always yield a sum graph, and the *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices that will achieve this result. A labelling that realises $G \cup \overline{K}_{\sigma(G)}$ as a sum graph is said to be *optimal*.

Vertices whose label equals the sum of the labels of two adjacent vertices are called *working vertices*. All connected graphs (except K_1) require additional isolates in order to support a sum labelling. Graphs for which the working vertices are confined to these extra isolates are called *exclusive* graphs.

Since the introductopn of sum graphs by Harary [4], optimal sum labellings have been described for many classes of graphs including complete graphs [1], complete bipartite graphs [5, 7, 12, ?], trees [2], cycles [4], wheels [6, 9], as well as partial results on more complex graphs such as multipartite graphs [11]. Many of these results (in particular cycles and trees) make use of a property stated in [8] that the minimum degree provides a lower bound for the sum number.

Ellingham ([2]) showed that all trees are unit graphs, that is, have sum number 1. He went on to show that any forest with all components greater than order 1 is also a unit sum graph ie can sustain a sum labelling with the addition of no more than 1 isolate. Motivated by this we investigate the disjoint union of more general graphs.

2 The Sum Labelling of a Disjoint Union of Graphs

Let G_1 be a um graph bearing a labelling

$$L = \{l_1, l_2, \ldots, l_m\}, \quad i < j \rightarrow l_i < l_j$$

and let G_2 be a sum graph with a labelling

$$K = \{k_1, k_2, \ldots, k_n\}, \quad i < j \rightarrow k_i < k_j.$$

Assume first that at least one of the labels from the labelling L is relatively prime to k_n. That is $(l_j, k_n) = 1$ for some l_j. Now multiply labelling L by k_n and K by l_j.

If z is a label in L (respectively K) then denote by z' the corresponding label in k_nL (respectively l_jK). Clearly if u, v, w are labels in L (respectively K) with $u + v = w$ then

$$k_nu + k_nv = k_nw \ \ \text{(respectively } l_ju + l_jv = l_jw)$$

so that any edges in the original disjoint union are preserved under the multiplication. Our first result is a proof of the converse.

Theorem 1 *If $u, v, w \in L \cup K$ and $u', v', w' \in k_nL \cup l_jK$ with $(l_j, k_n) = 1$ then any working vertex w' (as in $u' + v' = w'$) represents and edge in the original disjoint union.*

Proof Clearly for $u', v', w' \in k_n L$ (respectively $l_j K$)

$$u' + v' = w' \rightarrow u + v = w$$

Therefore we need only consider cases in which u', v' and w' are in different labellings. There are four such cases;

CASE 1: $u', v' \in k_n L,\ w' \in l_j K$.

$$u' + v' = w' \text{ becomes } k_n u + k_n v = l_j w$$

indicating that k_n divides w, but since k_n is the largest label in the set containing w, then $w = k_n$. Dividing the equation by k_n yields $u + v = l_j$ corresponding to an edge in the original graph G_1.

CASE 2: $u', v' \in l_j K,\ w' \in k_n L$.

$$u' + v' = w' \text{ becomes } l_j u + l_j v = k_n w$$

giving that l_j divides w. Putting $w = ml_j$ gives the equation $u + v = mk_n$ forcing m to be 1 (since u and v are less than or equal to k_n) and reflecting that u and v are adjacent in the original graph G_2.

CASE 3: $u', w' \in k_n L,\ v' \in l_j K$.

$$u' + v' = w' \text{ becomes } k_n u + l_j v = k_n w$$

so k_n divides v, hence $v = k_n$. Dividing by k_n gives $u + l_j = w$ which indicates an edge in the original graph G_1.

CASE 4: $v', w' \in l_j K,\ u' \in k_n L$. This case does not occur since

$$u' + v' = w' \text{ becomes } k_n u + l_j v = l_j w$$

and l_j divides u. So let $u = ml_j$ and the equation becomes $mk_n + v = w$ which is impossible since $k_n > w$. $\square$

The advantage of this labelling is the repeated use of the label $l_j k_n$ which occurs as a label in both sets thus reducing the cardinality of the union by 1. Since k_n is the largest label in the labelling set K, this reduction is of an isolate of G_2. So we have

Theorem 2 *$\sigma(G_1 \cup G_2) \leq \sigma(G_1) + \sigma(G_2) - 1$ provided that there exists labellings of G_1 and G_2 such that there is an element of one labelling that is relatively prime to the largest element of the other labelling.*

The condition of finding an element of one labelling that is relatively prime to the largest element of the other labelling is fulfilled if 1 is an element of either labelling. In particular we have,

Corollary 1

$$\sigma(\cup_{i=1}^{p} G_i) \leq \sum_{i-1}^{p} \sigma(G_i) - (p-1)$$

provided that at least none of the p disjoint graphs has 1 as an element of its labelling.

The concept of a sum labelling containing the label 1 was introduced in [11] and such a labelling was given the name *a minimal labelling*. The importance of a minimal labelling is reflected in the existance of the bound in Corollary 1. To date the question of whether all sum graphs bear a minimal labelling is open and while many labellings are not minimal, no graph has been found to be unable to bear such a labelling. One class of summable graphs known to *always* support a minimal labelling are the exclusive graphs. A labelling that restricts working vertices to the isolates is thus called a *exclusive sum labelling*.

In [10] it was shown that exclusive sum labellings are invariant under a linear transformation with integer coefficients. This means that the minimal label of any exclusive sum graph may be set to 1, so we have

Corollary 2

$$\sigma(\cup_{i=1}^{p} G_i) \leq \sum_{i=1}^{p} \sigma(G_i) - (p-1)$$

provided at least one of the p disjoint graphs is an exclusive sum graph.

The authors of [10] looked at exclusively labelling graphs whose optimal sum labelling is not exclusive. For this purpose they defined the *exclusive sum number* $\epsilon(G)$ of a graph as the minimum number of isolates for a graph to bear an exclusive sum labelling. Clearly $\epsilon(G) \geq \sigma(G)$ which leads to

Corollary 3

$$\sigma(\cup_{i=1}^{p} G_i) \leq \sum_{i \neq j}^{p} (G_i) - (p-1) + \epsilon(G_j) \quad \textit{for any } j \in \{1, \ldots, p\}.$$

3 Open Problems

1. Find the exclusive sum number for certain classes of graphs such as trees, complete bipartite graphs.

2. Find the sum number of disjoint families of graphs.

3. Find the exclusive sum number of disjoint families of graphs.

4. Gould & Rödl ([3]) showed that there exist graphs that require a number of isolates of the order of n^2 in order to support a sum labelling. Are there graphs that require fewer than order n^2 isolates to support a sum labelling but need order n^2 isolates to bear an exclusive sum labelling?

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