GRAPH DIAMETER PROBLEMS RELATED TO NETWORK ANALYSIS & DESIGN

William Fennell Smyth

This document is presented as part of the requirements for the award of the Degree of Doctor of Philosophy of the Curtin University of Technology.

June 1989

GRAPH DIAMETER PROBLEMS RELATED TO NETWORK ANALYSIS & DESIGN

William Fennell Smyth

SUMMARY

The main focus of this thesis is on a particular class of graphs called diameter-critical graphs. It is proved that every graph of maximum size is in fact diameter-critical. As a consequence, it becomes possible to establish a number of new relations among important graph parameters: order, size, minimum degree, diameter, and connectivity (edge-connectivity). In particular,

(1) K-connected and K-edge-connected graphs of given diameter and maximum size are completely characterized;

(2) K-connected diameter-critical graphs of minimum size are completely characterized;

(3) K-edge-connected diameter-critical graphs of minimum size and sufficiently large order are characterized;

(4) a sharp upper bound is given on the diameter of a graph of specified order, size, minimum degree, and connectivity;

(5) diameter-critical graphs of given order, size, minimum degree, and connectivity are partially characterized.

Applications of these new results to problems of network analysis and design, especially to the determination of the diameter, are discussed.

CERTIFICATION

I certify that this thesis is entirely my own work, and that it has not previously been submitted, in whole or in part, for any academic award at Curtin university or elsewhere.

William Fennell Smyth

William Fennell Smyth

5 July 1989

TABLE OF CONTENTS

DEDICATION

To my father, who didn't live to see it,

and to my mother, who did.

ACKNOWLEDGEMENTS

Above all, I wish to express my appreciation of the support of my friend and colleague, Lou Caccetta, who in his role as Thesis Supervisor taught me much about both research and mathematics. My thanks also to Dennis Moore, Chairman of my Thesis Committee, whose idea it was, and to K. Vijayan, Associate Supervisor, for his valuable comments. I am indebted as well to Julie Aizlewood, who did a marvellous job of word-processing my squiggles, and to Kerstin Baxter, for programming the messy results of Chapter 5. Finally, a salute to Bill Perriman, Head of the School of Mathematics and Statistics, whose energy and enterprise attracted me to Curtin University in the first place.

CHAPTER 1

INTRODUCTION

A network is perhaps most simply and naturally thought of as a set of points joined by lines. Networks can represent numerous real world objects, such as

* communications networks (global, national, or local);
* transportation (distribution) networks;
* computer networks;
* computer (VLSI) chips;
* multiprocessor configurations;
* neural connections;
* critical path (CPM/PERT) networks;
* flows (of gas, water, information);
* engineering structures (such as multistorey buildings, steel transmission towers).

In each of these examples of a network, a line joining two points has a significance determined by the real world object being represented. Thus, in a transportation network, a line might represent the highway connecting two cities; whereas in an engineering structure, a line might represent a steel beam connecting two nodes. In recognition of this representational role, a line in a network is often assigned one or more numerical values: in a transportation network, interesting values could be the distance between the two cities, or the time or cost involved in travelling from one to the other (not necessarily the same in each direction!); in an engineering structure, interesting values could be the length, the moment of inertia, or the stiffness of the beam. Similarly, values may also be assigned to the points of the network: in both the transportation network and engineering structure examples, interesting values associated with points could be their coordinates. Thus the solution of a very wide range of real world problems can be reduced to the solution of abstract problems defined on networks — that is, arrangements of points and lines in which values, usually non-negative integer values, are associated with the given points and/or lines.

In fact, it turns out that use of an even simpler model is often of great benefit in dealing with some of the problems enumerated above. Consider, for example, a communications network in which direct transmission from one node to another is possible if and only if there exists a line (communications link) joining them. Hence, to send a message from node A to another node B not adjacent to A requires determining a "shortest path" from A to B. Since usually the transmission time between adjacent nodes is very rapid, and since the major delays occur due to queuing and storage at intermediate nodes, it follows that a very good measure of transmission time is provided by a simple count of the least number of lines which need to be traversed in order to go from A to B. A network in which every line is assigned the single value 1 is called a <u>graph</u>, and the least number of lines which need to be traversed in order to go from a point A of the graph to another point B is called the <u>distance</u> from A to B and written d(A,B). In Figure 1.1, d(A,B) = 2. If there is no path from a point A to point B, then by convention d(A,B) = ∞.

Example of a graph in which d(A,B) = 2

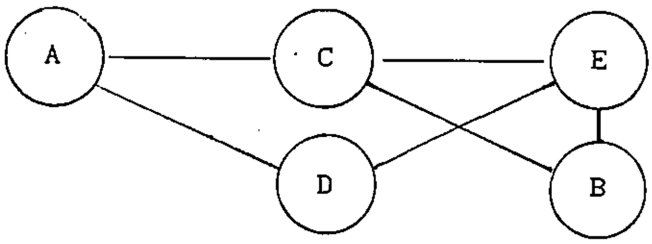

Figure 1.1

Graphs then provide a useful model for problems on communications networks and also, for similar reasons, for problems on computer networks, computer chips, multiprocessor configurations, and neural nets as well. For less obvious reasons, graphs are also fundamental to the computer-based analysis and design of all large engineering structures (George & Liu 1981, 1989) as well as to a myriad of other problems (Caccetta & Vijayan 1987; Caccetta 1989). Indeed, it turns out that the most elementary properties of graphs are of decisive importance to the solution of many of these problems — this is not to say that the problems are necessarily simple, only that they can be stated in terms of simple graph parameters.

In this chapter, then, a non-technical, hence slightly imprecise, introduction is given to the simple parameters used in this thesis to identify graphs, and an outline of the main results is provided which attempts to set them in the context of related work. A precise mathematical treatment begins only in Chapter 2.

The most obvious parameters associated with a graph G are the number of points and the number of lines. The number of points of G is called its order and denoted by n; the number of lines of G is called its size and denoted by m. Thus for the graph of Figure 1.1, $n = 5$ and $m = 6$. Another obvious property of a point of a graph is its degree; that is, the number of lines connecting it to other points. The degree of a point X is usually denoted deg(X), and the minimum degree δ of G is just the minimum of the degrees over all points of G. Similarly the maximum degree Δ is the maximum of the degrees over all points of G. For the graph of Figure 1.1, $\delta = 2$ (achieved by A, B, and C) and $\Delta = 3$ (achieved by D and E). Observe that n, m, δ, and Δ can be determined for a given graph G as a byproduct of reading the graph into the main memory of a computer; that is, in time O(m). (O(x) means "bounded above by cx, where c is some positive constant value".) . One further property of a graph is introduced at this point: the diameter D of G is just the maximum over all distances which occur in the graph. It is not difficult to see that the diameter of the graph of Figure 1.1 is

2, since every pair of points is distance either 1 or 2 apart; but observe that, in order to establish this fact, up to $\binom{n}{2}$ pairs of points may need to be considered.

Making use only of the simple parameters (n, m, δ, Δ, and D) defined so far, it is possible to state a famous, extremely important, widely studied, and unsolved problem of graph theory. Suppose that a graph G represents a communications network. Then the diameter D represents the maximum time required to transmit messages in the network; it is a measure of the efficiency of the network. On the other hand, the maximum degree Δ represents the maximum number of "ports" available at each node of the network; since clearly $m \leq n\Delta/2$, Δ is a measure of the cost per node of the network. In the design of communications networks, the following question then immediately arises:

> What is the maximum order n of a graph (communications network) of specified diameter D (efficiency) and maximum degree Δ (unit cost)?

Solutions to this problem are known only in certain very special cases (for D = 1, for Δ = 2, and for D = 2 with Δ = 3 or 7); it is the subject of much current research, and is discussed in several recent surveys of graph diameter problems (Bermond & Bollobás 1981; Chung 1984; Chung 1987). It is closely related to the "cage" problem (Wong 1982; Chartrand & Lesniak 1986 pp35-45). There is no intention to deal further here with the $n(\Delta,D)$ problem, as it is called; it has been introduced to persuade the reader that naturally-arising simply-stated graph diameter problems are of major current interest in computer science. Indeed, the $n(\Delta,D)$ problem is only one of many such problems, as the surveys referenced above make clear.

Since graph diameter problems are important, it becomes of interest to have an efficient means of determining the diameter D of a given graph. Unfortunately, as discussed in Section 5.2, even though a good estimate of the diameter can usually or often be computed in time $O(m)$, there exists no algorithm which guarantees determination of D in time less than $O(n^3)$. It has been a fundamental objective of the research described here to work toward an improved graph diameter algorithm in two main ways:

(a) clarifying the relationship among D and other basic graph parameters;

(b) determining an improved (sharper) upper bound on D in terms of other graph parameters.

Two of the most important graph parameters which relate to these objectives have not yet been defined: connectivity and edge-connectivity, both denoted by K throughout this work. Roughly speaking, the <u>connectivity</u> (respectively, <u>edge-</u>

connectivity) of a graph is the least number of points (respectively, lines) whose removal disconnects the graph (that is, yields at least two points X and Y not joined by any path). Thus the graph of Figure 1.1 is said to be 2-connected (K=2) because at least two points need to be removed in order to disconnect it; it is also 2-edge-connected. For more precise definitions of these terms, see Section 2.1.

Before an overview of the new results described in this thesis is given, it may be of interest to look at some of the main trends of previous work related to the objectives (a) and (b). To begin with, a great deal of work has been done which relates the four parameters n, δ, D, and K (where K denotes connectivity rather than edge-connectivity). Perhaps the definitive result is due to Klee and Quaife (1976), who give a lower bound $n_*(\delta,D,K)$ on the order n in terms of δ, D, and K, extending an earlier result due to Moon (1965). With a little effort, this bound translates into an upper bound $D^*(n,\delta,K)$ on the diameter D (see Theorem 2.20). The result was later rediscovered by Seidman (1983) and by Amar, Fournier and Germa (1983), and an alternate derivation was given by Myers (1980). Special cases of the Klee/Quaife result were found by Kane and Mohanty (1978) and by Goldsmith, Manvel and Faber (1981), while more precise forms of the result for particular classes of graphs were elucidated by Klee (1980), Myers (1981), and Bhattacharya (1985).

A slightly different problem arises in connection with the three parameters n, m, and D. Bosák, Rosa & Znám (1968) determined an upper bound $m^*(n,D)$ on the size m of a graph, later rediscovered by Smyth (1987) as an upper bound $D^*(n,m)$ on the diameter D. Klee and Larman (1981) and Bollobás (1981), treating the size $m = m(n)$ and the diameter $D = D(n)$ as functions of n, were able to specify conditions under which almost every graph of order n and size m had diameter D. More specifically, their results made it clear that "most" graphs had small diameters, so that their orders provided good lower bounds for $n(\Delta,D)$ in the communications network problem discussed above. In fact, Bollobás and de la Vega (1982) applied the results to random regular graphs (those for which $\delta = \Delta = 2m/n$) and were able to prove the existence of graphs of order $n = n(\Delta,D)$ much larger than those which could actually be constructed. Thus the curious situation arises that, while on the one hand extremal results make clear that a great many graphs of large order $n(\Delta,D)$ exist, on the other hand nobody has been able to construct them. This parallels the fact, mentioned earlier, that while the diameter can usually be estimated efficiently, it has not so far been possible to find a means of computing it exactly in worst case time less than $O(n^3)$.

This state of affairs becomes even more poignant when graphs of small diameter are considered. It is easy to see that $D = 1$ if and only if $\delta = n-1$. Further, it is well-known that if

$$(n-1)/2 \leq \delta < n-1,$$

then D = 2, so that for half the possible range of values of δ, the diameter can be immediately specified. Moreover, other graphs of diameter 2 can be described in terms of their complements. (The complement of a graph G is a graph $\bar{G}$ whose points are the points of G and whose lines are exactly the lines not in G. Figure 1.2 shows the complement of the graph of Figure 1.1.)

Complement of Figure 1.1

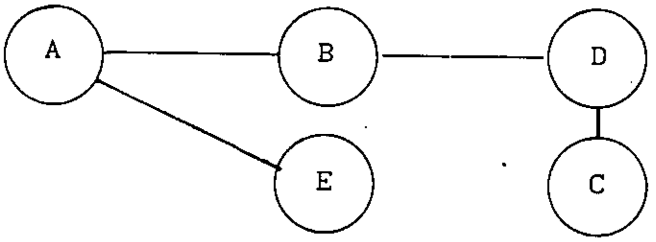

Figure 1.2

Then the following result, put together by a sequence of researchers (Sachs 1962; Ringel 1963; Harary & Robinson 1985; Straffin 1986; Bloom, Kennedy & Quintas 1987) can be proved:

A graph G of order n has diameter D = 2 if any one of the following conditions holds:

(a) $(n-1)/2 \le \delta < n-1$;

(b) diam $(\overline{G}) \ge 4$;

(c) diam $(\overline{G}) = 3$ and $\overline{G}$ is regular.

Despite the simplicity of these conditions, no corresponding results are known for graphs of diameter $D \ge 3$.

The above discussion has not been intended to be exhaustive or comprehensive, but merely to provide motivation and context for the work described in subsequent chapters. This work establishes the following main results:

(1) For given parameters n, D, and K (either connectivity or edge-connectivity), graphs of maximum size $m^* = m^*(n,D,K)$ are completely characterized. This characterization depends first on the definition of a <u>diameter-critical</u> graph (that is, a graph which has the property that the addition of any line necessarily decreases the diameter); then on the easily-proved fact that every graph of maximum size must be diameter-critical.

(2) For given parameters n, D, and K, diameter-critical graphs of minimum size $m_* = m_*(n,D,K)$ are completely characterized when K signifies connectivity, and characterized for sufficiently large n when K signifies edge-connectivity.

(3) For given parameters n, D, and connectivity K, diameter-critical graphs are partially characterized.

(4) For given parameters n, m, δ, and connectivity K, the maximum diameter $D^* = D^*(n,m,\delta,K)$ is determined, thereby sharpening the result of Klee and Quaife (1976). On the way to computing D^*, the maximum size $m^* = m^*(n,\delta,D,K)$ for given n, δ, D, and K is also determined.

These results are presented in four chapters. Chapter 2 gives a technical summary of the main results broken down into four main sections:

* a discussion of diameter-critical graphs;
* results for K-connected graphs;
* results for K-edge-connected graphs;
* determination of an upper bound on the diameter.

Then Chapters 3-5 give detailed results and proofs corresponding to the summaries provided in Sections 2.2-2.4 respectively. The reader may find it convenient to go through the thesis in a non-linear fashion; that is, for $2 \le i \le 4$, to follow the reading of Section 2.i by the reading of Chapter i+1. A glossary of terms and symbols is provided to facilitate the look-up of definitions.

CHAPTER 2
A SUMMARY OF THE MAIN RESULTS

In this chapter the concept of a "diameter-critical" graph is introduced, together with some closely-related ideas and terminology. Then the main results of this dissertation are summarized under the topic areas shown in Table 2.1. These results will be proved in the chapters indicated.

Main Topic Areas

Topic	Chapter	Main References
K-Connected Diameter-Critical Graphs	3	Ore (1986), Caccetta & Smyth (1986b, 1989d)
K-Edge-Connected Diameter-Critical Graphs	4	Caccetta & Smyth (1987a, 1987b, 1988a, 1988b, 1989a)
An Upper Bound on the Diameter of a Graph	5	Klee & Quaife (1976), Caccetta & Smyth (1989b, 1989c)

Table 2.1

2.1 Diameter-Critical Graphs

Throughout this document, unless explicitly stated to the contrary, the term <u>graph</u> will refer to a finite, non-empty, connected, simple, undirected graph. In general, notation and terminology follow Bondy & Murty (1977); in particular, the five qualifiers of "graph" in the previous sentence are used in accordance with this standard reference. A graph will generally be denoted by $G = (V,E)$ with vertex set V of cardinality $n = |V|$

(the order of G), and edge set E of cardinality $m = |E|$ (the size of G). The minimum (respectively, maximum) degree of G will be denoted by δ (respectively, Δ). For any two distinct vertices $u, v \in V$, the distance $d(u,v)$ from u to v is defined to be the number of edges on a shortest path from u to v; thus, if $(u,v) \in E$, $d(u,v) = 1$. If no shortest path from u to v exists (G disconnected), $d(u,v) \equiv \infty$; for every $u \in V$, $d(u,u) \equiv 0$. The diameter $D = D(G)$ of a graph G is then defined to be

$$D = \max_{u,v \in V} d(u,v).$$

For a non-complete graph G, consider all subsets $X \subseteq V$ (respectively, $X \subseteq E$) such that removing X from G yields a disconnected graph; let K_* be the minimum cardinality of any such subset X; then K_* is called the connectivity (respectively, edge-connectivity) of G. The connectivity (respectively, edge-connectivity) of a complete graph of order n is defined to be n-1. Then for every integer K satisfying $0 \leq K \leq K_*$, G is said to be K-connected (respectively, K-edge-connected). It is well known (Bondy & Murty 1977, p43) that every K-connected graph is K′-edge-connected, for some integer $K' \geq K$, and has minimum degree $\delta \geq K'$.

In general, the work described here may be thought of as an investigation of the relationships among the six integer parameters n, m, δ, Δ, D, and K (where for the moment K may be thought of as representing either the connectivity or the edge-connectivity). The trivial inequalities which these parameters must satisfy are as follows:

$$0 \le K \le \delta \le \Delta \le n-1; \qquad \ldots(2.1)$$

$$\max\{n-1, n\delta/2\} \le m \le n\Delta/2 \le \binom{n}{2}; \qquad \ldots(2.2)$$

$$0 \le D \le n-1. \qquad \ldots(2.3)$$

Of fundamental importance for this work is the idea of a <u>rooted level structure</u> (Arany, Smyth & Szóda 1971) or hierarchy: corresponding to any arbitrarily chosen vertex $u \in V$, this is an arrangement of the elements of V into subsets $L_i(u)$, $i = 0,1,\ldots$, consisting of the vertices distance exactly i from u. Each $L_i(u)$ is called a <u>level</u>, and its cardinality is denoted by $n_i = |L_i(u)|$. Then a <u>vertex sequence</u> corresponding to u is given by

$$S = S_d(u) = (n_0, n_1, \ldots, n_d), \qquad \ldots(2.4)$$

where $n_0 = 1$ and d is the largest integer such that $n_d > 0$; d is called the <u>length</u> of S, and if G has diameter D, it follows that $1 \le d \le D$. If $d = D$, the vertex u is said to be <u>peripheral</u>. Subsequences of S of length $k \ge 1$ are referred to as <u>k-tuples</u>; in particular, for k = 1, 2, or 3, as <u>terms</u>, <u>doubles</u>, or <u>triples</u>, respectively. A k-tuple $(n_i, n_{i+1}, \ldots, n_{i+k-1})$ is <u>internal</u> if $1 < i < d-k$ and <u>terminal</u> if $i = 0$ or $d-k+1$; thus the terms n_1 and n_{D-1} are neither internal nor terminal. The <u>order</u> of a k-tuple starting at n_i is just the sum of its terms, and its size is just the number of edges in the subgraph of G induced by the levels $L_i, L_{i+1}, \ldots, L_{i+k-1}$. If a double (x,y) has size $\binom{x+y}{2}$, then it is said to be complete; similarly, if every double of a k-tuple is complete, then the k-tuple itself will be called complete. The notation

$$(x_1, \ldots, x_h)^r$$

will be used to denote $r \geq 0$ consecutive occurrences of a given h-tuple $(x_1, \ldots, x_h)$. A k-tuple $(x_1, \ldots, x_k)$ is said to be h-recurring if $k > h$ and h is the least integer such that $x_{i+h} = x_i$ for every $1 \leq i \leq k-h$; accordingly, a vertex sequence (2.4) is said to be h-recurring if it contains an internal h-recurring (d-3)-tuple.

In connection with vertex sequences, observe that if (2.4) is complete, then within isomorphism it determines a graph G. As indicated below (Lemmas 2.2 and 2.4), the graphs discussed in this thesis will all have the property that they give rise to a complete vertex sequence. Thus, when a complete vertex sequence S is determined corresponding to a graph G, a one-one correspondence is thereby established between S and G. This property is exploited throughout and, in general, all vertex sequences considered here will be complete.

Another fundamental idea is that of a property-critical graph. Suppose that $\mathcal{P}$ is a property (for example, diameter, edge-connectivity, chromatic number, minimum degree) of a graph G which is measured by a real number P; this relationship may be expressed by writing $\mathcal{P}(G) = P$. Then G will be said to be <u>lower $\mathcal{P}$-critical</u> (respectively, <u>lower $\mathcal{P}$-edge-critical</u>) if the removal of any vertex (respectively, edge) from G yields a graph G′ such that $\mathcal{P}(G') \neq P$; similarly, G will be said to be <u>upper $\mathcal{P}$-edge-critical</u> if the addition of any edge to G yields a graph G′ such that $\mathcal{P}(G') \neq P$. (This use of "lower" and "upper" is not unprecedented; it occurs, for example, in Parthasarathy & Srinivasan (1984).) Lower diameter-critical graphs (Boesch, Harary & Kabell 1981; Bondy & Hell 1983; Caccetta 1984; Usami 1985) and lower diameter-edge-critical graphs (Chung & Garey 1984; Schoone, Bodlaender & van Leeuwen 1987; Fan 1987), as well as variations of such graphs, have been the subject of considerable study; these graphs are closely related to the $n(\Delta, D)$ problem described in Chapter 1, and tend, not surprisingly, to be very difficult to characterize. The main thrust of the development to be described here, however, relates to upper diameter-edge-critical graphs. To avoid this mouthful, the convention will be adopted that, unless the contrary is explicitly stated, <u>diameter-critical</u> will mean "upper diameter-edge-critical"; further, <u>D-critical</u> will mean "diameter-critical of diameter D". These terms will sometimes be used to refer, not only to a graph, but also to a vertex sequence $S_D(u)$ corresponding to a peripheral vertex u of a D-critical graph. Observe that, since all graphs are simple, these terms are of interest only if $D \geq 2$.

The first elementary result, due to Ore (1968), may now be stated:

<u>Lemma 2.1</u> A graph G is D-critical if and only if every peripheral vertex gives rise to a vertex sequence (2.4) such that

(a) each terminal term is of order 1;

(b) every double is complete. ∎

It is an easy consequence of Lemma 2.1 that every D-critical graph contains exactly two peripheral vertices. Moreover, it is not difficult to see that a graph can give rise to a complete vertex sequence corresponding to at most two peripheral vertices. Hence, an immediate corollary of Lemma 2.1 is the following:

<u>Lemma 2.2</u> A graph G is D-critical if and only if every peripheral vertex of G gives rise to a complete vertex sequence. ∎

It is now possible to define the main classes of graph which will be considered below. These classes are specified in terms of given values of the parameters n, m, δ, D, and K:

$\mathcal{G}_v(n,m,\delta,D,K)$: all K-connected graphs of order n, size m, minimum degree δ, and diameter D;

$\mathcal{G}_e(n,m,\delta,D,K)$: all K-edge-connected graphs of order n, size m, minimum degree δ, and diameter D;

$\mathscr{C}_v(n,m,\delta,D,K)$: all D-critical graphs of $\mathscr{G}_v(n,m,\delta,D,K)$;

$\mathscr{C}_e(n,m,\delta,D,K)$: all D-critical graphs of $\mathscr{G}_e(n,m,\delta,D,K)$.

Observe that a class of graphs will be non-empty if and only if there exists at least one graph whose parameters take the specified values; in particular, if any of (2.1)-(2.3) are violated no graph exists, and therefore (2.1)-(2.3) will always be assumed to hold. To simplify notation, it will be supposed in the remainder of this thesis that the class subscript "c" means "c = v,e"; that is, that both K-connected and K-edge-connected graphs are independently referenced. It will further be supposed that the replacement of a parameter by "*" indicates that the parameter is unspecified. Thus, for example, $\mathscr{G}_c(n,*,*,D,K)$ specifies two classes of graphs:

* all K-connected graphs of order n and diameter D;
* all K-edge-connected graphs of order n and diameter D.

Similarly, $\mathscr{C}_v(*,*,*,D,K)$ is the class of all K-connected D-critical graphs.

Observe that every non-D-critical graph can be made D-critical by the addition of edges. Hence, to determine whether or not $\mathscr{G}_c(n,*,*,D,K)$ is empty, it suffices to consider the corresponding class of D-critical graphs. To discover more about D-critical graphs, a new definition is required. Given a class $\mathscr{F}$ of graphs, a distinguished graph $G \in \mathscr{F}$ of size m is said to be <u>edge-maximal</u> (respectively, <u>edge-minimal</u>) <u>over $\mathcal{F}$</u> if no graph of $\mathcal{F}$ has size greater than (respectively, less than) m; when the context is clear, the qualifier "over $\mathcal{F}$" will be omitted. Further, both of these terms will sometimes be applied by extension to vertex sequences of G, or to tuples within vertex sequences. Now consider a peripheral vertex u of an arbitrary graph $G \in \mathcal{G}$, and for $1 \le i \le D-1$, call a vertex $v \in L_i(u)$ <u>exceptional</u> if its degree $deg(v) < n_{i-1} + n_i + n_{i+1} - 1$. A fundamental result can now be stated and proved:

<u>Lemma 2.3</u> For $D \ge 3$, no edge-maximal graph of $\mathcal{G}_c = \mathcal{G}_c(n,*,\delta,D,K)$ contains an exceptional vertex.

<u>Proof</u> Suppose on the contrary that an edge-maximal graph G contains the exceptional vertices $v_1, v_2, \ldots, v_t$. Let $S_D(u) = (1, n_1, n_2, \ldots, n_D)$ denote a vertex sequence of G corresponding to a peripheral vertex u. Note that if $n_1 = \delta$, then by the edge-maximality of G, $t = 0$. Suppose then that $n_1 > \delta$ and note further that $t \ge 2$. Then the edge-maximality of G implies that the exceptional vertices belong either to one level or to adjacent levels of G. Further, the only vertices of G that can have degree δ are the exceptional vertices. Suppose without loss of generality that $deg(v_1) = \delta$. Then v_1 is not adjacent to any v_i, $2 \le i \le t$, and the vertices $v_2, v_3, \ldots, v_t$ form a clique (if this were not so, an edge could be added to yield a graph of $\mathcal{G}_c(n,*,\delta,D,K)$ having more edges than G).

If $L_1(u)$ contains no exceptional vertex, then a graph $G' \in \mathcal{G}_c(n,*,\delta,D,K)$ with vertex sequence

$$S'_D = (1,\delta,n_1+n_2-\delta,n_3,\ldots,n_D)$$

has more edges than G. Hence the exceptional vertices of G are contained in $L_1(u) \cup L_2(u)$. If $v_1 \notin L_1(u)$, then the vertices of $L_1(u)$ form a clique and a graph G′ can be constructed as above having more edges than G. Hence $v_1 \in L_1(u)$. Now the graph induced by the vertices of $L_1(u) \cup L_2(u) \setminus \{v_1\}$ must be complete, since otherwise G could not have been edge-maximal. But if the edges (v_1,v_i), $1 \le i \le t$, are added and if $n_1-\delta$ vertices are then transferred from $L_1(u)$ to $L_2(u)$, a graph $G'' \in \mathcal{G}_c(n,*,\delta,D,K)$ will be formed having at least t-1 more edges than G. This contradiction establishes the lemma. ∎

Lemma 2.3 of course also holds for graphs of $\mathcal{G}_c(n,*,*,D,K)$. Further, let $S_D(u) = (1,n_1,n_2,\ldots,n_D)$ be a vertex sequence corresponding to a peripheral vertex u of a graph G edge-maximal over $\mathcal{G}_c(n,*,\delta,D,K)$. Observe from the proof of Lemma 2.3 that it may be supposed that $n_1 = \delta$. It follows then that the size of n_D is $\binom{n_D}{2}$, hence by Lemma 2.3 that every peripheral vertex of an edge-maximal graph gives rise to a complete vertex sequence. But then by Lemma 2.2 G must be D-critical. Hence:

Lemma 2.4 For $D \geq 3$, G is edge-maximal over $\mathcal{G}_c = \mathcal{G}_c(n,*,\delta,D,K)$ if and only if G is edge-maximal over $\mathcal{C}_c = \mathcal{C}_c(n,*,\delta,D,K)$. ∎

Lemma 2.4 has two very important consequences: first, it makes clear that the search for edge-maximal graphs of $\mathcal{G}_c$ can be restricted to a search of $\mathcal{C}_c$; second, it allows results established for D-critical graphs (such as Lemmas 2.1, 2.2, 2.6, and 2.7 of this section) to be used in the proof of theorems characterizing edge-maximal graphs. Note also that Lemma 2.4 extends naturally to the classes $\mathcal{G}_c(n,*,*,D,K)$ and $\mathcal{C}_c(n,*,*,D,K)$.

The class $\mathcal{G}_v(*,*,\delta,D,K)$ has been studied by Klee and Quaife (1976), who derive expressions for the minimum order $n_* = n_*(\delta,D,K)$ attainable by any graph $G \in \mathcal{G}_v(*,*,\delta,D,K)$. The same expressions (Amar, Fournier & Germa 1983), or some of them (Seidman 1983), or special cases of them (Goldsmith, Manved & Faber 1981; Moon 1965) have been discovered or rediscovered by others.

This section ends with three simple but useful technical lemmas related to vertex sequences, all of them essentially due to Ore (1968). First, however, consider a vertex sequence $S_d(u)$ generated by any vertex u in a graph $G \in \mathcal{G}_c(*,*,\delta,*,K)$. Observe that a lower bound on the order of an internal triple of $S_d(u)$ is given by

$$M_c = \max\{\delta+1, 3K\}, \quad \text{when } c = v;$$
$$= \delta+1 \quad , \quad \text{when } c = e. \qquad \ldots(2.5)$$

Further, recalling (2.1), observe that when $G \varepsilon \mathscr{G}_c(*,*,*,*,K)$, the lower bound occurs when δ is set equal to K in (2.5); this is equivalent to choosing $G \varepsilon \mathscr{G}_c(*,*,K,*,K)$. In this context, then, an internal triple of $S_D(u)$ is said to be lean if its order is exactly M_c; otherwise fat. The lemmas may now be stated as follows:

Lemma 2.5 Suppose that $\beta = (x,y,z)$ is a lean internal triple contained in a vertex sequence of a graph $G \varepsilon \mathscr{G}_c(*,*,\delta,*,K)$. If β is embedded in a 5-tuple (u,β,v), then

(a) $u \geq z$; (b) $v \geq x$. ∎

Lemma 2.6 Let $\beta = (x,y,z)$ be a triple of a vertex sequence $S_D(u)$ corresponding to a peripheral vertex u of a graph $G \varepsilon \mathscr{C}_c(*,*,*,D,*)$. Then the size of β is $\binom{x+y+z}{2} - xz$. ∎

Lemma 2.7 Suppose there exist vertex sequences of length D of graphs in $\mathscr{C}_c(*,*,*,D,*)$ which contain the 4-tuples

$$\beta_1 = (w,x,y,z), \quad \beta_2 = (w,x-a,y+a,z), \quad a \geq 0.$$

Then the size of β_2 exceeds the size of β_1 by $a(z-w)$. ∎

2.2 K-Connected D-Critical Graphs $\mathscr{C}_v(n,*,*,D,K)$

In this section the main results for edge-maximal and edge-minimal graphs $G \in \mathscr{C}_v(n,*,*,D,K)$ are presented; then a methodology is described which permits the determination of whether the class $\mathscr{C}_v(n,m,*,D,K)$ is empty or not, and, if not, the construction of a graph in the class. Proofs of these results and further details will be found in Chapter 3.

Observe first (Kane & Mohanty 1978) that for any graph $G \in \mathscr{G}_v(n,*,\delta,D,K)$, it must be true that $n \geq n_*$, where $n_* = (D-3)K + 2(\delta+1)$. When δ is not specified, so that $G \in \mathscr{G}_v(n,*,*,D,K)$, n_* takes the form $(D-1)K + 2$, by virtue of the relation (2.1). In either case, G is said to have <u>excess</u> $a = n-n_*$. Independent of the choice of the parameter D, the following lemma is fundamental for K-connected graphs:

<u>Lemma 2.8</u> For every vertex sequence of a graph $G \in \mathscr{G}_v(n,*,\delta,*,K)$,

(a) every non-terminal term has order at least K;

(b) every terminal double has order at least $\delta+1$;

(c) every non-terminal triple has order at least $M_v = \max\{\delta+1, 3K\}$.

<u>Proof</u> Condition (a) follows from the K-connected constraint, (b) from the minimum degree constraint, and (c) from both constraints taken together. ∎

Note that for graphs of $\mathcal{G}_v(n,*,*,*,K)$, Lemma 2.8 still holds, but, again by (2.1), with δ replaced by K in conditions (b) and (c).

In view of Lemma 2.8(a), the terms of any vertex sequence (1.4) of a graph $G \in \mathcal{C}_v(n,*,*,D,K)$ can be expressed in the form

$$n_k = K + \delta_k, \qquad \delta_k \geq 0,$$

for every $1 \leq k \leq D-1$. Then corresponding to a peripheral vertex u of G, the vertex sequence takes the form

$$S_D(u) = (1, K+\delta_1, K+\delta_2, \ldots, K+\delta_{D-1}, 1), \qquad \ldots(2.6)$$

where now the excess

$$a = \sum_{1 \leq k \leq D-1} \delta_k. \qquad \ldots(2.7)$$

In fact the excess vertices can be isolated in the (D-1)-tuple

$$T_{D-1} = (\delta_1, \delta_2, \ldots, \delta_{D-1}), \qquad \ldots(2.8)$$

whose size is given by

$$i = \sum_{1 \leq k \leq D-2} \left[\binom{\delta_k}{2} + \delta_k \delta_{k+1} \right] + \binom{\delta_{D-1}}{2}. \qquad \ldots(2.9)$$

Let

$$b = \delta_1 + \delta_{D-1}. \qquad \ldots(2.10)$$

Then the following basic result can be stated:

<u>Lemma 2.9</u> For $D \geq 3$, suppose that a graph $G \in \mathscr{E}_v(n,*,*,D,K)$ has a vertex sequence (2.6); let a, i, and b be defined by (2.7), (2.9), and (2.10), respectively. Then the size of G is

$$m = K[(3D-5)K - (D-6a-5)]/2 + i - (K-1)b. \qquad \ldots(2.11)$$

<u>Proof</u> The size of the vertex sequence $S_1 = (1,K,K,\ldots,K,1)$ is

$$m_1 = \binom{K}{1}(D-1) + K^2(D-2) + 2K,$$

and the size of the (D-1)-tuple $S_2 = (\delta_1,\delta_2,\ldots,\delta_{D-1})$ is i. The edges induced between S_1 and S_2 are given by

$$m_2 = b(2K+1) + 2(a-b)K.$$

Then $m = m_1 + m_2 + i$, and the result is a matter of algebra. ∎

This important lemma can be used to establish the first main result of this section: Ore's characterization of edge-maximal

K-connected D-critical graphs (Ore 1968), which states essentially that the excess $a = n-n_*$ vertices are concentrated in at most two adjacent levels.

<u>Theorem 2.1</u> A graph $G \varepsilon \mathscr{C}_v(n,*,*,D,K)$, $D \geq 4$, is edge-maximal if and only if it has a vertex sequence $(1,n_1,n_2,\ldots,n_{D-1},1)$ which satisfies both of the following conditions:

(a) Let $j' = \min\{2,K\}$ and let $n_* = (D-1)K+2$. Then there exists an integer j satisfying $j' \leq j \leq D-j'$ such that

$$n_j \geq K,\ n_{j+1} \geq K,\ n_j + n_{j+1} = 2K + (n-n_*).$$

(b) For every integer i satisfying $1 \leq i < j$ or $j+1 < i \leq D-1$, $n_i = K$.

<u>Proof</u> Observe that (2.11) is maximized by choosing $i = \left\lceil \frac{a}{2} \right\rceil$ and, for $K > 1$, $b = 0$. It follows that the excess vertices must occur in at most two adjacent non-terminal levels which can include L_1 and L_{D-1} only when $K = 1$. ■

It is easy to see that for $2 \leq D \leq 3$, every graph $G \varepsilon \mathscr{C}_v(n,*,*,D,K)$ has the same size, and thus is both edge-maximal and edge-minimal. For $D \geq 4$, the details of the characterization of edge-minimal graphs turn out to be rather complicated, and are therefore left to be spelled out in Chapter 3. Nevertheless, the *idea* of the characterization is equally clear: for edge-minimal K-connected D-critical graphs, the first 2(K-1) of the excess vertices are divided evenly between the two levels L_1 and L_{D-1}, with any remaining excess being spread as uniformly as possible over non-adjacent non-terminal levels chosen from $L_1,\dots,L_{D-1}$.

On the assumption that edge-maximal and edge-minimal graphs of $\mathcal{C}_v(n,*,*,D,K)$ have been characterized, it is natural now to enquire which classes $\mathcal{C}_v(n,m,*,D,K)$ are non-empty, and if so, how to construct graphs in these classes. Observe first from (2.11) that, for sufficiently large K (for example, greater than $\lceil\frac{a}{2}\rceil$+b), there must be values of m which cannot be realized. (2.11) may also be used, in conjunction with Theorem 2.1 and the characterization of edge-minimal graphs, to compute bounds on m. For D ≥ 4, the upper bound is computed from (2.11) to be

$$m^* = [(3D-5)K^2 - (D-6a-5)K + a(a-1)]/2, \qquad \dots(2.12)$$

while expressions for the lower bound are found, after some computation, to be

$$m_* = [(3D-5)K^2 - (D-4a-5)K + (\lfloor a/2\rfloor^2+\lceil a/2\rceil^2+a)]/2, \qquad \dots(2.13)$$

$$\text{for } D \le 5 \text{ or } a \le 2(K-1);$$

$$= \left[(3D+5)K^2-(D-6a'+5)K\right]/2+\binom{a''}{2}\lfloor D/2\rfloor+a''\left(a' \bmod \lfloor D/2\rfloor\right), \quad \ldots(2.14)$$

for $D \geq 6$ and $a \geq 2(K-1)$;

where $a' = a-2K+2$ and $a'' = \lfloor a'/\lfloor D/2\rfloor\rfloor$. Then for $D \geq 4$, $\mathscr{E}_v(n,m,*,D,K)$ can possibly be non-empty only for

$$m_* \leq m \leq m^*, \quad \ldots(2.15)$$

where m_* and m^* are defined by (2.12)-(2.14).

Observe that the tuple (2.8) is slightly different from the tuples originally defined, in that possibly $\delta_k = 0$, for any $1 \leq k \leq D-1$, but Lemmas 2.1(b), 2.6, and 2.7 continue to hold, and therefore the previously-defined terminology can safely be used. Observe also that the expression (2.11) depends not only on the given parameters n, D, and K (hence a), but also on b and i, which are functions only of the δ_k. The dependence on the given parameters can be removed by writing

$$\begin{aligned} m' &= m - K\left[(3D-5)K - (D-6a-5)\right]/2 \\ &= i - b(K-1), \end{aligned} \quad \ldots(2.16)$$

and it is clear that the size m is feasible if and only if the δ_k of (2.8) can be chosen to yield b and i satisfying (2.16). Thus the feasibility/construction problem reduces to a consideration of the properties of tuples (2.8). The key lemmas are the following, which define a tuple's minimum size $\sigma(a, \lceil k/2\rceil)$ and establish its monotonicity.

Lemma 2.10 For any integer $k \geq 1$, the least size of a k-tuple of order $a \geq 0$ is

$$\sigma(a,k') = (a \bmod k')\lfloor a/k' \rfloor + k'\left\lceil \frac{\lfloor a/k' \rfloor}{2} \right\rceil,$$

where $k' = \lceil k/2 \rceil$. This size is attained by the k-tuple

$$T_*(a,k) = \left[(\lfloor a/k' \rfloor + 1, 0)^{a \bmod k'},\ (\lfloor a/k' \rfloor, 0)^{\lfloor k/2 \rfloor - a \bmod k'},\ \lfloor a/k' \rfloor^{k \bmod 2} \right].$$ ∎

(Throughout this document, the notation x mod y is used for integers $x \geq 0$, $y > 0$, to mean the remainder when x is divided by y: $x - y\lfloor x/y \rfloor$. This is consistent with Pascal and many other computer languages.)

Lemma 2.11 For integers $a \geq 0$, $k \geq 1$,

(a) $\sigma(a,k) - \sigma(a,k+1) \geq \left\lceil \frac{\lfloor a/(k+1) \rfloor + 1}{2} \right\rceil$;

(b) $\sigma(a+1,k) - \sigma(a,k) = \lfloor (a+1)/k \rfloor + \lceil [(a+1) \bmod k]/k \rceil - 1.$ ∎

Based on these lemmas, the main results can then be proved:

<u>Theorem 2.2</u> For given integers $a \ge 0$ and $k \ge 4$, there exists a k-tuple of size i for every integer i satisfying

$$\sigma\left(a, \lfloor k/2 \rfloor\right) \le i \le \binom{a}{2}. \quad \blacksquare$$

<u>Theorem 2.3</u> For $k \ge 8$, let $\mathcal{T} = \mathcal{T}(a,b,k)$ denote the set of all k-tuples $(\delta_1, 0, \delta_3, \delta_4, \ldots, \delta_{k-2}, 0, \delta_k)$ of order $a \ge b$ such that $\delta_1 + \delta_k = b \ge 0$. Then for $k' = \lfloor (k-4)/2 \rfloor$, every size i in the range

$$\left[\sigma(a-b, k') + \binom{\lfloor b/2 \rfloor}{2} + \binom{\lceil b/2 \rceil}{2},\ \binom{a-b}{2} + \binom{b}{2}\right]$$

is achieved by some element of $\mathcal{T}$, provided

$$b \le \hat{b} = (a-k'+k'') - \sqrt{k''(2a-2k'+k''-2)},$$

where $k'' = k'/(k'-1)$. When $k' \mid (a-b)$, this upper bound becomes

$$\hat{b} = \left\lfloor (a+k'') - \sqrt{k''(2a+k''-2)} \right\rfloor,$$

and is least possible. $\blacksquare$

<u>Theorem 2.4</u> For $k \ge 8$, let $\mathcal{T}_3 = \mathcal{T}_3(a,b,k)$ denote the set of all k-tuples $(\delta_1, \delta_2, \delta_3, 0, 0, \ldots, 0, \delta_{k-2}, \delta_{k-1}, \delta_k)$ of order $a \ge b$ such that $\delta_1 + \delta_k = b > 0$ and

$$\delta_2 + \delta_3 + \delta_{k-2} + \delta_{k-1} = a - b.$$

Then, for $x = \delta_1 + \delta_2 + \delta_3$, the size i of an element of $\mathcal{T}_3$ is given by

$$i = \binom{a}{2} - x(a-x) - \delta_1\delta_2 - \delta_{k-2}(b-\delta_1). \qquad \blacksquare$$

Theorems 2.2-2.4 form the basis of an algorithm to determine whether or not a given class $\mathscr{C}_v = \mathscr{C}_v(n,m,*,D,K) = \phi$, and, if not, to construct an element $G \in \mathscr{C}_v$. This algorithm is complicated and is therefore not described here; see Chapter 3 for details. The algorithm will deal efficiently with many cases (for example, the upper bound $\hat{b}$ of Theorem 2.3 is within $O(\sqrt{a})$ of a); but due to the fact that Theorems 2.2-2.4 do not provide a complete characterization of all sizes achievable by graphs $G \in \mathscr{C}_v$, there remain certain cases which can only be dealt with by "brute force" -- that is, by an exhaustive (and correspondingly time-consuming) inspection of every possible k-tuple of the specified order (as suggested by (2.9)). For example, achievable sizes specified by Theorems 2.3 and 2.4 do not exhaust all possible sizes of k-tuples of order a such that $\delta_1 + \delta_k = b$; an example given in Chapter 3 makes this clear. Further, there are certain values of b not covered by Theorem 2.3 which may give rise to feasible sizes $m = i - (K-1)b$: if for some value b, the corresponding size

$$i = m + (K-1)b$$

turns out to be not feasible, then, corresponding to b+1, it is necessary to try to achieve the size

$$i' = m + (K-1)(b+1) = i + (K-1).$$

If K > 1, and if the use of brute force is excluded, there is at present no sure methodology to determine whether size i′ is achievable or not.

2.3 K-Edge-Connected D-Critical Graphs $\mathcal{C}_e(n,*,*,D,K)$

In this section characterizations of edge-maximal and edge-minimal graphs $G \in \mathcal{C}_e(n,*,*,D,K)$ are provided. These characterizations are complete for edge-maximal graphs, but for edge-minimal graphs omit certain cases when n is small with respect to the product DK. Proofs of the results presented here, together with further details, will be found in Chapter 4. It follows from Lemma 2.8 that, when δ is not specified, a K-connected graph is fully characterized by the requirement that every non-terminal term of its vertex sequence be at least K. For K-edge-connected graphs, however, no such easy characterization is possible: as the next lemma makes clear, both doubles and triples need to be specified. Further, as will be apparent later, the doubles and triples conditions interact in a complex manner; it is precisely this interaction which makes the analysis of K-edge-connected graphs so much more difficult. In accordance with these observations, a new parameter

$$\alpha = \lceil 2\sqrt{K} \rceil \qquad \ldots(2.17)$$

is now introduced; as the following result shows, α is just the minimum order of a double in any vertex sequence of a K-edge-connected graph.

<u>Lemma 2.12</u> For every vertex sequence of a graph $G \in \mathcal{G}_e(n,*,\delta.*,K)$,

(a) every terminal double has order at least $\delta+1$;

(b) the product of the terms of every non-terminal double is at least K;

(c) every non-terminal triple has order at least $\delta+1$.

<u>Proof</u> Conditions (a) and (c) follow immediately from the minimum degree constraint. To prove (b), observe that in order to maintain K-edge-connectivity, there must be at least K edges between adjacent levels. ■

We shall say that a vertex sequence is <u>feasible</u> if it satisfies Lemma 2.12. This lemma of course applies also to graphs of $\mathcal{C}_e = \mathcal{C}_e(n,*,*,D,K)$, with $\delta = K$; in this case, it follows that, for $D \geq 3$, a necessary condition that $\mathcal{C}_e \neq \phi$ is given by

$$n \geq r(K+1) + r', \qquad \ldots(2.18)$$

where $r = \lfloor D/3 \rfloor + 1$, and

$$r' = D \bmod 3, \quad \text{for } D \bmod 3 \le 1;$$
$$= \alpha \quad , \quad \text{otherwise.}$$

In fact, for D mod 3 ≠ 1, it is easy to see that the condition (2.18) is also sufficient. For D mod 3 = 1 and D ≥ 7, however, suppose that equality holds in (2.18). This implies the existence of an internal term x = 1, which therefore by Lemma 2.12(b) must have at least one internal neighbouring term of order K. If x has two internal neighbours of order K, it follows that n > r(K+1) + K, a contradiction; but if x has just one internal neighbour of order K, it follows from Lemma 2.12(b) again that x has a neighbouring internal triple of order at least K + α, so that by Lemma 2.12(c), n ≥ r(K+1) + α, also a contradiction. Hence condition (2.18) is not sufficient when D mod 3 = 1. This little demonstration illustrates the difficulty of reasoning about the classes $\mathscr{C}_e$, whose vertex sequences satisfy doubles and triples conditions, but no non-trivial single term condition. The same point arises in the first main theorem, which once again gives a necessary condition only:

<u>Theorem 2.5</u> For D ≥ 6 and K ≥ 8, every edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence in which every internal triple except possibly $(n_{D-4}, n_{D-3}, n_{D-2})$ is lean. ■

This result corresponds closely to Theorem 2.1 for K-connected graphs and expresses a similar condition: that the excess vertices are concentrated in one level (either n_2 or n_{D-2}). For K-edge-connected graphs, however, the exact value of the excess can be difficult to specify, as the preceding discussion has shown. To see that the condition of Theorem 2.5 is not sufficient, consider two graphs $G_1, G_2 \in \mathscr{C}_e(153,*,*,6,50)$ with corresponding vertex sequences

$$S_1 = (1,50,7,36,8,50,1), \quad S_2 = (1,50,36,5,10,50,1);$$

neither S_1 nor S_2 contains a fat triple, but G_2 has 5765 edges, while G_1 has only 4519. Finally, an even more serious deficiency of Theorem 2.5 is that it does not determine the structure of the graph G: as Tables 2.4 and 2.5 at the end of this section show, the lean internal triple which determines maximum size is not fixed, but can vary widely as a function of n for fixed D and K.

For $2 \le D \le 3$, it has already been noted in Section 2.2 that every graph of $\mathscr{C}_e$ has the same size, while for $4 \le D \le 5$, edge-maximal graphs of $\mathscr{C}_e$ are exactly those whose terminal doubles are of order K+1. Vertex sequences for these edge-maximal graphs are illustrated in Table 2.2.

Edge-Maximal Vertex Sequences for $\mathscr{C}_e(n,*,*,D,K)$, $2 \le D \le 5$

D	Vertex Sequence	Conditions
2	(1,K+a,1)	a≥0
3	(1,K+a,K+b,1)	a≥0, b≥0
4	(1,K,a,K,1)	a≥1
5	(1,K,a,b,K,1)	a≥0, ab≥K

Table 2.2

To express more precise results for $D \ge 6$, further definitions are required. First, the definition of "lean" is extended to doubles: in any vertex sequence of a graph of $\mathscr{C}_e$, an internal double is said to be lean if its order is exactly α; otherwise, fat. Then a graph of $\mathscr{C}_e$ is said to be vertex-minimal if it has a vertex sequence of length D for which one of the following conditions is satisfied:

(for $K \le 7$) Every internal double is lean.

(for $K \ge 8$) Every internal triple is lean.

By extension, a vertex sequence satisfying one of these conditions is also said to be vertex-minimal. Note that, for given D and K, the fact that a vertex sequence is vertex-minimal does not fix n: for example, for $D = 7$ and any $K \ge 9$, both $(1,K,3,\lceil K/3\rceil,K-\lceil K/3\rceil-2,3,K,1)$ and $(1,K,4,\lceil K/4\rceil,K-\lceil K/4\rceil-3,4,K,1)$ are vertex-minimal. Moreover, for $K = 2$ or 3, there may even exist vertex sequences which are not vertex-minimal but which have the same order and size as a vertex-minimal vertex sequence: compare, for example,

$$S = (1,2,1,2,1,2,2,1) \text{ and } S' = (1,2,1,2,2,1,2,1),$$

or $$S = (1,3,3,1,3,3,1) \text{ and } S' = (1,3,1,3,3,3,1).$$

A basic result can now be stated:

Lemma 2.13 Every vertex-minimal vertex sequence of a graph of $\mathscr{C}_e(n,*,*,D,K)$ is h-recurring, where

$$h = 2, \quad \text{for } K \le 7;$$
$$= 3, \quad \text{for } K \ge 8.$$

Proof An immediate consequence of the definitions. ∎

Lemma 2.13 deals with vertex-minimal vertex sequences for all values of K, while Theorem 2.5 deals with arbitrary vertex sequences for K ≥ 8. For K ≤ 7, then, in order to present a complete picture, a counterpart to Theorem 2.5 is required. It turns out that, as a result of the conflict between the doubles and the triples conditions (Lemma 2.12(b) and 2.12(c), respectively), the corresponding theorem for K ≤ 7 deals with numerous special cases and is therefore complicated to state. For this reason, the statement of the theorem for K ≤ 7 is delayed until Section 4.1: note however that, except for very special cases, the result is essentially the same as Theorem 2.5: excess vertices are concentrated either in n_2 or n_{D-2}.

In order to facilitate further discussion, several new quantities are now defined. Let α_1 be the least integer such that $\alpha_1(\alpha-\alpha_1) \geq K$. Then set $\alpha_2 = \alpha-\alpha_1$, so that α_1 and α_2 are, respectively, the least and greatest terms of a lean double. In order to have, for $K \geq 8$, a corresponding representation of a lean triple, let $K_\alpha = K+1-\alpha$. The quantities α_1, α_2, and K_α are important throughout the discussion of K-edge-connected graphs.

Turning now to edge-maximal graphs for $K \geq 8$, observe that by Theorem 2.5 every edge-maximal graph of $\mathscr{C}_e$ has a vertex sequence

$$S_D = (1, K, (x, y, z)^{r-1}, \beta, K, 1), \qquad \ldots(2.19)$$

where as above $r = \lfloor D/3 \rfloor - 1$, (x,y,z) is a lean triple, and β denotes the tuple

$(x, y, z+a)$, for $D \bmod 3 = 0$;
$(x, y, z, x+a)$, for $D \bmod 3 = 1$;
$(x, y, z, x, y+a)$, otherwise;

where the <u>excess</u> $a \geq 0$. The maximum number $m^* = m^*(n,D,K)$ of edges represented by (2.19) may therefore be expressed as follows:

$$m^* = (r+2)\binom{K+1}{2} + \binom{n^*}{2} + f^*(x,y,z;a), \qquad \ldots(2.20)$$

where $n^* = n - (r+2)(K+1)$ and f^* is a function determined by choosing x, y, and z (hence a) so as to maximize

$$f(x,y,z;a) = \begin{cases} K(x+z+a)+a(y+z)-xz, & \text{if } D \bmod 3 = 0; & \ldots(2.21) \\ K(2x+a)+az, & \text{if } D \bmod 3 = 1; & \ldots(2.22) \\ K(x+y+a), & \text{otherwise.} & \ldots(2.23) \end{cases}$$

From (2.23) it follows immediately that, since $n^* = x+y+a$ for $D \bmod 3 = 2$, m^* is constant. Hence in this case the condition of Theorem 2.5 is also sufficient:

Theorem 2.6 For $D \geq 6$, $D \bmod 3 = 2$, and $K \geq 8$, a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-maximal if and only if G has a vertex sequence in which every internal triple except possibly $(n_{D-4}, n_{D-3}, n_{D-2})$ is lean. ∎

Moreover, in the case $D \bmod 3 = 1$, it follows from (2.22) that $f(x,y,z;a) = 2Kx = 2Kn^*$ for $a = 0$, so that m^* is once again constant. This remark justifies condition (a) of the following result; condition (b) is proved in Section 4.2.

Theorem 2.7 For $D \geq 6$ and $K \geq 8$, a vertex-minimal graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-maximal if and only if one of the following conditions is satisfied:

(a) $D \bmod 3 \neq 0$;

(b) G has a vertex sequence (2.19) where

$$(x,y,z) = \begin{cases} (\alpha_2, \alpha_1, K_\alpha), & \text{if } K \neq 16, 19, 20, 22\text{–}24, 26, 27, 36; \\ (\alpha_2+3, \alpha_1-2, K_\alpha-1), & \text{if } K = 36, 49 \\ (\alpha_2+2, \alpha_1-1, K_\alpha-1), & \text{otherwise.} \end{cases}$$ ■

Since the result for D mod 3 = 0 is complex for vertex-minimal graphs, it is not surprising that it turns out to be even more complex for $a > 0$:

<u>Theorem 2.8</u> For $D \geq 6$, $D \bmod 3 = 0$, and $K \geq 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $a \leq \alpha-1$. Suppose further that integers a^* and $a^\ddagger$ are given, where $a^* = a^\ddagger = 0$ except as shown in the following table:

K	a^*	$a^\ddagger$
16	2	2
19	2	2
20	1	2
22	2	2
23	1	2
24	1	1
26	1	2
27	1	1
36	2	2

Table 2.3

Then G is edge-maximal if and only if the vertex sequence (2.19) satisfies one of the following conditions:

(a) $0 \le a \le a^*$ and (x,y,z) is specified by Theorem 2.7(b);

(b) $a^\ddagger \le a < \alpha-1$ and $(x,y,z) = (\alpha_2, \alpha_1, K_\alpha)$;

(c) $a = \alpha-1$ and $z = K_\alpha$. ∎

In order to state the result for $a \ge \alpha$ economically, it is convenient to introduce the idea of a <u>transformation</u> which carries a k-tuple of a vertex sequence into another k-tuple of the same order. A transformation is said to be <u>feasible</u> if both the original vertex sequence and the transformed one are feasible.

<u>Theorem 2.9</u> For $D \ge 6$, $D \bmod 3 = 0$, and $K \ge 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $a \ge \alpha$. Then G is edge-maximal if and only if there exists no feasible transformation

$$\tau : (x,y,z) \longrightarrow (x',y',z')$$

of (2.19) satisfying all of the following conditions:

(a) $x' < x$, $y' > y$, $z' < z$;

(b) $y' = \lceil K/x' \rceil$;

(c) $z+a > [K(y'-y)+x'(z'-z)]/(x-x')$. ∎

In Chapter 4 a more computationally useful form of Theorem 2.9 is derived. Here the final result for edge-maximal graphs is the following:

<u>Theorem 2.10</u> For $D \geq 6$, $D \bmod 3 = 1$, and $K \geq 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $n^* > K_\alpha$. Let j^* and k^* be the values of j and k which maximize

$$(i+j)\left\{a-i\left[(K-\alpha_2)/(i+j) - 1\right]\right\}$$

over $0 \leq j \leq \alpha_1-3$ and $1 \leq k \leq 2$, where

$$\begin{aligned} i = i(j,k) &= \lceil K/(\alpha_1-j)\rceil-\alpha_2-j, \quad \text{for } k = 1; \\ &= K_\alpha - \lceil K/(\alpha_1-j)\rceil, \quad \text{for } k = 2. \end{aligned}$$

Let $i^* = i(j^*,k^*)$. Then G is edge-maximal if and only if

$$(x,y,z) = (K_\alpha-i^*,\alpha_1-j^*,\alpha_2+i^*+j^*).$$

Further, there exist integers $a_* \leq \alpha-1$ and $a^* \geq \alpha-1$ such that

(a) for $a \leq a_*$, $(x,y,z) = (K_\alpha,\alpha_1,\alpha_2)$;

(b) for $a \geq a^*$, $(x,y,z) = (\alpha_2,\alpha_1,K_\alpha)$. ∎

To conclude the discussion of edge-maximal graphs, and to convey some appreciation of the significance of Theorems 2.6-2.10, Tables 2.4 and 2.5 give numerical examples of edge-maximal vertex sequences for $K = 16$ ($\alpha = 8$, $\alpha_1 = \alpha_2 = 4$, $K_\alpha = 9$) and $K = 70$ ($\alpha = 17$, $\alpha_1 = 7$, $\alpha_2 = 10$, $K_\alpha = 54$), respectively.

Edge-Maximal Vertex Sequences, K = 16, D mod 3 = 1

a	(x,y,z)
0-5	(9,4,4)
5-7	(8,3,6)
7-11	(6,3,8)
≥11	(4,4,9)

Table 2.4

Edge-Maximal Vertex Sequences, K = 70, D mod 3 = 0

a	(x,y,z)
0	(12,6,53)
0-15	(10,7,54)
16	(9,8,54)
	(8,9,54)
	(7,10,54)
17-80	(7,10,54)
80-82	(6,12,53)
82-216	(5,14,52)
216-356	(4,18,49)
≥356	(3,24,44), for D > 6
356-706	(3,24,44), for D = 6
706-2392	(2,35,34), for D = 6
≥2392	(1,70,0), for D = 6

Table 2.5

What has been presented above amounts to a complete characterization of edge-maximal graphs of $\mathscr{C}_e$, hence by Lemma

2.4 of $\mathscr{C}_e$. For edge-minimal graphs, two main results are presented, which provide a characterization for "larger" values of n. Recall that for $2 \le D \le 3$ every graph in $\mathscr{C}_e$ is edge-minimal. Then the first of these results depends upon the observation that, for even $D \ge 4$ and sufficiently large n, $\mathscr{C}_e(n,*,*,D,K) \subseteq \mathscr{C}_v(n,*,*,D,1)$.

<u>Theorem 2.11</u> For even $D \ge 4$ and $n \ge D(K+1)/2 + 1$, a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-minimal if and only if G is edge-minimal over $\mathscr{C}_v(n,*,*,D,1)$. ∎

For even D, then, the characterization depends essentially on Theorems 3.1 and 3.2. For odd D, however, the dependence is indirect and more complicated:

<u>Theorem 2.12</u> For odd $D \ge 5$ and $n \ge (D-1)K + 2$, a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-minimal if and only if it has a vertex sequence which satisfies all of the following conditions:

(a) $n_j = 1$, for $j = D-1$ and $j = 2,4,\ldots,D-5$;

(b) $\min\{n_{D-4}, n_{D-5}\} \ge K$;

(c) for every $1 \le j \le (D-5)/2$,

  (i) $n_{2j} \ge 2K-1$;

  (ii) for every $1 \le j' \le (D-5)/2$,

$$|n_{2j-1}-n_{2j'-1}| \le 1;$$

  (iii) $|n_{2j-1}-n_{D-1}| \le 1$;

  (iv) $|n_{2j-1}-n_{D-4}-n_{D-3}| \le 1$. ∎

These two results are proved in Section 4.3. From (2.18) it follows that the missing values of n lie in relatively narrow ranges:

$$\left(\lfloor D/3\rfloor+1\right)(K+1) + r' \le n \le (D/2)(K+1) + 1, \text{ n even;}$$

$$\left(\lfloor D/3\rfloor+1\right)(K+1) + r' \le n \le (D-1)K + 2, \text{ n odd.}$$

No work has been done on the existence or construction of graphs of $\mathscr{C}_e(n,m,*,D,K)$, as described in Section 2.2 for graphs of $\mathscr{C}_v$. Since there is no single term condition for K-edge-connected graphs, it is clear that to deal with this problem, a quite different methodology would be required. In general, the problem appears to be very difficult.

2.4 Upper Bounds on the Diameter of Graphs of $\mathscr{G}_v(n,m,\delta,*,K)$

The results described in this section derive ultimately from the realization that diameter-critical graphs are essentially graphs of large diameter, and further that, by applying the methodology developed for the class $\mathscr{C}_e(n,*,*,D,K)$, it should be possible to characterize edge-maximal graphs of $\mathscr{G}_v(n,*,\delta,D,K)$; hence to determine a sharp upper bound on the diameter of any graph of $\mathscr{G}_v(n,m,\delta,*,K)$. Recall from Section 1 that for graphs of $\mathscr{G}_v(*,*,\delta,*,K)$, a lean triple has order $M_v = \max\{\delta+1, 3K\}$. Then the basic result is as follows (compare Theorem 2.5):

<u>Theorem 2.13</u> For $D \geq 6$, there exists an edge-maximal graph of $\mathcal{G}_v(n,*,\delta,D,K)$ with a vertex sequence in which every internal triple except possibly $(n_{D-4}, n_{D-3}, n_{D-2})$ is lean. ∎

For $D \leq 5$ the edge-maximal graphs can be determined by inspection, as shown in the following table:

Edge-Maximal Graphs of $\mathcal{G}_v(n,*,\delta,D,K)$, $D \leq 5$

D	Vertex Sequence S_D	$m^*(n,\delta,D,K)$
1	$(1,n-1)$, $n-1 = \delta = K$	$\binom{n}{2}$
2	$(1,\delta,n-\delta-1)$	$\binom{n-1}{2} + \delta$
3	$(1,n-(x+2),x,1)$, $\delta = x$ or $n-(x+2)$	$\binom{n-1}{2}$
4	$(1,\delta,n-2(\delta+1),\delta,1)$, $n-2(\delta+1) \geq K$	$\binom{n-2}{2} - \delta(\delta-2)$
5	$(1,\delta,x,n-2(\delta+1)-x,\delta,1)$, $K \leq x \leq n-2(\delta+1)-K$	$n(n-2\delta-5)/2 + (\delta+1)(\delta+3)$

Table 2.6

For $D \geq 6$, however, in order to make use of Theorem 2.13, a derivation is required very similar to that described in Section 2.3 for edge-maximal graphs of $\mathcal{G}_e(n,*,*,D,K)$, with the slight additional complication that the minimum degree δ must now be considered. Accordingly, generalizing (2.19), let

$$S_D = (1, \delta, (x,y,z)^{r-1}, \beta, \delta, 1) \qquad \ldots(2.24)$$

be a vertex sequence of an edge-maximal graph of $\mathcal{G}_v(n,*,\delta,D,K)$, where $r = \lfloor D/3 \rfloor - 1$, (x,y,z) is a lean triple, β denotes the tuples

$$(x,y,z+a), \quad (x,y,z,x+a), \quad (x,y,z,x,y+a),$$

according as D mod 3 = 0, 1, or 2, respectively, and the excess $a \geq 0$. Then the maximum number of edges $m^* = m^*(n,\delta,D,K)$ corresponding to (2.24) is given by

$$m^* = r\binom{M_v}{2} + 2\binom{\delta+1}{2} + \binom{n^*}{2} + f^*(x,y,z;a), \qquad \ldots(2.25)$$

where $n^* = n - rM_v - 2(\delta+1)$ and f^* is a function determined by choosing x, y, and z (hence a) to maximize

$$f(x,y,z;a) = \begin{cases} \delta(x+z+a) + a(y+z) - xz, & D \bmod 3 = 0; \qquad \ldots(2.26) \\ \delta(2x+a) + az, & D \bmod 3 = 1; \qquad \ldots(2.27) \\ \delta(x+y+a), & \text{otherwise.} \qquad \ldots(2.28) \end{cases}$$

After a calculation similar to that outlined in Section 2.3, the following result is derived:

<u>Theorem 2.14</u> For $D \geq 5$, the size m^* of an edge-maximal graph of $\mathcal{G}_v(n,*,\delta,D,K)$ is given by (2.30), where

(a) for $D \bmod 3 = 0$,

$$f^* = (\delta+K+K')n^* + (K'\delta+\delta K-KK');$$

(b) for $D \bmod 3 = 1$,

(i) $f^* = 2\delta n^*$,

for $a \leq 2K-1$ and $a \leq K'-K$, <u>or</u>

for $2K \leq a \leq K'-K-2$;

(ii) $f^* = 2\delta n^* - (\delta-K)(n^*-K')$,

for $a \leq 2K-1$ and $a > K'-K$;

(iii) $f^* = 2\delta n^* - (\delta-K')(n^*-K)$,

for $a \geq 2K$ and $a \geq K'-K-1$;

(c) for $D \bmod 3 = 2$, $f^* = \delta n^*$;

and $K' = \delta+1-2K$. ∎

From this result, together with Table 2.6, it is tedious but straightforward to prove

<u>Theorem 2.15</u> For fixed n, δ, and K, the function $m^*(n,\delta,D,K)$ specified by (2.25) is monotone decreasing in D.

This monotonicity property, together with the computational results of Theorem 2.14, makes possible the determination of an upper bound on the diameter of a graph of $\mathcal{G}_v(n,m,\delta,*,K)$. To express this bound, consider a set of quadratic equations

$$F_j(r) = r^2 - [b_j+2(n-1)]r + [c_j+n(n-3)-m] = 0, \quad 1 \le j \le 5, \qquad \ldots(2.29)$$

whose coefficients b_j and c_j are given by the following table.

Coefficients of $F_j(r)$ (where $\gamma = \delta+1$)

j	b_j	c_j
1	$-2\gamma+M_v-2K$	$n(-2\gamma+2M_v-2K) + \gamma(2\gamma-2M_v+2K+4) - K(2M_v-4K-2) - 2M_v$
2	$-M_v-2$	$-2n - \gamma(2\gamma-8)$
3	$-2\gamma-M_v+2K$	$n(-2\gamma+2K) - \gamma(2\gamma+2M_v-8K+4) - K(2M_v-4K-4) - 2M_v$
4	$-2\gamma+M_v-4K$	$n(-2\gamma+2M_v-4K) + \gamma(2\gamma-4M_v+10K+4) - K(2M_v-4K+2)$
5	$-2\gamma-M_v$	$-2\gamma n + \gamma(2\gamma+4)$

Table 2.7

Suppose now that each of the equations (2.29) gives rise to real roots, and let r_j, $1 \le j \le 5$, denote the lesser of each pair. Then set

$$D_j^* = 3\lfloor r_j/M_v \rfloor + \lceil (j-1)/3 \rceil + 3, \quad 1 \le j \le 5. \qquad \ldots(2.30)$$

<u>Theorem 2.16</u> For a graph of $\mathcal{G}_v(n,m,\delta,*,K)$, where $m < m^*(n,\delta,5,K)$, the maximum diameter

$$D^* = D^*(n,m,\delta,K) = \min(D_1^*, D_{j'}^*, D_5^*),$$

where $j' = 2$, 3, or 4 according as

$$a = n - (\lfloor D/3 \rfloor - 1)M_v - 2\gamma - K,$$

satisfies Theorem 2.14(b) (i), (ii), or (iii), respectively; and the D_j^*, $1 \le j \le 5$, are specified by (2.29), (2.30), and Table 2.7. ∎

It is not clear what, if any, algorithmic impact the bound expressed by Theorem 2.16 will have. Although the bound is sharp for the class $\mathcal{G}_v(n,m,\delta,*,K)$, it can nevertheless be much larger than the diameter of an individual graph in the class; although this effect is reduced substantially for graphs of higher connectivity, the difficulty then arises that the connectivity K of a graph is generally not known and is roughly as difficult to determine as the diameter D. Thus, while Theorem 2.16 does represent an advance in quantifying the relationship among the parameters n, m, δ, D, and K, there is at the present time no specific algorithm to which it directly relates. Future work in this general area might focus on any of the following topics:

* the determination of the maximum diameter of a graph of $\mathcal{G}_e(n,m,\delta,*,K)$;

* algorithms for determining a lower bound on the (edge-) connectivity of a graph of $\mathcal{G}_c(n,m,\delta,D,*)$;

* fast recognition algorithms for graphs of "large" or "small" diameter (generalizing, for example, the well-known result that if $(n-1)/2 \le \delta < n-1$, then $D = 2$);

* numerical experimentation based on Theorems 2.16 and 2.17 (see below).

This survey is concluded by quoting a form of a result mentioned in Section 1, due to Klee and Quaife (1976). In contrast to the calculation described here of an upper bound on the size of graphs of $\mathcal{G}_V(n,*,\delta,D,K)$, Klee and Quaife determine a lower bound on the order of graphs of $\mathcal{G}_V(*,*,\delta,D,K)$; just as the upper bound found here was inverted to yield an upper bound on D, so also can Klee and Quaife's lower bound be inverted, to yield

<u>Theorem 2.17</u> For a graph of $\mathcal{G}_V(n,*,\delta,*,K)$, where $\delta < (n-1)/2$, the maximum diameter D^* is given by

$$D^* = \lfloor [n+3K-2(\delta+1)]/K \rfloor, \quad \delta \le 3K - 1;$$
$$= 3\lfloor n/(\delta+1) \rfloor + \min\left\{ \lfloor [n \bmod (\delta+1)]/K \rfloor, 2 \right\} - 3,$$

otherwise. ∎

CHAPTER 3

K-CONNECTED D-CRITICAL GRAPHS $\mathscr{C}_v(n,*,*,D,K)$

In this chapter edge-minimal graphs of $\mathscr{C}_v = \mathscr{C}_v(n,*,*,D,K)$ are characterized (the characterization of edge-maximal graphs was completed in Section 2.2), and then results are proved which partially characterize graphs of $\mathscr{C}'_v = \mathscr{C}_v(n,m,*,D,K)$. Based on these results, an algorithm is described which determines, for given n, m, D, and K, whether or not $\mathscr{C}'_v$ is empty; and if not, constructs a graph $G \in \mathscr{C}'_v$. The main references for this chapter are Ore (1968) and Caccetta & Smyth (1988b, 1989d); the main results of this chapter are summarized in Section 2.2.

3.1 Edge-Minimal Graphs over $\mathscr{C}_v(n,*,*,D,K)$

As indicated in Section 2.2, the main results of this section can be summarized by the statement that, for $D \geq 4$, a graph G of $\mathscr{C}_v$ is edge-minimal if and only if the first 2(K-1) of its excess vertices are partitioned as uniformly as possible between levels L_1 and L_{D-1} of the level structure corresponding to G, and the remaining excess vertices distributed as uniformly as possible over non-adjacent non-terminal levels. However, the technical details of a precise characterization became somewhat complicated. In this section these details are spelled out, making use of preliminary results which have application also in the development of Section 3.2.

Recall that in Section 2.2, in order to discuss the construction of graphs of $\mathscr{C}'_v$, it was convenient to consider a (D-1)-tuple (2.8) in which any term δ_j, $1 \le j \le D-1$, could be zero. Here this idea is generalized slightly to denote by

$$T(a,k) = (\delta_1, \delta_2, \dots, \delta_k) \qquad \dots(3.1)$$

a k-tuple, $k \ge 1$, of order $a \ge 0$. Similarly, the use of edge-minimal is generalized to describe a k-tuple T(a,k) of size not greater than any other k-tuple of order a. A k-tuple will be called partitionable if for every j satisfying $1 \le j \le \lceil k/2 \rceil - 1$, δ_{2j} and δ_{2j+1} are not both non-zero. A partition of a partitionable k-tuple is the set of $\lfloor k/2 \rfloor$ doubles $(\delta_{2j-1}, \delta_{2j})$, $1 \le j \le \lfloor k/2 \rfloor$, together with the single term δ_k when k is odd. Each of the $\lceil k/2 \rceil$ elements of a partition is called a clump. With this nomenclature a fundamental lemma can now be proved:

Lemma 3.1 For $a \ge 0$ and $k \ge 1$, a k-tuple $T = T(a,k)$ on a vertices is edge-minimal if and only if

(a) T is partitionable;

(b) the orders of each pair of clumps of T differ by at most one.

Proof Suppose first that $T = (\delta_1, \delta_2, \dots, \delta_k)$ is not partitionable, and observe that therefore the sum

$\sum^* = \sum_{1\le j\le \lceil k/2\rceil -1} \delta_{2j}\delta_{2j+1} > 0$. Now consider the k-tuple

$$T^* = T^*(a,k) = (\delta'_1, \delta'_2, \ldots, \delta'_k),$$

where

$$\begin{aligned} \delta'_j &= \delta_j + \delta_{j+1}, \quad \text{for } j \text{ odd;} \\ &= 0 \qquad\qquad\;\; , \quad \text{otherwise.} \end{aligned}$$

Observe then that the size of T exceeds the size of T^* by exactly $\sum^*$, so that T cannot be edge-minimal.

Suppose next that T is partitionable, but that condition (b) is false. Hence there exist two clumps of orders c_1 and c_2 such that $c_1 - c_2 \ge 2$. But then T cannot be edge-minimal, because the transformation

$$\mathcal{T}_1 : c_1 \rightarrow c_1 - 1, \quad c_2 \rightarrow c_2 + 1$$

would yield a k-tuple on a vertices whose size was less than that of T by

$$\binom{c_1}{2} - \binom{c_1-1}{2} + \binom{c_2}{2} - \binom{c_2+1}{2} = c_1 - c_2 - 1.$$

This proves necessity. To prove sufficiency, suppose that both conditions (a) and (b) are true, and observe that any rearrangement of T incompatible with (a) and (b) would either make some $\delta_j > \lfloor a/k' \rfloor + 1$ or make some zero term non-zero, or both; since all of these changes necessarily increase the size, it follows that T must be edge-minimal. This completes the proof. ∎

Let a double (x_1, x_2) such that $x_1 > 0$, $x_2 > 0$, be called a <u>non-trivial</u> double. Then Lemma 3.1 has the following immediate corollary:

<u>Lemma 3.2</u> For $a \geq 0$, and $k \geq 1$, an edge-minimal k-tuple $T(a,k)$ contains

(a) no non-trivial doubles, if k is odd;

(b) at most one non-trivial double, if k is even. ∎

Lemmas 3.1 and 3.2 deal with the problem of minimizing the size of an unconstrained k-tuple, but from Lemma 2.9 it is clear that for $K > 1$ the problem of determining an edge-minimal graph of $\mathscr{C}_v$ will require the minimization of the quantity

$$g(i,b) = i - (K-1)b.$$

This minimization problem is fortunately simplified by the first main characterization theorem:

<u>Theorem 3.1</u> Suppose that $K \geq 2$, $n_* = (D-1)K+2$, and $n \geq n_*$. Then for $4 \leq D \leq 5$, or for $D \geq 6$ and $n \leq (D+1)K$, a graph $G \in \mathscr{C}_v(n,*,*,D,K)$ is edge-minimal if and only if it has a vertex sequence

$$(1, K+\delta_1, K, K, \ldots, K, K+\delta_{D-1}, 1)$$

of length D in which $\delta_1 + \delta_{D-1} = n-n_*$ and $|\delta_1-\delta_{D-1}| \leq 1$.

<u>Proof</u> Since $n \leq (D+1)K$, it follows that the excess $a = n-n_* \leq 2(K-1)$. By Lemma 2.8(c), the minimum order of a non-terminal triple is 3K, greater by K-1 than the minimum order of each terminal triple. Then for $a \leq 2(K-1)$, the excess vertices must be distributed only to δ_1 and δ_{D-1}: any k-tuple such that $\delta_j > 0$, for some $1 < j < D-1$, must have size greater than any other k-tuple of the same order, where δ_j is reduced by one, and either $\delta_1 < 2K-1$ or $\delta_{D-1} < 2K-1$ is correspondingly increased by one. Further, the distribution to δ_1 and δ_{D-1} must be as uniform as possible, so that $|\delta_1-\delta_{D-1}| \leq 1$, since otherwise the size could be increased by transferring a vertex from larger to smaller. Thus for $K \geq 2$ and $n \leq (D+1)K$, uniform distribution to δ_1 and δ_{D-1} is a necessary and sufficient condition that G be edge-minimal. ∎

Theorem 3.1 has been stated to apply only for $K \geq 2$ in order to avoid certain trivial special cases which arise for $K = 1$. In fact, it is easy to see that Theorem 3.1 holds for $D \geq 6$, $n \leq (D+1)K$, and $K = 1$, since in this case the excess is zero. Moreover, for $4 \leq D \leq 5$, the condition is sufficient when $K = 1$; that it is not necessary is a consequence of the following possibilities:

* when $n = D+2 = n_*+1$, the single excess vertex may be added to any level $L_1, L_2, \ldots, L_{D-1}$;

* when $D = 5$ and $n-n_* \geq 2$, the excess vertices may be distributed uniformly over levels L_1 and L (rather than L_1 and L_3 as the theorem states).

As a preliminary to the second main theorem, three more lemmas are proved:

<u>Lemma 3.3</u> Suppose $D \geq 4$ and let $S = (n_0, n_1, \ldots, n_D)$ denote a vertex sequence of an edge-minimal graph $G \in \mathscr{C}_v(n,*,*,D,K)$. Then for $K \geq 2$, $n_2 = n_{D-2} = K$; for $K = 1$, S has the same size as an edge-minimal vertex sequence S′ of the form $(1, n_1', 1, n_3', n_4', \ldots, n_{D-3}', 1, n_{D-1}', 1)$.

Proof By Lemma 2.7, if $n_2 = K+a$, $a > 0$, then the size of the 4-tuple

$$\beta_2 = (1, n_1, K+a, n_3)$$

exceeds that of

$$\beta_1 = (1, n_1+a, K, n_3)$$

by $a(n_3-1) \geq a(K-1)$. For $K \geq 2$, this is a contradiction; for $K=1$, the size difference is zero. Similarly for n_{D-2}. ∎

Lemma 3.4 For $n \geq (D+1)K$, every vertex sequence of every edge-minimal graph of $\mathscr{C}_v(n,*,*,D,K)$ has $n_1 \geq 2K-1$, $n_{D-1} \geq 2K-1$.

Proof The result is trivial for $K = 1$. Then suppose that $K > 1$ and further that the lemma is false. Then some $\delta_j > 0$, $1 < j < D-1$, and as in the proof of Theorem 3.1, it is clear that the size must be increased by reducing δ_j by one and at the same time increasing either δ_1 or δ_{D-1} by one, a contradiction. ∎

<u>Lemma 3.5</u> For $t = n-(D+1)K \geq 0$, the size of an edge-minimal graph $G \in \mathscr{C}_V(n,*,*,D,K)$ is

$$m = K\left[(3D+5)K - (D-6t+5)\right] + i,$$

where i is the minimum size of a (D-1)-tuple of order t.

<u>Proof</u> By Lemmas 3.3 and 3.4, m may be computed from the vertex sequence

$$(1, 2K-1+\delta_1, K, K+\delta_3, K+\delta_4, \ldots, K+\delta_{D-3}, K, 2K-1+\delta_{D-1}, 1),$$

where $t = \sum_{1 \leq j \leq D-1} \delta_j$. Hence

$$m = \binom{K}{2}(D-3) + K^2(D-4) + 2\binom{2K}{2} + 2K(2K-1) + 3tK + i,$$

from which the result follows. ∎

From this last result, it follows that Lemma 3.1 may be applied to characterize edge-minimal graphs of $\mathscr{C}_V$. In order to state the characterization theorem precisely, further definitions are required. In the spirit of the definition of "lean" for triples, let a non-terminal term be called <u>lean</u> if it has order K; otherwise, <u>fat</u>. For integers h and k, let $\mathscr{S}(h,k)$ denote the set of all distinct k-tuples consisting of h non-adjacent fat terms of order K+1 and k−h lean terms. Note that $\mathcal{S}(h,k)$ is non-empty if and only if $k \geq 1$, $0 \leq h \leq \lceil k/2 \rceil$. For example,

$$\mathcal{S}(2,4) = \Big\{(K,K+1,K,K+1),(K+1,K,K+1,K),(K+1,K,K,K+1)\Big\}.$$

Then for $k \geq 6$, define the sets of (k+1)-tuples

$$\mathcal{S}_1(h,k) = \Big\{(1,2K-1,K,S,K,2K-1,1) \mid S \in \mathcal{S}(h,k-5)\Big\},$$
$$\mathcal{S}_2(h,k) = \Big\{(1,2K-1,K,S,K,2K,1) \mid S \in \mathcal{S}(h-1,k-5)\Big\},$$
$$\mathcal{S}_3(h,k) = \Big\{(1,2K,K,S,K,2K,1) \mid S \in \mathcal{S}(h-2,k-5)\Big\},$$

and

$$\mathcal{S}^*(h,k) = \mathcal{S}_1(h,k) \cup \mathcal{S}_2(h,k) \cup \mathcal{S}_3(h,k). \qquad \ldots(3.2)$$

Next, for any integer $h \geq 2$, consider the particular (2h+1)-tuple

$$S = (1,2K,K,(K+1,K)^{h-2},2K,1) \in \mathcal{S}_3(h,2h),$$

and for any integer $t \geq 0$, let

$$\mathcal{S}^{\ddagger}(t,2h) \qquad \ldots(3.3)$$

denote the set of all distinct (2h+1)-tuples obtained by spreading t vertices as evenly as possible over the fat terms of S. Now let $\mathscr{S}_1^{\#}(t,2h+1)$ denote the set of all distinct 2(h+1)-tuples obtained by spreading $t \geq 0$ vertices as uniformly as possible over the fat terms of every element of

$$\mathscr{S}_3(h,2h+1) = \left\{(1,2K,(K,K+1)^{h_1},K,K,(K+1,K)^{h-h_1-2},2K,1) \mid 0 \leq h_1 \leq h-2\right\}.$$

Observe that every element $S \in \mathscr{S}_1^{\#}(h,2h+1)$ contains exactly one of the 4-tuples (K,K+a,K,K) or (K,K,K+a,K), for some $a \geq 1$. If in fact $a \geq 2$, then each of these 4-tuples can be transformed into a-1 corresponding 4-tuples

$$(K,K+b,K+c,K),$$

where $b > 0$, $c > 0$, $b+c = a$, and by Lemma 2.7 the size of the 4-tuple is unchanged. Hence let $\mathscr{S}_2^{\#}(t,2h+1)$ denote the set of all distinct 2(h+1)-tuples obtained by carrying out these transformations on every element of $\mathscr{S}_1^{\#}(t,2h+1)$, and then let

$$\mathscr{S}^{\#}(t,2h+1) = \mathscr{S}_1^{\#}(t,2h+1) \cup \mathscr{S}_2(t,2h+1). \qquad \ldots(3.4)$$

Finally, based on (3.2)-(3.4), the main characterization result can be stated:

Theorem 3.2 For $D \geq 6$ and $t = n-(D+1)K > 0$, a graph $G \in \mathscr{C}_V(n,*,*,D,K)$ is edge-minimal if and only if it has a vertex sequence S_D satisfying one of the following conditions:

(a) $t < \lfloor D/2 \rfloor$ and $S_D \in \mathscr{S}^*(t,D)$;

(b) D even, $t \geq D/2$, and $S_D \in \mathscr{S}^{\ddagger}(t-D/2,D)$;

(c) D odd, $t \geq \lfloor D/2 \rfloor$, and $S_D \in \mathscr{S}^{\#}(t-\lfloor D/2 \rfloor,D)$;

(d) D odd, $t \geq \lfloor D/2 \rfloor$, K=1, and $S_D = (1,S)$ or $(S,1)$, where $S \in \mathscr{S}^{\ddagger}(t-\lfloor D/2 \rfloor+2,D-1)$. ∎

Proof For case (a), observe that by Lemma 3.5, the (D-1)-tuple

$$S_0 = (\delta_1,0,\delta_3,\delta_4,\ldots,\delta_{D-3},0,\delta_{D-1})$$

determines the size of an edge-minimal graph. By Lemma 3.1, S_0 must contain t non-adjacent ones and D-1-t zeros, and hence specifies exactly the vertex sequences of $\mathscr{S}^*(t,D)$.

For case (b), observe that since D is even and $t \geq D/2$, the vertex sequences are by Lemmas 3.1, 3.2(a), and 3.5 restricted to the forms specified by $\mathscr{S}^{\ddagger}(t-D/2,D)$.

For case (c), observe that for D odd and $t \geq \lfloor D/2 \rfloor$, the vertex sequences of $\mathscr{S}^{\#}(t-\lfloor D/2 \rfloor,D)$ are, by Lemmas 3.1, 3.2(b), and 3.5, the only sequences possible for $K \geq 2$;

for K = 1, however, the vertex sequences specified by (c) and (d) exhaust all possible cases. This completes the proof. ∎

3.2 A Partial Characterization of Graphs of $\mathscr{C}_v(n,m,*,D,K)$

In this section, results are presented which, for given integers n, m, D, and K, provide partial answers to the following problems:

* $\mathscr{C}_v(n,m,*,D,K) = \phi$?
* Construct a graph $G \in \mathscr{C}_v(n,m,*,D,K)$.

First recall Lemmas 2.10 and 2.11, stated in Section 2.2, which specify a sharp lower bound $\sigma(a,k)$ on the size of a k′-tuple, $2k-1 \le k' \le 2k$, on a vertices, and establish its monotonicity properties. Lemma 2.10 is now seen to be an easy corollary of Lemmas 2.9 and 3.1, but Lemma 2.11 is restated here and proved.

<u>Lemma 2.11</u> For integers $a \ge 0$, $k \ge 1$,

(a) $\sigma(a,k) - \sigma(a,k+1) \ge \left\lceil \dfrac{\lfloor a/(k+1) \rfloor + 1}{2} \right\rceil$

(b) $\sigma(a+1,k) - \sigma(a,k) = \lfloor (a+1)/k \rfloor + \lceil [(a+1) \bmod k]/k \rceil - 1.$

<u>Proof</u> To prove (a), observe that the transformation of a (2k+2)-tuple of minimum size into a 2k-tuple of minimum size may be thought of as a sequence of steps by which a term initially of order $\beta = \lfloor a/(k+1) \rfloor$ is reduced to zero: at each step β is decreased by one, while another term $\alpha \geq \lfloor a/(k+1) \rfloor$ is increased by one. Consider the identity

$$\binom{\alpha+1}{2} - \binom{\alpha}{2} + \binom{\beta-1}{2} - \binom{\beta}{2} = \alpha-\beta+1, \qquad \ldots(3.5)$$

true for all integers $\alpha \geq 0$, $\beta \geq 1$. From this expression it is clear that the increase in size at step h is

$$\alpha - \beta + 1 \geq h.$$

Then in exactly $\lfloor a/(k+1) \rfloor$ such steps a 2k-tuple of minimum size can be created, where by construction

$$\sigma(a,k) - \sigma(a,k+1) \geq \sum_{1 \leq h \leq \lfloor a/(k+1) \rfloor} h = \binom{\lfloor a/(k+1) \rfloor + 1}{2},$$

as required.

To prove (b), observe that the removal of a single vertex from a minimum 2k-tuple of order a + 1 affects a term of order either $\lfloor (a+1)/k \rfloor + 1$ (when $(a+1) \bmod k > 0$) or $\lfloor (a+1)/k \rfloor$ (when $(a+1) \bmod k = 0$). ∎

Lemma 2.11 implies that for fixed a and for $k \le a-1$, $\sigma(a,k)$ is strictly monotone decreasing in k; while for fixed k and for $a \ge k$, $\sigma(a,k)$ is strictly monotone increasing in a. It is now possible to prove Theorem 2.2, one of the main results of this section:

<u>Theorem 2.2</u> For given integers $a \ge 0$ and $k \ge 4$, there exists a k-tuple of size i for every integer i satisfying

$$\sigma\left(a, \lfloor k/2 \rfloor\right) \le i \le \begin{bmatrix} a \\ 2 \end{bmatrix}. \qquad \blacksquare$$

<u>Proof</u> Suppose that k is even, and let $k' = k/2$, $a' = \lfloor a/k' \rfloor$. Then recall from Lemma 2.10 that for even k the minimum size $\sigma(a,k')$ is achieved by the k-tuple

$$T_*(a,k) = \left[(a'+1,0)^{a \bmod k'}, (a',0)^{k'-a \bmod k'}\right].$$

The maximum size $\begin{bmatrix} a \\ 2 \end{bmatrix}$ is achieved by the k-tuple $(a,0,0,\ldots,0)$. For $\sigma(a,k') < i < \begin{bmatrix} a \\ 2 \end{bmatrix}$, the proof proceeds by construction.

Consider the rightmost 4-tuple, say $(x,0,y,0)$, of $T_*(a,k)$. By Lemma 2.1 this 4-tuple has size $\begin{bmatrix} x \\ 2 \end{bmatrix} + \begin{bmatrix} y \\ 2 \end{bmatrix}$. Observe that the order of the rightmost two terms can be reversed, yielding the 4-tuple

$$T(x+y,4) = (x,0,0,y),$$

without changing the size of $T_*(a,k)$. Suppose without loss of generality that $x \le y$. It will be demonstrated that $T(x+y,4)$ can be modified to yield any size i' satisfying

$$\binom{x}{2} + \binom{y}{2} \le i' \le \binom{x+y}{2},$$

so that the corresponding size i of the k-tuple satisfies

$$i = \sigma(a,k) + i' - \binom{x}{2} - \binom{y}{2}.$$

Consider the modified 4-tuple

$$T_{h,r}(x+y,4) = (x-h-1, 1, r, y+h-r),$$

where $0 \le h \le x-1$, $0 \le r \le y+h$. This 4-tuple has size

$$\begin{aligned}\sigma_{h,r}(x+y,4) &= \binom{x-h}{2} + \binom{y+h}{2} + r \\ &= \binom{x}{2} + \binom{y}{2} + h(y-x+h) + r.\end{aligned}$$

Observe that for fixed h, $\sigma_{h,r}(x+y,4) - \binom{x}{2} - \binom{y}{2}$ takes every integer value in the closed range

$$\left[h(y-x+h), U_h\right],$$

where

$$U_h = h(y-x+h) + (y+h)$$
$$= (h+1)y - hx + h(h+1)$$
$$\geq (h+1)(y-x+h+1)$$

for all allowable values of h. Thus, as h varies from 0 to x-1, $\sigma_{h,r}(x+y,4) - \binom{x}{2} - \binom{y}{2}$ takes every integer value in every closed range

$$[h(y-x+h), (h+1)(y-x+h+1)].$$

Since for h = x-1, U_h = xy, it follows that $\sigma_{h,r}(x+y,4)$ takes every integer value from $\binom{x}{2} + \binom{y}{2}$ to

$$\binom{x}{2} + \binom{y}{2} + xy = \binom{x+y}{2}$$

at least once.

It has been shown that the rightmost 4-tuple of $T_*(a,k)$ can be rearranged so that its size i takes every value in the range

$$[\sigma(a,k'), \sigma(a,k')+xy].$$

If k=4, $\sigma(a,k') = \binom{x}{2} + \binom{y}{2}$, so that

$$\sigma(a,k') + xy = \binom{x+y}{2},$$

and the proof is complete. For $k \geq 6$, however, the construction described above can be repeated on the $(k-2)$-tuple $T(a,k-2)$ which is identical to $T_*(a,k)$ in the first $k-4$ positions and whose rightmost double is $(x+y,0)$. After the construction has been carried out altogether $k'-1$ times, it is clear that the size i must have taken every value in the range

$$\left[\sigma(a,k'), \binom{a}{2}\right],$$

as required.

To complete the proof, observe that, by virtue of the monotonicity of $\sigma(a,k')$ (Lemma 2.11), the size range $\sigma(a,k') \leq i \leq \binom{a}{2}$ can be achieved by any j-tuple, $j \geq k$, and in particular, by $j = k+1$. ∎

A comparison of this result with Lemma 2.10 shows that, for even k, every size between minimum and maximum is in fact achieved by some k-tuple; thus, in this case, the specified range provides both a necessary and sufficient condition for existence of the k-tuple. For odd k, however, the minimum of the range specified by Theorem 2.24 is $\sigma(a, \lfloor k/2 \rfloor)$, while by Lemma 2.10 there exists a k-tuple of size $\sigma(a, \lceil k/2 \rceil) < \sigma(a, \lfloor k/2 \rfloor)$. Hence, for odd k, the range

$$\left[\sigma(a, \lceil k/2 \rceil),\ \sigma(a, \lfloor k/2 \rfloor)\right] \qquad \ldots(3.6)$$

remains to be investigated. The next result shows that for $k = 3$, this range in general contains gaps; that is, sizes which are achieved by no triple $T(a,3)$ on a vertices.

Lemma 3.6 For given integers $a \geq 0$ and i, there exists a triple on a vertices of size i if and only if both of the following conditions are satisfied:

(a) $\sigma(a,2) \leq \binom{\lfloor a/2 \rfloor}{2} + \binom{\lceil a/2 \rceil}{2} \leq i \leq \binom{a}{2} = \sigma(a,1)$;

(b) $i = \binom{a}{2} - p$, where $p \leq \lfloor a/2 \rfloor \lceil a/2 \rceil$ is a non-negative integer, and some factorization $p = xy$ satisfies $x + y \leq a$.

Proof By Lemma 2.10, no size outside the range specified by condition (a) is achievable. Further, by Lemma 2.6, the achievable sizes necessarily take the form

$$i = \binom{a}{2} - xy, \qquad \ldots(3.7)$$

where $0 \leq x \leq a$, $0 \leq y \leq a-x$. Since the minimum value of this expression is

$$\sigma(a,2) = \binom{a}{2} - \lfloor a/2 \rfloor \lceil a/2 \rceil,$$

it is clear that $xy \leq \lfloor a/2 \rfloor \lceil a/2 \rceil$. Since the condition on p stated in (b) is equivalent to the condition on x and y stated for (3.7), the result is proved. ∎

From Lemma 3.6 it follows that no triple T(a,3) can achieve a size $\binom{a}{2}$ - p, where p is a prime satisfying $a \le p \le \lfloor a/2 \rfloor \lceil a/2 \rceil$. But exceptional values p are not restricted to primes; for example, no triple T(11,3) can achieve a size $\binom{11}{2} - 22 = 33$ or $\binom{11}{2} - 27 = 28$, even though $\binom{11}{2} - 24 = 31$ and $\binom{11}{2} - 25 = 30$ are both achievable. It is natural then to enquire whether odd k-tuples, $k \ge 5$, necessarily achieve every size in the range (3.6), since if so, the problem of the existence of a graph of $\mathscr{C}_V(n,m,*,D,K)$ would be greatly simplified. Unfortunately, no definite answer can be given to this enquiry, and the problem appears to be difficult, as the following examples indicate.

Consider tuples T(a,k) with the property that a = k′a′ for $k' = \lceil k/2 \rceil$ and some positive integer a′. Then the edge-minimal such tuple is

$$T_*(a,k) = (a',0,a',0,\ldots,a',0,a')$$

with size $\sigma(k'a',k') = k'\binom{a'}{2}$. Observe that, in order to form sizes

$$i = \sigma(k'a',k') + i',$$

where $1 \le i' \le a'-1$, only k-tuples of the form

$$(a_1, 0, a_2, 0, \ldots, a_{k'-1}, 0, a_{k'}) \qquad \ldots (3.8)$$

need be considered; that is, k-tuples generated by transferring vertices among the non-zero terms of $T_*(a,k)$.

Now suppose $k = 5$. By considering the possible 5-tuples (3.8), it is not difficult to see that sizes

$$i = 3\binom{a'}{2} + i',$$

for $i' = 2,5,6,8,10,11,14,15$, and an infinite number of larger values, cannot be achieved. Thus, for a' sufficiently large, there must exist sizes in the range $[\sigma(3a', a),\ \sigma(3a', 2)]$ which are not achievable by any 5-tuple $T(3a', 5)$, and in fact this negative result extends easily to more general 5-tuples $T(b, 5)$, where b is not necessarily a multiple of $k' = 3$.

A similar calculation for $k = 7$ yields the result that

$$i = 4\binom{a'}{2} + i'$$

is achievable by some 7-tuple (3.5) when $1 \le i' \le 13$ or $15 \le i' \le 21$, but not for $i' = 14$. Thus for $a' \ge 16$ $(a \ge 64)$ there exists at least one size in the range

$[\sigma(4a',4),\sigma(4a',3)]$ which is not achievable by any 7-tuple $T(4a',7)$. For $k = 9$ it is not known whether any such non-achievable sizes exist, but it appears likely that they do. In fact, the following conjecture is stated:

<u>Conjecture 3.1</u> For given positive integers $k' \geq 2$ and λ, there exists an integer $a = a(k',\lambda)$ such that exactly λ sizes in the range $[\sigma(a,k'),\sigma(a,k'-1)]$ are achieved by no tuple $T(a,2k'-1)$.

If it is correct, this conjecture is discouraging. On the other hand, let $i^* = i^*(a,k')$ be the largest integer such that for $\sigma(a,k') \leq i \leq i^*$, every size i is achievable by some tuple $T(a,2k'-1)$. It seems likely that i^* grows rapidly with k', so that, if i^* could somehow be computed, the existence of many long tuples of specified order and size could be determined. Attempts to compute or estimate i^* have however not been successful.

So far the investigation has dealt with the sizes of k-tuples which are unconstrained in any way. However, as expression (2.16) indicates, for $K > 1$ the size i to be achieved by a given k-tuple on a vertices depends on the sum b of the first and last terms:

$$i = m' + b(K-1),$$

where, as explained in Section 2.2, m' is the size obtained from the given size m by subtracting out the components based solely on the given parameters n, D, and K. The results obtained so far, then, relate to the cases b = 0 or K = 1. To deal with the more general case, for $k \geq 5$ let

$$\mathcal{T}(a,b,k) \qquad \ldots(3.9)$$

denote the set of all k-tuples $(\delta_1, \delta_2, \ldots, \delta_k)$ of order $a \geq b \geq 0$ such that $\delta_1 + \delta_k = b \geq 0$: these k-tuples will be called <u>b-constrained</u>. Further, let $\mathcal{T}'(a,b,k)$ denote the subset of $\mathcal{T}(a,b,k)$ characterized by tuples in which $\delta_2 = \delta_{k-1} = 0$. The second main result of this section can now be proved:

<u>Theorem 2.3</u> For given integers $a \geq b \geq 0$, $k \geq 8$, and $k' = \lfloor (k-4)/2 \rfloor$, every size i in the range

$$\left[\sigma(a-b,k') + \binom{\lfloor b/2 \rfloor}{2} + \binom{\lceil b/2 \rceil}{2} \;,\; \binom{a-b}{2} + \binom{b}{2}\right] \qquad \ldots(3.10)$$

is achieved by some element of $\mathcal{T}' = \mathcal{T}'(a,b,k)$, provided

$$b \leq \hat{b} = (a-k'+k'') - \sqrt{k''(2a-2k'+k''-2)} \;,$$

where $k'' = k'/(k'-1)$. When $k' \mid (a-b)$, this upper bound becomes

$$\hat{b} = \lfloor (a+k'') - \sqrt{k''(2a+k''-2)} \rfloor ,$$

and is least possible.

<u>Proof</u> Observe first that the upper boundary of the range (3.10) is the largest size actually achieved by any element of $\mathcal{T}'$; also that, by Theorem 2.2, the lower boundary is achieved by an element of $\mathcal{T}'$; further that, when k is even, the lower boundary is by Lemma 2.10 the least size of any element of $\mathcal{T}'$.

Let $\delta_1 = x$ be fixed, so that $\delta_k = b-x$. Then by Theorem 2.2, every size in the range

$$[L_x, U_x] = \left[\sigma(a-b,k')+\binom{x}{2}+\binom{b-x}{2},\ \binom{a-b}{2}+\binom{x}{2}+\binom{b-x}{2}\right]$$

is achieved by some element of $\mathcal{T}'$. Both the lower and upper boundaries of this range are monotone increasing as x takes the values $\lfloor b/2 \rfloor, \lfloor b/2 \rfloor -1, \ldots, 0$. Then every size in the entire range (3.10) will be achieved if and only if $U_x \geq L_{x-1}$ for every $\lfloor b/2 \rfloor \geq x \geq 1$; that is, using (3.5),

$$\binom{a-b}{2} - \sigma(a-b,k') \geq \binom{b-x+1}{2} - \binom{b-x}{2} + \binom{x-1}{2} - \binom{x}{2}$$

$$= b - 2x + 1. \qquad \ldots(3.11)$$

It follows that every size in (3.10) will be achieved if and only if

$$\begin{bmatrix} a-b \\ 2 \end{bmatrix} - \sigma(a-b,k') \geq b-1. \qquad \ldots(3.12)$$

Now suppose that $k' \mid (a-b)$, so that $a' = (a-b)/k'$ is an integer and $\sigma(a-b,k') = k' \begin{bmatrix} a' \\ 2 \end{bmatrix}$. After some algebra, the expression (3.12) then becomes

$$b^2 - 2(a+k'')b + (a^2+2k'') \geq 0.$$

Replacing the inequality yields a quadratic equation in b, whose solution $\hat{b}$ is the largest value of b for which (3.12) holds. It is straightforward to verify that the expression thus obtained is the one given in the statement of the theorem.

In case $k' \nmid (a-b)$, set $a' = \lceil (a-b)/k' \rceil - 1$. It follows then from (3.12) that

$$\begin{bmatrix} a-b \\ 2 \end{bmatrix} - \sigma(a-b,k') > \begin{bmatrix} k'a' \\ 2 \end{bmatrix} - k' \begin{bmatrix} a' \\ 2 \end{bmatrix} \geq b-1$$

is a necessary condition that every size in the range (3.10) is achievable. As before, this inequality can be transformed into a quadratic equation in b, with the solution $\hat{b}$ given in the statement of the theorem. ∎

Since for large values of a the upper bound $\hat{b}$ is roughly $O(a-\sqrt{a})$, it follows that for most values of b, every size in the range (3.10) will be achievable by an element of $\mathcal{T}'(a,b,k)$. In this sense Theorem 2.3 is encouraging. On the other hand, the result also makes clear that there always exist larger values of b which will give rise to gaps in the range (3.10) — that is, sizes which are not achieved by any k-tuple of $\mathcal{T}'(a,b,k)$. For those values of b which give rise to gaps, (3.11) may be used in the form

$$x \geq \left[\sigma(a-b,k') + b - \binom{a-b}{2} - 1\right]/2 \qquad \ldots(3.13)$$

to determine the critical values $\delta_1 = x$, and the omitted sizes may then be specified. This calculation is included in the algorithm of Section 3.3.

The preceding theorem provides a mechanism for determining the sizes achievable by k-tuples which are subject to two restrictions: $\delta_1 + \delta_k = b$ and $\delta_2 = \delta_{k-1} = 0$. The second of these restrictions ensures that no sizes greater than $\binom{a-b}{2} + \binom{b}{2}$ can be achieved by any k-tuple of $\mathcal{T}'(a,b,k)$. For sizes ranging up to $\binom{a}{2}$, the following theorem provides a partial characterization:

Theorem 2.4 For $k \geq 8$, let $\mathcal{T}_3 = \mathcal{T}_3(a,b,k)$ denote the set of all k-tuples $(\delta_1,\delta_2,\delta_3,0,0,\ldots,0,\delta_{k-2},\delta_{k-1},\delta_k)$ of order $a \geq b$ such that $\delta_1 + \delta_k = b > 0$ and

$$\delta_2 + \delta_3 + \delta_{k-2} + \delta_{k-1} = a - b.$$

Then, for $x = \delta_1 + \delta_2 + \delta_3$, the size i of an element of $\mathcal{T}_3$ is given by

$$i = \begin{bmatrix} a \\ 2 \end{bmatrix} - x(a-x) - \delta_1\delta_2 - \delta_{k-2}(b-\delta_1).$$

<u>Proof</u> An immediate consequence of Lemma 2.6. ∎

To see that this result does not provide complete coverage of all sizes which may arise, consider $\mathcal{T}_4 = \mathcal{T}_4(a,b,k)$, the set of all b-constrained k-tuples

$$(\delta_1, \delta_2'', \delta_3'', \delta_4'', 0, 0, \ldots, 0, \delta_{k-3}'', \delta_{k-2}'', \delta_{k-1}'', \delta_k),$$

$k \geq 9$, of order a such that $\delta_1 + \delta_2'' + \delta_3'' + \delta_4'' = x$ and $\delta_{k-3}'' + \delta_{k-2}'' + \delta_{k-1}'' + \delta_k = a-x$. By Lemma 2.6, the size of an element of $\mathcal{T}_4$ is

$$i'' = \begin{bmatrix} a \\ 2 \end{bmatrix} - x(a-x) - \delta_1(x-\delta_1-\delta_2'') - \delta_k(a-x-\delta_{k-1}''-\delta_k) - \delta_2''\delta_4'' - \delta_{k-1}''\delta_{k-3}''.$$

If this quantity is compared with the expression of Theorem 2.4, it is found after some manipulation that

$$i''-i' = \delta_1(\delta_2''-\delta_2) + \delta_k(\delta_{k-1}''-\delta_{k-1}) - \delta_2''\delta_4'' - \delta_{k-1}''\delta_{k-3}'',$$

representing the size difference between elements of $\mathcal{T}_4$ and $\mathcal{T}_3$ which have the same values of a, b, and x. In particular, let

$$i_*' = \binom{\lfloor (a-b)/2 \rfloor}{2} + \binom{\lceil (a-b)/2 \rceil}{2} + \binom{\lfloor b/2 \rfloor}{2} + \binom{\lceil b/2 \rceil}{2}$$

denote the minimum edge count of $\mathcal{T}_3$ for given a and b, where $\delta_2 = \delta_{k-1} = 0$. Then

$$i'' - i_*' = \delta_2''(\delta_1-\delta_4'') - \delta_{k-1}''(\delta_k-\delta_{k-3}''),$$

and it follows that the minimum edge count i_*'' of $\mathcal{T}_4$ will be less than i_*' if the choices

$$\delta_2'' = 1,\ \delta_{k-1}'' = 1,\ \delta_4'' > \delta_1,\ \text{and}\ \delta_{k-3}'' > \delta_k$$

are made: this can always be done provided $(a-b)/2 - 1 - b/2 > 0$; that is, $a > 2b+2$. Thus, for fixed a and b, it can happen that elements of $\mathcal{T}_4$ achieve sizes less than the minimum of $\mathcal{T}_3$. On the other hand, at the upper end of the range for i'', observe from Theorem 2.4 that the maximum size $\binom{a}{2}$ is achieved for $x = \delta_1 = \delta_{k-2} = 0$; while the next largest size is either

$$\binom{a}{2} - b,$$

corresponding to $x = \delta_1 = 0$ and $\delta_{k-2} = 1$, or

$$\begin{bmatrix} a \\ 2 \end{bmatrix} - (a-1),$$

corresponding to $x = 1$ and $\delta_1 = \delta_{k-2} = 0$. Thus, for given a and b, there are gaps in the sizes achieved by elements of $\mathcal{T}_4(a,b,k)$.

The analysis of this section has shown that there are possible gaps in the sizes achievable by graphs $G \in \mathcal{C}_V(n,*,*,D,K)$; that is, gaps in the values of m for which $\mathcal{C}_V(n,m,*,D,K)$ is non-empty. These gaps are of two main kinds:

(a) for unconstrained k-tuples $T(a,k)$ on a vertices, where k is odd, there appear always to be sizes in the range $[\sigma(a,\lceil k/2\rceil),\ \sigma(a,\lfloor k/2\rfloor)]$ which are achieved by no k-tuple;

(b) for b-constrained k-tuples $T(a,b,k)$ on a vertices, there appear always to be sizes in the range $\left[\sigma(a-b,\lfloor k-4)/2\rfloor) + \begin{bmatrix} \lfloor b/2\rfloor \\ 2 \end{bmatrix} + \begin{bmatrix} \lceil b/2\rceil \\ 2 \end{bmatrix},\ \begin{bmatrix} a \\ 2 \end{bmatrix}\right]$ which are achieved by no k-tuple.

In general, although the feasibility of a given size m can be efficiently determined in many cases, there always exist sizes which cannot be handled by any means other than some form of exhaustive search ("brute force"), as suggested for example by the expression (2.9). In the next section, the algorithmic consequences of the analysis conducted here are explored.

3.3 The Existence/Construction of Graphs of $\mathscr{C}_V(n,m,*,D,K)$

In this section, based on the results of Section 3.2, an algorithm for determining whether or not the class $\mathscr{C}'_V = \mathscr{C}_V(n,m,*,D,K)$ exists is outlined. In order to make the flow of the algorithm clear, a number of trivial or algebraically complicated details are omitted. In particular, the details of the construction of graphs of $\mathscr{C}'_V$ are not explicitly considered; it will be clear that, by virtue of the constructive nature of the proofs of Lemma 3.6 and Theorems 2.2-2.4, a graph $G \in \mathscr{C}'_V$ can be constructed in a straightforward manner once it has been established that $\mathscr{C}'_V \neq \phi$.

In order to state the algorithm, a further result is required, a counterpart for b-constrained k-tuples to Lemma 3.1:

<u>Lemma 3.7</u> For $a \geq b \geq 0$, $k \geq 2$, and $k' = \lceil k/2 \rceil$, a b-constrained k-tuple $T = T(a,b,k)$ on a vertices is edge-minimal if and only if

(a) T is partitionable;

(b) the orders of non-terminal clumps of T differ by at most one;

(c) (i) for odd k, or for even k and $a < k'b$, the orders of terminal clumps differ by at most one;

(ii) for even k and $a > k'(b-1)$, one terminal term is a clump of order b.

<u>Proof</u> For odd k, the proof is a straightforward extension of the proof of Lemma 3.1. Then suppose that k is even. If $a > k'(b-1)$, every tuple with a terminal term of order b and $k'-1$ other non-terminal clumps satisfying condition (b) is edge-minimal; whereas, for $a \le k'(b-1)$, no such tuple can be edge-minimal. On the other hand, if $a > k'b$, no tuple with a non-zero terminal term less than b can be edge-minimal; while for $a \le k'b$, every tuple satisfying condition (b) together with

$$\delta_1 = \max\left\{\lfloor a/k' \rfloor, \lceil b/2 \rceil\right\}, \ \delta_{k-1} + \delta_k = b-\delta_1$$

or

$$\delta_k = \max\left\{\lfloor a/k' \rfloor, \lceil b/2 \rceil\right\}, \ \delta_1 + \delta_2 = b-\delta_k$$

is edge-minimal. Observe that these latter tuples can always be constructed. Then, by an argument similar to that of Lemma 3.1, it follows that, also when k is even, the given conditions are both necessary and sufficient for T to be edge-minimal. ∎

With this result, it is clear that Lemma 3.2 applies also to tuples of $\mathcal{T}(a,b,k)$, and further that the following counterpart of Lemma 2.10 holds:

<u>Lemma 3.8</u> For $a \ge b \ge 0$, $k \ge 5$, and $k' = \lceil k/2 \rceil$, an edge-minimal tuple of $\mathcal{T}(a,b,k)$ has size

$$i_* = \sigma(a-b,k'-2) + \binom{\lfloor b/2 \rfloor}{2} + \binom{\lceil b/2 \rceil}{2},$$

if k is odd, or if k is even and $a \le k' \lfloor b/2 \rfloor$;

$$= \sigma(a,k'),$$

if k is even and $k' \lfloor b/2 \rfloor < a < k'b$;

$$= \sigma(a-b,k'-1) + \binom{b}{2}, \text{ otherwise.}$$

<u>Proof</u> A direct consequence of Lemma 3.7. ∎

It is now possible to embark on a structured description of an algorithm which determines whether or not $\mathcal{C}'_V = \phi$. As mentioned at the beginning of this section, the extension of the algorithm to specify a graph $G \in \mathcal{C}'_V$ is straightforward. The algorithm is presented in Figure 3.1 as a Boolean function EXIST, which returns TRUE if $\mathcal{C}'_V \ne \phi$; otherwise, FALSE. The input parameters n, m, D, and K are integers satisfying (2.1)-(2.3) together with the inequalities

$$D \ge 2, \; K \ge 1, \; n \ge (D-1)K+2.$$

EXIST is expressed in a Pascal-like pseudocode that makes use of the feature

return(X),

An Algorithm to Determine the Existence of a Graph in $\mathscr{C}_V(n,m,*,D,K)$

```
function EXIST (N,m,D,K) : boolean;

a := n - (D-1)K - 2;                                  {Compute excess.}
if D = 2 then m' := m - K(K+4a+3)/2 + a               {Compute m' using
       else m' := m - K[(3D-5)K - (D-6a-5)]/2;              Lemma 2.9.}
if m' < 0 then return (FALSE); if m' = 0 then return (a=0);

{For a = 0 or D ≤ 3, 𝒞_V(n,*,*,D,K) contains a unique graph.}
if a = 0 or D ≤ 3 then return [m' + (K-1)a = (a choose 2)];

{K = 1 gives rise to the special case i = m'.}
if K = 1 then return (EXIST1(a,m',D-1));

{The bounds on i (Theorem 2.2) imply bounds on b.}
b_* := max{0, ⌈[σ(a, ⌈(D-1)/2⌉)-m']/(K-1)⌉};
b^* := min{a, ⌊[ (a choose 2)-m']/(K-1)⌋};
if b_* > b^* then return (FALSE);           {Ensure i satisfies its bounds.}
{For D = 4 or 5, there exist only 1 or 2 internal terms: handle as special
if D ≤ 5 then return (EXIST45(a,m'D,K));                          cases.}

{b = 0 gives rise to the special case i = m'.}
if b_* = 0 then
    if EXIST1(a,m',D-3,K) then return (TRUE) else b_* := 1;

{D = 6 is an application of Lemma 3.6.}
if D = 6 then return (EXIST6(a,m',D,K));

for b := b_* to b^*                            {Each valid b gives rise to
    i := m' + (K-1)b;                             a corresponding size i.}
    i_* := MINSIZE(a,b,D-1);                          {Using Lemma 3.8.}
    if i = i_* then return (TRUE)
    elseif i > i_* then                        {i<i_* invalid: try next b.}
        {Handle D = 7,8 as special cases using Lemmas 2.10 and 3.6.}
        if D ≤ 8 and EXIST78(a,b,i,D-1) then return (TRUE);

        {Deal with the general case using Theorems 2.3 and 2.4.}
        elseif EXISTG(a,b,i,D-1) then return (TRUE)

next b;             {If the current b didn't yield TRUE, try the next one.}

{If all else fails, do an exhaustive search based on expression (2.9).}
if D ≤ 8 then return (FALSE) else return (BRUTEFORCE(a,m',D,K)).
```

FIGURE 3.1

which assigns the value of X to the function F currently being executed and then exits from F. This feature is in fact often useful in practice, and is recommended to the attention of future computer language developers.

EXIST calls seven other functions, whose roles are described in Table 3.1. Of these functions, one is the inefficient but straightforward BRUTEFORCE algorithm, called only when more efficient methods have failed; and three (EXIST45, EXIST6, and EXIST78) deal in a fairly obvious manner with graphs of small diameter. The remaining three functions (EXIST1, MINSIZE, and EXISTG) are however more interesting: pseudocode is given for them in Figures 3.2-3.4, and further discussion of EXISTG, by far the most complex of the three, may be found in the next paragraph. Apart from purely mathematical functions such as $\lfloor X \rfloor$, $\lceil Y \rceil$, $\binom{X}{2}$, $\sigma(X,Y)$, $L(X,a,b,k)$, and $U(X,a,b)$, these three functions call three other subfunctions, which are described in Table 3.2.

Theorem 2.3 deals with tuples of $\mathcal{T}' = \mathcal{T}'(a,b,k)$; these tuples have the property that $\delta_2 = \delta_{k-1} = 0$, while $\delta_1 = x$ and $\delta_k = b-x$ for $0 \le x \le b$. However, Theorem 2.3 does not take into account the fact that, when $x = 0$ or b, δ_2 or, respectively, δ_{k-1} may be non-zero. This special case has however been taken into account in EXISTG, by allowing k' to be set equal to $\lfloor (k-3)/2 \rfloor$ when $x = 0$ rather than $\lfloor (k-4)/2 \rfloor$. EXISTG then makes use of the value

$\hat{b}$ specified in Theorem 2.3, but also uses expression (3.13) to compute a value x^* which has the property that, for every $x^* \le x < \lfloor b/2 \rfloor$, the sub-ranges $[L_x, U_x]$ and $[L_{x+1}, U_{x+1}]$ overlap; thus the inspection of the sub-ranges is required only when they do not overlap.

This concludes the discussion of the algorithm EXIST and Chapter 3.

Functions Called by EXIST(n,m,D,K)

FUNCTION (all parameters positive integers)	TYPE	DESCRIPTION
EXIST1(a,m',k), $k \geq 3$	boolean	For K=1 or b=0, determines whether or not m' is a size achievable by T(a,k): using Lemma 3.6 if k=3, and Theorem 2.2 otherwise. May call a specialized brute force function BABYBRUTE when k is odd. See Figure 3.2.
EXIST45(a,m',D,K), $4 \leq D \leq 5$, $K \geq 2$	boolean	Uses a and m' to determine whether or not m is a size achievable by a graph of $\mathscr{C}_V = \mathscr{C}_V(n,*,*,D,K)$.
EXIST6(a,m',K), $K \geq 2$	boolean	For D=6, uses a and m' together with Lemma 3.6 to determine whether or not m is a size achievable by a graph of $\mathscr{C}_V$.
MINSIZE(a,b,k), $k \geq 6$	integer	Computes the minimum size i_* of T(a,b,k) corresponding to Lemma 3.8. See Figure 3.3.
EXIST78(a,b,i,k), $6 \leq k \leq 7$	boolean	Determines whether or not i is a size achievable by T(a,b,k).
EXISTG(a,b,i,k), $k \geq 8$	boolean	Returns TRUE if, according to Theorems 2.3 and 2.4, i is a size achievable by T(a,b,k); otherwise, returns FALSE. See Figure 3.4.
BRUTEFORCE(a,m',D,K), $D \geq 9$, $K \geq 2$	boolean	Using expression (2.9) and the variables a and m', performs an exhaustive search to determine whether or not m is a size achievable by a graph of $\mathscr{C}_V$. Also calls BABYBRUTE.

Table 3.1

Function EXISTI (a,m′,k)

```
function EXIST1 (a,m′,k) : boolean;

{To determine whether or not the size m′ is achievable by T(a,k).}

{Return FALSE if m′ is outside the bounds of Lemma 2.10 and Theorem 2.2.}
if m′ < σ(a, ⌈k/2⌉) or m′ > (a choose 2) then return (FALSE);

{Use Lemma 3.6 to deal with the special case of a triple.}
if k = 3 then return (EXIST3(a,m′));

{Return TRUE if m′ is the exact minimum or inside the range specified by
                                                          Theorem 2.2.}
if m′ = σ(a, ⌈k/2⌉) or m′ ≥ σ(a, ⌊k/2⌋) then return (TRUE);

{If all else fails, do an exhaustive search of the range
                                     [σ(a, ⌈k/2⌉), σ(a, ⌊k/2⌋)].}
return (BABYBRUTE(a,m′,k)).
```

Figure 3.2

Function MINSIZE (a,b,k)

```
function MINSIZE (a,b,k) : integer;

{A straightforward application of Lemma 3.8.}

a′ := a - b; b′ := ⌊b/2⌋; k′ := ⌈k/2⌉;
if odd (k) or a ≤ b′k′ then return [σ(a′,k′-2) + (b′ choose 2) + (b-b′ choose 2)];
if a > b′k′ and a < bk′ then return (σ(a,k′));
return [σ(a′,k′-1) + (b choose 2)].
```

Figure 3.3

Function EXISTG (a,b,i,k)

```
function EXISTG (a,b,i,k) : boolean;

{To determine whether, on the basis of Theorems 2.3 and 2.4, a given
size i is achievable by T(a,b,k).  In accordance with Theorem 2.3 and
the discussion in the text, the following functions are defined:
```

$$\hat{b}(a,k) := (a''+k'') - \sqrt{k''(2a''+k''-2)},$$

$$L(x,a,b,k) := \sigma(a',k') + \binom{x}{2} + \binom{b-x}{2},$$

$$U(x,a,b) := \binom{a'}{2} + \binom{x}{2} + \binom{b-x}{2},$$

where
$k' := \lfloor (k-4)/2 \rfloor$, if x is undefined or if $x \neq 0$;
$\quad := \lceil (k-3)/2 \rceil$, if $x = 0$;
$k'' := k'/(k'-1)$;
$a' := a-b$;
$a'' := a$, if $k' \mid (a-b)$;
$\quad := a-k'$, otherwise.}

```
{If i is outside the range covered by Theorem 2.3, try Theorem 2.4.}
if i > U(0,a,b) then return (EXIST2.6 (a,b,i,k));

{If i is within the range for Theorem 2.3 and b ≤ b̂, then TRUE.}
if i ≥ min{L(0,a,b,k), L(⌊b/2⌋, a,b,k)} and b ≤ b̂(a,k)
   then return (TRUE);

{Using expression (3.13), determine the value of x above which intervals
                                                               overlap.}
```

$$x^{*} := \left\lfloor \left[\sigma(a,k') - \binom{a'}{2} + b - 1 \right] / 2 \right\rfloor;$$

```
if i ≤ U(x*+1,a,b) and i ≥ L(⌊b/2⌋,a,b,k) then return (TRUE);

{Below x*, inspect each interval individually.}
for x := x* down to 0
   if i ≤ U(x,a,b) and i ≥ L(x,a,b,k) then return (TRUE).
```

Figure 3.4

Functions Called by EXIST1 and EXISTG

FUNCTION (all parameters positive integers)	TYPE	DESCRIPTION
EXIST3 (a,m′)	boolean	Uses Lemma 3.6 to determine whether or not m′ is a size achievable by a triple T(a,3).
BABYBRUTE (a,m′,k), $k \geq 4$	boolean	Performs an exhaustive search of the range $[\sigma(a, \lceil k/2 \rceil), \sigma(a, \lfloor k/2 \rfloor)]$, k odd, to determine whether or not m′ is an achievable size of T(a,k).
EXIST2.6 (a,b,i,k), $k \geq 8$	boolean	Uses Theorem 2.4 to determine whether or not i is a size achievable by T(a,b,k).

Table 3.2

CHAPTER 4

K-EDGE-CONNECTED D-CRITICAL GRAPHS $\mathscr{C}_e(n,*,*,D,K)$

In this chapter edge-maximal graphs of $\mathscr{C}_e = \mathscr{C}_e(n,*,*,D,K)$ (hence, by Lemma 2.4, $\mathscr{G}_e(n,*,*,D,K)$) are completely characterized, and edge-minimal graphs of $\mathscr{C}_e$ are characterized for n sufficiently large with respect to the product DK. The references for this work are Caccetta & Smyth (1987a, 1987b, 1988a, 1988b, 1989a), and the main results are summarized in Section 2.3. Section 4.1 is devoted to a proof of Theorem 2.5, which for $D \ge 6$ and $K \ge 8$ specifies the form of vertex sequences of edge-maximal graphs; then in Section 4.2, smaller values of D and K are considered, and Theorem 2.5 is applied to yield a more precise determination of edge-maximal graphs (Theorems 2.6-2.10); finally, in Section 4.3, edge-minimal graphs are considered.

4.1 Vertex Sequences of Edge-Maximal Graphs over $\mathscr{C}_e(n,*,*,D,K)$

In this section a rather lengthy sequence of lemmas is proved, leading eventually to the establishment of Theorem 2.5; that is, to the conclusion that for $D \ge 6$ and $K \ge 8$ every edge-maximal graph of $\mathscr{C}_e$ gives rise to a vertex sequence in which all excess vertices are confined to a single term. The result is thus essentially the same as Theorem 2.1, established in Section 2.2 for edge-maximal graphs of $\mathscr{C}_v$; the proof however is much more difficult, and the result itself is less useful, in that it does not lead easily to a specification of the structure of the edge-maximal graph.

Recall first the definitions of the terms feasible, lean and fat (applied both to doubles and triples), transformation, and feasible transformation introduced in Section 2.3. One further definition will be useful: a vertex u of a graph $G \varepsilon \mathscr{C}_e(n,*,*,D,K)$ is said to be removable if $G-\{u\} \varepsilon \mathscr{C}_e(n-1,*,*,D,K)$. By extension, if the removable vertex u occurs in level L_i of a rooted level structure R of G, then L_i will sometimes be said to contain u; by further extension, since $n_i = |L_i|$ is a term in the vertex sequence induced by R, n_i may similarly be said to contain u. The first lemma, valid for $\mathscr{C}_v$ as well as for $\mathscr{C}_e$, may then be stated and proved:

Lemma 4.1 Let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \varepsilon \mathscr{C}_c(n,*,*,D,K)$. If any two terms n_i and n_j of S contain a removable vertex of G, then $|i-j| \leq 1$.

Proof Suppose $|i-j| \geq 2$, and observe that by Lemma 2.1 no removable vertex is terminal. Then the transformations

$$\tau_1 : n_i \longrightarrow n_i-1,\ n_j \longrightarrow n_j+1,$$
$$\tau_2 : n_i \longrightarrow n_i+1,\ n_j \longrightarrow n_j-1,$$

change the size of S by

$$-(n_{i-1} + n_i + n_{i+1}) + 1 + (n_{j-1} + n_j + n_{j+1}),$$
$$-(n_{j-1} + n_j + n_{j+1}) + 1 + (n_{i-1} + n_i + n_{i+1}),$$

respectively. Since one of these size changes is positive, G cannot be edge-maximal, a contradiction. ■

An immediate consequence of Lemma 4.1 is that no two non-adjacent terms of an edge-maximal vertex sequence can exceed K. Hence, for c = v, the result provides an alternative means of proving Theorem 2.1. For c = e, however, a great deal more effort is required:

<u>Lemma 4.2</u> For $D \geq 7$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph of $\mathcal{C}_e(n,*,*,D,K)$. If for some i, $2 \leq i \leq D-2$, $n_i < \sqrt{K}$, then

(a) for $3 \leq i \leq D-3$, $n_i < \min(n_{i-2}, n_{i+2})$;

(b) for $i = 2$, $n_i < n_{i+2}$;

(c) for $i = D-2$, $n_i < n_{i-2}$.

<u>Proof</u> Result (a) will be proved; the proofs of (b) and (c) use an almost identical argument. Suppose then that (a) is false, so that

$$\min\{n_{i-2}, n_{i+2}\} = n_i - x, \ x \geq 0. \qquad \ldots(4.1)$$

Then without loss of generality, it may be supposed that $n_{i+2} = n_i - x$. Hence S contains the subsequence

$$(n_{i-1}, n_i, \ldots, n_{i+3}) = (\lceil K/n_i \rceil + a, n_i, \lceil K/n_i \rceil + b, n_i - x, \lceil K/(n_i - x) \rceil + c),$$

where a, b and c are non-negative integers. Since by the hypothesis of the lemma, $n_i < \lceil K/n_i \rceil$, and since by Lemma 2.12(c),

$$2n_i + \lceil K/n_i \rceil + b - x \geq K + 1 \qquad \ldots(4.2)$$

it follows that the triples (n_{i-1}, n_i, n_{i+1}) and $(n_{i+1}, n_{i+2}, n_{i+3})$ are both fat. Hence, if $b > 0$, the transformation

$$(n_{i+1}, n_{i+2}) \longrightarrow (n_{i+1}-1, n_{i+2}+1)$$

is feasible and increases the size of S by

$$\lceil K/(n_i - x) \rceil + c - n_i > 0,$$

a contradiction. Therefore $b = 0$ and so, since $n_{i+1}n_{i+2} \geq K$, $x = 0$. Thus for $K \geq 8$, (4.2) can hold only if $n_i = 1$. Lemma 4.1 then implies that at least one of a or c, say a, is zero. But then the transformation

$$(n_{i-1}, n_i, \ldots, n_{i+3}) = (K, 1, K, 1, K+c) \longrightarrow (K, 2, K-2, K+c)$$

is feasible and increases the size of S by $2K + c - 3 > 0$, again a contradiction. Therefore the original assumption (4.1) must be false, and (a) is true. This completes the proof. ∎

<u>Lemma 4.3</u> For $D \geq 7$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then no internal term of S is one.

<u>Proof</u> Suppose on the contrary that S contains an internal term $n_i = 1$. Lemma 2.12(b) together with Lemma 4.1 implies then that

$$\min\left\{n_{i-1}, n_{i+1}\right\} = K.$$

First consider the case $3 \leq i \leq D-3$. Lemma 4.2(a) implies that $n_{i-2} > 1$ and $n_{i+2} > 1$. Hence the transformations

$$\tau_1 : (n_{i-1}, n_i, n_{i+1}) \longrightarrow (n_{i-1}-1, n_i+1, n_{i+1}),$$

$$\tau_2 : (n_{i-1}, n_i, n_{i+1}) \longrightarrow (n_{i-1}, n_i+1, n_{i+1}-1),$$

are feasible and alter the edge count by $(n_{i+1}-n_{i-2})$ and $(n_{i-1}-n_{i+2})$, respectively. Therefore, since S is maximal, $n_{i-2} \geq n_{i+1}$ and $n_{i+2} \geq n_{i-1}$. Thus

$$(n_{i-2}, n_{i-1}, \ldots, n_{i+2}) = (K+a, K+b, 1, K+c, K+d)$$

with $a \geq c \geq 0$ and $d \geq b \geq 0$. Since, as pointed out above, Lemma 4.1 implies that no two non-adjacent terms can exceed K, it follows that

$$b = c = 0 \text{ and } \min\{a, d\} = 0.$$

Without loss of generality, take $a = 0$. Then the transformation

$$\tau_3 : (n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = (K, K, 1, K, K+d)$$
$$\longrightarrow (K, 2, K-2, K+1, K+d)$$

is feasible and increases the edge count by $K + d - 2 > 0$. This proves that for $3 \le i \le D-3$, $n_i > 1$.

The remaining cases are $n_2 = 1$ and $n_{D-2} = 1$, which by symmetry are equivalent. Suppose then without loss of generality that $n_2 = 1$. It follows from the edge-maximal property of G that $n_4 \ge K$, since $n_1 = K$ and the transformation

$$\tau_4 : (n_2, n_3) \longrightarrow (n_2+1, n_3-1)$$

is feasible and by Lemma 2.7 alters the edge count by $K - n$. Therefore, without loss of generality, it may be supposed that

$$(n_0, n_1, n_2, n_3, n_4) = (1, K, 1, K, K+a),$$

where $a \ge 0$. Observing that $n_D = 1$, consider the sequence

$$S' = (n_2, n_3, \dots, n_D, n_1, n_0)$$

formed by a feasible transformation of S. This sequence has the same size as S, and is therefore edge-maximal. By Lemma 4.2(c), $n_{D-2} > 1$. Since the transformation

$$\tau_5 : (n_{D-2}, n_{D-1}, n_D, n_1) = (n_{D-2}, K, 1, K) \longrightarrow (n_{D-2}, K-1, 2, K)$$

is feasible and increases the size by $K - n_{D-2}$, it follows that $n_{D-2} \geq K$. Observe then that $n_{D-3} \geq 2$, since otherwise by Lemma 2.7 the size could be increased by the feasible transformation

$$\tau_6 : (n_{D-2}, n_{D-1}) \longrightarrow (n_{D-2}-1, n_{D-1}+1).$$

Hence for $D \geq 7$ the transformations

$$\tau_7 : n_4 \longrightarrow n_4-1 \quad \text{and} \quad n_{D-2} \longrightarrow n_{D-2}+1,$$

$$\tau_8 : n_4 \longrightarrow n_4+1 \quad \text{and} \quad n_{D-2} \longrightarrow n_{D-2}-1,$$

are both feasible. Thus the size can again be increased, again a contradiction, so that the only remaining possibility is $D = 7$. In this case S can, without loss of generality, be taken to be

$$S = (1, K, 1, K, K, K+b, K, 1),$$

where $b \geq 0$. But the sequence

$$(1, K, 2, K-2, K, K+b+1, K, 1)$$

yields a graph with more edges, a contradiction. This completes the proof of the lemma. ∎

Observe that, for this result, the assumption $D \geq 7$ is necessary, since for $D = 6$ the sequence

$$(1, K, 1, K, K+a, K, 1)$$

will in fact be maximal for sufficiently large a: see Table 2.4, for example. The next result establishes a lower bound of 3 for the internal terms of an edge-maximal vertex sequence:

<u>Lemma 4.4</u> For $D \geq 7$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then every internal term of S exceeds two.

<u>Proof</u> Suppose on the contrary that S is an edge-maximal vertex sequence with an internal term $n_i = 2$; further that $n_j > 2$ for $1 \leq j \leq i-1$. Two cases need to be considered:

Case 1 (i=2)

Here for some $a \geq 0$, $n_3 = \lceil K/2 \rceil + a$ and by Lemma 4.2(b), $n_4 > 2$. If $n_5 = 2$, then $n_4 \geq \lceil K/2 \rceil$ and $n_6 \geq \lceil K/2 \rceil$. In this case, it may be assumed, without loss of generality, that S contains the subsequence

$$(n_1, n_2, \ldots, n_6) = (K, 2, \lceil K/2 \rceil, \lceil K/2 \rceil, 2, \lceil K/2 \rceil + b),$$

where $b \geq 0$. But then the transformation

$$\tau_1 : n_2 \longrightarrow n_2+1 \quad \text{and} \quad n_3 \longrightarrow n_3-1 \qquad \ldots(4.3)$$

is feasible and increases the size. This contradiction establishes that $n_5 > n_2 = 2$. Hence (n_3, n_4, n_5) is a fat triple, so that the transformation (4.3) is feasible and by Lemma 2.7 alters the size by $K-n_4$. Therefore, since S is edge-maximal, $n_4 \geq K$. Now since $n_5 > 2$, it follows from Lemma 2.7 and the edge-maximality of S that $a = 0$; hence

$$(n_1, n_2, \ldots, n_5) = (K, 2, \lceil K/2 \rceil, K+b, n_5),$$

where $b \geq 0$. If $n_5 \leq K-2$, the transformation

$$\tau_2 : n_2 \longrightarrow n_2 + n_5 + b \quad \text{and} \quad n_4 \longrightarrow K-n_5$$

is feasible and increases the size by $2(n_5+b)$, an impossibility. Hence $n_5 = K-1 + c$, $c \geq 0$. Now since $D \geq 7$ and the transformations

$$\tau_3 : (n_4, n_5) \rightarrow (n_4 \pm 1, n_5 \mp 1)$$

are feasible, it follows from Lemma 2.7 that $n_6 = \lceil K/2 \rceil$. But then the transformation

$$\tau_4 : (n_1, n_2, \ldots, n_6) = (K, 2, \lceil K/2 \rceil, K+b, K-1+c, \lceil K/2 \rceil)$$
$$\rightarrow (K, K+b+c, \lceil K/2 \rceil, 2, K-1, \lceil K/2 \rceil)$$

is feasible and increases the edge count by $K+b+c-2 > 0$, a contradiction. This proves that $n_2 > 2$ and also (because of symmetry) that $n_{D-2} > 2$.

Case 2 $(3 \leq i \leq D-3)$

Since $D \geq 7$ and since S could be considered in reverse order, it may without loss of generality be supposed that $i < D-3$. Since $n_i = 2$, it follows that

$$n_{i-1} = \lceil K/2 \rceil + a \quad \text{and} \quad n_{i+1} = \lceil K/2 \rceil + b,$$

where $a \geq 0$, $b \geq 0$, and Lemma 4.2(a) implies that

$$n_{i-2} > 2 \quad \text{and} \quad n_{i+2} > 2.$$

Since, by assumption, $n_j > 2$ for $2 \leq j \leq i-1$, the triple

$(n_{i-3}, n_{i-2}, n_{i-1})$ is fat. Hence the transformation

$$\tau_5 : n_{i-1} \rightarrow n_{i-1}-1 \quad \text{and} \quad n_i \rightarrow n_i+1$$

is feasible and alters the edge count by $n_{i+1}-n_{i-2}$. It follows therefore that

$$n_{i-2} = \lceil K/2 \rceil + c,$$

where $c \geq b$.

It is now shown by contradiction that $n_{i+3} \geq 3$. Suppose that $n_{i+3} < 3$. Then, since $K \geq 8$, it follows from Lemma 2.12(a) that n_{i+3} is internal, and therefore from Lemma 4.3 that $n_{i+3} = 2$. Hence

$$(n_i, n_{i+1}, \ldots, n_{i+4}) = (2, \lceil K/2 \rceil + b, \lceil K/2 \rceil + x, 2, \lceil K/2 \rceil + y)$$

for some $x \geq 0$, $y \geq 0$. By Lemma 2.7, it may without loss of generality be supposed that $b = 0$. Now if $x > 0$, the transformation

$$\tau_6 : n_{i-2} \rightarrow n_{i-2}+x \quad \text{and} \quad n_{i+2} \rightarrow n_{i+2}-x$$

is feasible and alters the size by

$$x(n_{i-3}+a+c-2) \leq 0. \qquad \ldots(4.4)$$

If $i = 3$, then $n_{i-3} = 1$, $c = \lfloor K/2 \rfloor$, and it follows from (4.4) that $a + \lfloor K/2 \rfloor \le 1$, an impossibility. For $i \ge 4$, $n_{i-3} \ge 3$ by hypothesis, so that by (4.4), $a + c + 1 \le 0$, again an impossibility. Hence $x = 0$, and the size can be increased by the feasible transformation

$$\tau_7 : (n_i, n_{i+1}, n_{i+2}, n_{i+3}) = (2, \lceil K/2 \rceil, \lceil K/2 \rceil, 2)$$
$$\rightarrow (3, \lceil K/2 \rceil - 1, \lceil K/2 \rceil - 1, 3),$$

a contradiction. This proves that $n_{i+3} > 2$ and hence that $(n_{i+1}, n_{i+2}, n_{i+3})$ is a fat triple.

The result of the preceding paragraph implies that the transformation

$$\tau_8 : n_i \rightarrow n_i + 1 \quad \text{and} \quad n_{i+1} \rightarrow n_{i+1} - 1$$

is feasible; furthermore, it increases the size by $(n_{i-1} - n_{i+2})$. It must therefore be true that

$$n_{i+2} = \lceil K/2 \rceil + d,$$

for some $d \ge a$. Moreover, since $n_{i+3} \ge 3$, it follows that $n_{i+1} = \lceil K/2 \rceil$; that is, $b = 0$. Hence

$$(n_{i-2}, n_{i-1}, \ldots, n_{i+2}) = (\lceil K/2 \rceil + c, \lceil K/2 \rceil + a, 2, \lceil K/2 \rceil, \lceil K/2 \rceil + d).$$

The next task is to prove that $a = 0$. If $a > 0$, then $(n_{i-3}, n_{i-2}, n_{i-1})$ is a fat triple; since $d \geq a$, G would have removable vertices in levels i-1 and i+2 unless $(n_{i+2}, n_{i+3}, n_{i+4})$ were a lean triple. Hence by Lemma 4.1, $(n_{i+2}, n_{i+3}, n_{i+4})$ must be lean, and

$$(n_{i+2}, n_{i+3}, n_{i+4}) = (\lceil K/2 \rceil + d, 2+e, \lceil K/2 \rceil - f)$$

for some $e > 0$, $f > 0$ such that

$$f-(e+d) = 2\lceil K/2 \rceil - K+1 > 0. \qquad \ldots(4.5)$$

Now the transformation

$$\tau_9 : (n_{i-1}, n_i, n_{i+1}) = (\lceil K/2 \rceil + a, 2, \lceil K/2 \rceil) \longrightarrow (\lceil K/2 \rceil + a+f-e, 2+e, \lceil K/2 \rceil - f)$$

is feasible and increases the edge count by

$$\begin{aligned}(f-e)(f+c) + f(a-d) &= f(f-e-d) + [f(a+c)-ec] \qquad \ldots(4.6)\\ &> 0,\end{aligned}$$

by (4.5). This contradiction establishes the fact that $a = 0$. Indeed, since the expression (4.6) is positive even when $a = 0$, it also establishes the fact that $(n_{i+2}, n_{i+3}, n_{i+4})$ is fat.

A similar argument shows that for $i \geq 4$, $n_{i-3} \geq 3$ and $(n_{i-4}, n_{i-3}, n_{i-2})$ is a fat triple. But then the edge-maximal graph G has removable vertices in levels i-2 and i+2, by Lemma 4.1 a contradiction. This establishes the lemma for $i \geq 4$.

Suppose then that $i = 3$. Since $n_{i+2} = \lceil K/2 \rceil + d$, $n_{i+3} \geq 3$, and $(n_{i+2}, n_{i+3}, n_{i+4})$ is a fat triple, it must be true that n_{i+2} contains a removable vertex. If $n_{i+4} > n_{i+1} = \lceil K/2 \rceil$, then since by Lemma 4.3 $n_{i+5} \geq 2$, it follows that n_{i+4} also contains a removable vertex, contrary to Lemma 4.1. Hence $n_{i+4} \leq n_{i+1}$. Let

$$n_{i+3} = 2+e, \quad n_{i+4} = \lceil K/2 \rceil - f,$$

for some $e \geq 1$, $f \geq 0$. Further, let

$$t = 2\lceil K/2 \rceil + d + e - f + 2 - K - 1. \qquad \text{...(4.7)}$$

Then n_{i+2} contains

$$r = \min\{d+1, t\}$$

removable vertices. The transformation

$$\tau_{10} : n_{i-1} \longrightarrow n_{i-1} + r \quad \text{and} \quad n_{i+2} \longrightarrow n_{i+2} - r$$

alters the edge count by

$$r(\lfloor K/2 \rfloor - d - e + r) \le 0 .$$

Hence

$$e + d - r \ge \lfloor K/2 \rfloor . \qquad \ldots(4.8)$$

If $D = 7$, then $n_{i+3} = K$ and the edge count can be increased by the transformation

$$\tau_{11} : (1, K, \lceil K/2 \rceil, 2, \lceil K/2 \rceil, \lceil K/2 \rceil + d, K, 1)$$
$$\rightarrow (1, K, \lceil K/2 \rceil - 2, 3, \lceil K/2 \rceil, \lceil K/2 \rceil + d + 1, K, 1).$$

Therefore $D > 7$. Now two subcases specified according to the value of r are considered.

(a) $r = t < d + 1$

Here equation (4.7) yields

$$d - t + e - f = K - 2\lceil K/2 \rceil - 1 < 0,$$

from which, since $d \ge t$, it follows that $e < f$. The transformation

$$\tau_{12} : (n_2, n_3, \ldots, n_5) = (\lceil K/2 \rceil, 2, \lceil K/2 \rceil, \lceil K/2 \rceil + d)$$
$$\rightarrow (\lceil K/2 \rceil + f + t - e, 2 + e, \lceil K/2 \rceil - f, \lceil K/2 \rceil + d - t)$$

is feasible and increases the edge count by

$$
\begin{aligned}
&(t+f-e)(\lfloor K/2 \rfloor+f+t) - d(f+t) \\
&\geq (d+1)(\lfloor K/2 \rfloor+f+t) - d(f+t) \\
&> 0,
\end{aligned}
$$

a contradiction. Consequently $t \geq d + 1$.

(b) $r = d + 1$

In this case, by (4.8),

$$n_{i+3} = 2 + e \geq \lfloor K/2 \rfloor + 3;$$

that is,

$$n_{i+3} = \lceil K/2 \rceil + y,$$

for some $y \geq 3$. Now since $n_{i+4} \geq 2$, G has removable vertices in level i + 3 whenever $(n_{i+3}, n_{i+4}, n_{i+5})$ is a fat triple. It may therefore, without loss of generality, be assumed that $(n_{i+3}, n_{i+4}, n_{i+5})$ is lean. Then for some $g \geq 0$,

$$n_{i+5} = \lceil K/2 \rceil - g, \quad n_{i+6} \geq n_{i+3}, \qquad \ldots(4.9)$$

$$y - f - g = K + 1 - 3\lceil K/2 \rceil. \qquad \ldots(4.10)$$

In view of (4.9), it may be assumed that n_{i+4} is as small as possible; that is,

$$(\lceil K/2\rceil - f - 1)(\lceil K/2\rceil - g + 1) < K.$$

Since $n_{i+4}n_{i+5} \geq K$, it follows that $f \geq g$ and hence that

$$(\lceil K/2\rceil - g)(\lceil K/2\rceil + y) \geq K.$$

Thus by (4.10) the quantity

$$\lambda = d + f + g = \lceil K/2\rceil - y + 2 \geq d+1,$$

and the transformation

$$\begin{aligned}\tau_{13} : (n_2, n_3, \ldots, n_5) &= (\lceil K/2\rceil, 2, \lceil K/2\rceil, \lceil K/2\rceil + d) \\ &\longrightarrow (\lceil K/2\rceil + \lambda, \lceil K/2\rceil + y, \lceil K/2\rceil - f, \lceil K/2\rceil - g)\end{aligned}$$

is feasible and increases the edge count by

$$(d+f+g)(\lambda - f) + \lambda\lfloor K/2\rfloor > \lambda(\lambda + \lfloor K/2\rfloor - f) > 0.$$

This contradiction completes the proof of the lemma. ∎

Lemma 4.4 establishes a crucial lower bound on the order of internal terms of a vertex sequence of an edge-maximal graph of $\mathscr{C}_e$. In order to establish a sharper upper bound than that implied by Lemma 4.1, recall the definition (2.17) of $\alpha = \lceil 2\sqrt{K} \rceil$ and the related definitions of α_1, α_2, and $K_\alpha = K+1-\alpha$ given in Section 2.3. Then the following result can be stated and proved as an application of Lemma 4.4.

Lemma 4.5 For $D \geq 7$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathcal{G}_e(n,*,*,D,K)$. Then if any internal term n_i of S exceeds K_α, n_i contains a removable vertex.

Proof Supose on the contrary that S contains a term

$$n_i = K_\alpha + a,$$

where $a > 0$ and $2 \leq i \leq D-2$. Suppose further that n_i contains no removable vertex. Then at least one of the following conditions must hold:

(i) (n_{i-2}, n_{i-1}, n_i) is fat;
(ii) (n_{i-1}, n_i, n_{i+1}) is fat;
(iii) (n_i, n_{i+1}, n_{i+2}) is fat;
(iv) $\min\{n_{i-1}, n_{i+1}\} = \lceil K/(K_\alpha+a) \rceil$.

Condition (i) cannot hold since it would imply that

$$n_{i-2} + n_{i-1} \leq \alpha - a,$$

hence that $n_{i-2}n_{i-1} < K$. Similarly, condition (iii) cannot hold. Now suppose (iv) is true, and observe that it suffices to consider only the case $n_{i-1} \le n_{i+1}$. Let

$$f(K,a) = K/(\bar{K_\alpha}+a-1).$$

For fixed K, f(K,a) attains its maximum at a = 1. Hence

$$n_{i-1} < f(K,a) \le f(K,1),$$

so that by Lemma 4.4, f(K,1) > 3; that is,

$$K < 3(\alpha-1)/2,$$

an inequality which does not hold for any $K \ge 8$. Hence condition (iv) cannot hold.

The only remaining possibility is condition (ii). Since (i) and (iii) are not possible, it follows that

$$n_{i-2} > n_{i+1} \quad \text{and} \quad n_{i+2} > n_{i-1}.$$

But then since (iv) does not hold, this means that the transformation

$$\tau : (n_{i-1},n_i) \longrightarrow (n_{i-1}+1,n_i-1)$$

is feasible and increases the size, a contradiction which completes the proof of the lemma. ∎

A sharper upper bound on internal terms of an edge-maximal vertex sequence is then a corollary:

<u>Lemma 4.6</u> For $D \geq 7$, $K \geq 8$, let $S = S_D(a)$ be a vertex sequence of an edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then no two non-adjacent internal terms exceed K_α.

<u>Proof</u> Suppose on the contrary that S contains terms

$$n_i > K_\alpha, \quad n_j > K_\alpha,$$

where $2 \leq i \leq D-2$, $2 \leq j \leq D-2$, and $|i-j| > 1$. But then by Lemma 4.5, both n_i and n_j contain a removable vertex, in contradiction to Lemma 4.1. ∎

A second consequence of Lemma 4.4 (also a consequence of Lemma 4.3) is the following fundamental property:

<u>Lemma 4.7</u> For $D \geq 6$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then $n_1 = n_{D-1} = K$.

<u>Proof</u> By Lemma 2.1(a), $n_0 = n_D = 1$; further, by Lemma 2.12(a), $n_1 \geq K$ and $n_{D-1} \geq K$. Then consider the 4-tuple

$$(1, n_1, n_2, n_3),$$

and suppose that $n_1 = K+a$ for some $a > 0$. By Lemma 2.7 the transformation

$$\tau_1 : (n_1, n_2) \longrightarrow (n_1-a, n_2+a)$$

increases the size of S by $\lambda = a(n_3-1)$, a positive quantity unless $n_3 = 1$. Since by Lemma 4.4, $n_3 \geq 3$ for $D \geq 7$, it may be supposed that $n_3 = 1$ and $D = 6$. It follows that

$$S = (1, K+a, K+b, 1, K+c, K+d, 1),$$

where $b \geq 0$, $c \geq 0$, $d \geq 0$. But by Lemma 4.1, the assumption $a > 0$ implies that $c = d = 0$. Hence the transformation

$$\tau_2 : S \longrightarrow (1, K, K+a+b, K, 1, K, 1)$$

increases the edge count by $(a+b)(K-1)$, a contradiction. Therefore $a = 0$ and $n_1 = K$, as required. The same argument shows that $n_{D-1} = K$. ∎

The basic properties of edge-maximal vertex sequences of $\mathscr{C}_e$ established by the above lemmas will now be applied in a further sequence of lemmas leading to the proof of Theorem 2.5. Much of this work will be concerned with the order of triples within a vertex sequence, hence the following definitions. A triple T of a vertex sequence S will be said to be maximal (respectively, minimal) if no triple of S has order greater (respectively, less) than that of T. Similarly, an internal triple T of S will be said to be maximal internal (respectively, minimal internal) if no internal triple of S has order greater (respectively, less) than that of T. Note that a minimal internal triple is not necessarily lean, and recall that an internal triple exists if and only if $D \geq 6$.

Lemma 4.8 For $D \geq 6$, $K \geq 8$, let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then if two adjacent terms n_i and n_{i+1} of S contain a removable vertex, $3 < i < D-3$.

Proof Observe first that by virtue of Lemmas 2.1(a) and 4.7, $1 < i < D-1$. Suppose then that $i = 2$, so that both n_2 and n_3 contain a removable vertex. But this implies that $n_4 = K$, since otherwise by Lemma 2.7 one of the feasible transformations

$$\tau_1 : (K,n_2,n_3,n_4) \longrightarrow (K,n_2+1,n_3-1,n_4),$$
$$\tau_2 : (K,n_2,n_3,n_4) \longrightarrow (K,n_2-1,n_3+1,n_4),$$

would increase the edge count. For $D = 6$, $n_4 = K$ implies that n_4 contains a removable vertex, in contradiction to Lemma 4.1; for $D \geq 7$, $n_4 = K$ contradicts Lemma 4.6. It follows that $i \neq 2$ and, by the same argument, that $i \neq D-2$.

Suppose therefore that $i = 3$. Applying Lemma 2.7 as in the preceeding paragraph then yields the conclusion that $n_2 = n_5$. Furthermore, $n_4 \geq K$, since otherwise the feasible transformation τ_1 would increase the edge count. Suppose that n_4 contains exactly λ removable vertices, and consider the feasible transformation

$$\tau_3 : (K, n_2, n_3, n_4, n_2) \longrightarrow (K, n_2+1, n_3+\lambda-1, n_4-\lambda, n_2),$$

which changes the edge count by $K - (n_4-\lambda)$. For $D = 6$, $n_2 = n_5 = K$, while for $D \geq 7$, $n_2 = n_5 \geq 3$ by Lemma 4.4; in either case it follows that $n_4 - \lambda < K$, so that τ_3 in fact increases the edge count. This contradiction forces the conclusion that $i \neq 3$, and the same argument shows that $i \neq D-3$. This completes the proof. ∎

Observe that this lemma establishes Theorem 2.5 for $D \leq 7$; in fact, for $D = 6$ it establishes the slightly stronger result that at most one term of an edge-maximal vertex sequence contains a removable vertex. The next lemma is really a more precise form of Lemma 4.1.

<u>Lemma 4.9</u> Suppose that an edge-maximal vertex sequence $S = S_D(u)$ of a graph $G \varepsilon \mathscr{C}_e(n,*,*,D,K)$ contains an internal term n_i which has $\lambda > 0$ removable vertices. Then the order t_i of the triple $T_i = (n_{i-1}, n_i, n_{i+1})$ satisfies

$$t_i \geq t_j + \lambda,$$

where t_j is the order of any triple $T_j = (n_{j-1}, n_j, n_{j+1})$ disjoint from T_i.

<u>Proof</u> A transformation which moves λ vertices from n_i to n_j changes the edge count by

$$Q_j = \binom{t_j}{2} - \binom{t_i - \lambda}{2}.$$

If for some value of j, $t_i < t_j + \lambda$, then $Q_j > 0$, contradicting the assumption that S is edge-maximal. ∎

The triple T_i of Lemma 4.9 is clearly a maximal triple. The next few lemmas establish important properties of minimal internal triples.

<u>Lemma 4.10</u> For $D \geq 6$, $K \geq 8$, let (x,y,z) be a minimal internal triple of an edge-maximal vertex sequence S. Then y does not contain a removable vertex.

Proof Suppose on the contrary that y does contain a removable vertex. Then by Lemma 4.9 (x,y,z) is a maximal triple, and so all internal triples of S are maximal. Consequently $n_4 \geq n_1 = K$, $n_{D-4} \geq n_{D-1} = K$, and $n_{i+3} = n_i$ for every $2 \leq i \leq D-5$; in other words, S is 3-recurring. Since by Lemma 4.1 y occurs at most once in S, it follows that $D \leq 8$; further, these conditions also imply that for $D = 6$ or 7, more than one term of S contains a removable vertex, contrary to Lemma 4.8. Then for $D = 8$,

$$S = (1,K,z,x,y,z,x,K,1),$$

where $y \geq K$ contains $\lambda > 0$ removable vertices. Then $y-\lambda \leq K$. If $y-\lambda = K$, it follows without loss of generality that $(z,x) = (K,1)$, hence that the feasible transformation

$$\tau_1 : S \longrightarrow (1,K,K+\lambda,K,1,K,1,K,1)$$

increases the edge count by $\lambda(K-1) > 0$; for $y-\lambda < K$, the feasible transformation

$$\tau_2 : S \longrightarrow (1,K,z+\lambda,x,y-\lambda,z,x,K,1)$$

increases the edge count by $\lambda(K-y+\lambda) > 0$. These contradictions force the conclusion that $D \neq 8$, and the proof is complete. ∎

<u>Lemma 4.11</u> For $D \geq 7$ and $K \geq 8$, let β be a minimal internal triple of an edge-maximal vertex sequence S. Then β contains a removable vertex if and only if β is fat.

<u>Proof</u> Necessity is obvious. To prove sufficiency, suppose that $\beta = (x,y,z)$ is of order $t > K+1$ and contains no removable vertex. Without loss of generality assume that $x \geq z$. By Lemma 2.12(b), $y \geq \lceil K/z \rceil$, and in fact $y = \lceil K/z \rceil$, since otherwise y would contain a removable vertex.

Now suppose that $x > \max(y,z)$, and observe that the term to the left of x in S must either be $n_1 = K$ or $w \geq z$ (since β is minimal). Hence x must contain a removable vertex, a contradiction. Then it must be true that $x \leq \max\{y,z\}$.

By Lemmas 4.3 and 4.4, every term of S is at least 3; in particular, since $K \geq 8$, it follows that $3 \leq z \leq \lceil K/2 \rceil - 1$ ($z \geq \lceil K/2 \rceil$ would contain a removable vertex). Therefore

$$2y + z = 2\lceil K/z \rceil + z \leq K+1 < x+y+z,$$

so that $y < x \leq \max\{y,z\}$. Since by assumption $x \geq z$ it must therefore be true that $x = z \leq \lceil K/2 \rceil - 1$. If $x < \lceil K/2 \rceil - 1$, then

$$t = 2x + \lceil K/x \rceil \le K+1,$$

in contradiction to the original assumption that β is fat. On the other hand, if $x = \lceil K/2 \rceil - 1$, then as above x must contain a removable vertex, also a contradiction. ∎

<u>Lemma 4.12</u> For $D \ge 7$ and $K \ge 8$, let $\beta = (x,y,z)$ be a minimal internal triple of an edge-maximal vertex sequence S. If S contains the 3-recurring 4-tuple (β, x), then β is lean.

<u>Proof</u> By Lemma 4.10 applied to the minimal internal triples (x,y,z) and (y,z,x), neither y nor z can contain a removable vertex; by Lemma 4.1, x cannot contain a removable vertex. Then β does not contain a removable vertex, and therefore, by Lemma 4.11, is lean. ∎

<u>Lemma 4.13</u> For $D \ge 7$ and $K \ge 8$, let β be a minimal internal triple of order t of an edge-maximal vertex sequence S which contains a maximal triple of order t'. Then $t' > t$.

<u>Proof</u> Since clearly $t' \ge t$, suppose that $t' = t$. Then every internal triple of S has order t'. Since by Lemma 2.1 the order of (n_0, n_1, n_2) exceeds K+1, it follows that $t' > K+1$, hence by Lemma 4.11 that every internal triple contains a removable vertex, in contradiction to Lemma 4.10. Then $t' > t$, as required. ∎

<u>Lemma 4.14</u> For $D \geq 7$ and $K \geq 8$, let $\beta = (x,y,z)$ be a minimal internal triple of an edge-maximal vertex sequence S. Then $xz \geq K$.

<u>Proof</u> Consider the 5-tuple (u,β,v) of S, and observe that if $u < z$, β can be minimal only if $u = n_1 = K$, so that $z > K$ and $xz > K$, as required. Similarly if $v < x$. Then suppose that $u \geq z$, $v \geq x$, and further that $xz < K$. But by Lemma 2.12(b) it must therefore be true that $u > z$, $v > x$, and hence, since S is edge-maximal, that neither of the transformations

$$\tau_1 : (x,y,z) \longrightarrow (x+1,y-1,z),$$
$$\tau_2 : (x,y,z) \longrightarrow (x,y-1,z+1),$$

is feasible. It follows that $z = x$ and $y = \lceil K/x \rceil$. Observe now that since $xz = x^2 < K$, $\lceil K/x \rceil > x$; furthermore, since $K \geq 8$,

$$x + y + z = 2x + \lceil K/x \rceil < K+1,$$

in contradiction to Lemma 2.12(c). This contradiction establishes the lemma. ∎

The next result is a near-converse of Lemma 2.5 and identifies an important criterion for determining whether or not a given triple is minimal internal. The proof depends on Lemmas 4.12-4.14.

<u>Lemma 4.15</u> For $D \geq 7$ and $K \geq 8$, suppose that an edge-maximal vertex sequence S contains a triple $\beta = (x,y,z)$ embedded in a 5-tuple (u,β,v), where $u \geq z$ and $v \geq x$. Then β is a minimal internal triple of S.

<u>Proof</u> Observe first that by virtue of the condition $u \geq z$ together with Lemmas 2.1 and 4.3, β is not the triple (n_1,n_2,n_3). Nor, by a similar argument, is it $(n_{D-3},n_{D-2},n_{D-1})$. Hence β is internal.

Suppose then that β is not minimal, and let $\beta' = (x',y',z')$ be the leftmost minimal internal triple of S. It follows that the term preceding x' in S is either $n_1 = K$ or else exceeds z'. Without loss of generality, it may be supposed that β' precedes β in S. Observe that if β' and β have two terms $y' = x$ and $z' = y$ in common, then since β is not minimal, $z > x'$, while by hypothesis, $u = x' \geq z$. Hence β' and β have at most one term in common.

Suppose then that $z' = x$, so that $u = y' \geq z$. Since β' is minimal, it follows that $x' < y$. Then let $y = x' + \lambda$, $y' = z + \mu$, where $\lambda > \mu \geq 0$. Consider the transformation

$$\tau_1 : (x',y',z',x'+\lambda,y'-\mu,v) \longrightarrow (x',y',z',x',y'+\lambda-\mu,v),$$

which by Lemma 2.7 increases the edge count by $\lambda(v-z')$. By Lemma 4.14, $x'z' \geq K$, so that τ_1 is feasible, and hence, to avoid contradicting the hypothesis that S is edge-maximal, it must be assumed that $v = z'$. It follows that the term $y' + \lambda - \mu$ in the transformed sequence S′ must contain $\lambda - \mu$ removable vertices. Since β' is minimal also in S′, it follows from Lemma 4.12 that β' is lean. But then the transformation

$$\tau_2 : y' + \lambda - \mu \longrightarrow y', \quad n_{D-2} \longrightarrow n_{D-2} + \lambda - \mu$$

must increase the edge count, a contradiction. Hence β and β' must be disjoint.

Now let t and $t' = t-s$, $s > 0$, be the orders of β and β', respectively, and imagine a transformation τ_3 of S as follows. First remove the triple β so that $u \geq z$ and $v \geq x$ are adjacent; then use t' of the removed vertices to form a new triple (x',y',z') adjacent to the existing β'. The result is a feasible vertex sequence S′. Next add the remaining s vertices to the middle term of a maximal triple of S′, yielding a vertex sequence with

$$s(t''-t') + (u-z)(v-x)$$

more edges than S, where t'' is the order of a maximal triple of S′. Since $u \geq z$, $v \geq x$, and since by Lemma

4.13, $t'' > t'$, S'' has more edges than S, a contradiction. It follows that β must be a minimal internal triple of S. ∎

Lemma 4.16 For $D \geq 7$ and $K \geq 8$, let S be an edge-maximal vertex sequence. Then $\beta = (n_2, n_3, n_4)$ (respectively, $\beta = (n_{D-4}, n_{D-3}, n_{D-2})$) is a minimal internal triple of S if and only if $n_5 \geq n_2$ (respectively, $n_{D-5} \geq n_{D-2}$).

Proof By symmetry the proof for (n_2, n_3, n_4) implies the result for $(n_{D-4}, n_{D-3}, n_{D-2})$. Hence suppose first that $n_5 < n_2$, and let the order of β be t. Then the order of (n_3, n_4, n_5) is less than t, so that β cannot be minimal internal. This proves necessity.

To prove sufficiency, suppose that $n_5 \geq n_2$ and further that β is not minimal internal. Then by Lemma 4.15, $n_4 > K$ and hence contains a removable vertex. Therefore by Lemma 4.10 a minimal internal triple β' of S is disjoint from β. Further, by Lemmas 4.1 and 4.11, β' can be fat only if $\beta' = (n_5, n_6, n_7)$ and n_5 contains a removable vertex. But in this case, since S is edge-maximal, it follows from Lemma 2.7 that $n_6 = n_3$, so that vertices can be transferred from n_5 to n_4 without affecting the edge count. Thus it may without loss of generality be supposed that β' is lean.

Now let $t = K+1+s$, where $s > n_4-K$, and imagine rearranging S as in the proof of Lemma 4.15. Remove β from S and use K+1 of the removed vertices to form a lean triple β' adjacent on the right to the existing β'; the remaining s vertices are added to n_5. The result is a feasible vertex sequence having

$$s(n_5+n_6-1) - (n_4-K)(n_5-n_2)$$
$$\geq (n_4-K)(n_2+n_6-1) > 0$$

more edges than S, a contradiction which completes the proof. ∎

In order to state the next lemma, further definitions are convenient. For every $2 \leq i \leq D-2$, let β_i denote the triple (n_{i-1},n_i,n_{i+1}). Then for $j-i \geq 2$ an internal $(j-i+1)$-tuple $S_{i,j} = (n_i,\ldots,n_j)$ of a vertex sequence S is said to be <u>compact</u> if every triple β_k, $i+1 \leq k \leq j-1$, is minimal internal; further, $S_{i,j}$ is said to be <u>maximal compact</u> if it is compact and neither β_i nor β_j is minimal internal. S is said to be <u>regular</u> if it contains exactly one maximal compact k-tuple.

<u>Lemma 4.17</u> For $D \geq 7$ and $K \geq 8$, every edge-maximal vertex sequence is regular.

Proof Suppose the contrary, and let S denote a non-regular vertex sequence which, over all non-regular edge-maximal vertex sequences for given n, D, and K, contains a maximal compact tuple $S^*_{i,j}$ of greatest length. Let $\beta = (x,y,z)$ denote the rightmost triple of $S^*_{i,j}$ and let $\beta' = (x',y',z')$ denote a minimal internal triple of S which is not in $S^*_{i,j}$. Without loss of generality, it may therefore be assumed that S contains the 5-tuples (u,β,v) and (u',β',v') where $v > x$ and $u' > z'$. Hence, if β and β' overlap, the only possibility is that $y = u'$, $z = x'$. In this case, for some $\lambda > 0$, S must contain the tuple

$$(u,x,y,z,x+\lambda,y-\lambda,v'),$$

which can be transformed to

$$(u,x,y,z,x,y,v')$$

yielding a feasible vertex sequence S′ with $v'-z \geq 0$ more edges than S. But this implies that $S^*_{i,j}$ was not of greatest length, in contradiction to the original assumption. Hence the triples β and β' must be disjoint.

Imagine therefore the following transformation of S. Remove β' so that u' and v' are adjacent, and then from the vertices of β' form a triple (x,y,z) and insert it between β and v. Since by Lemma 4.14, $xz \geq K$, this transformation yields a feasible vertex sequence S′ with $(u'-z')(v'-x') \geq 0$ more edges than S. But as above this means that $S^*_{i,j}$ was not of greatest length, a contradiction which proves the lemma. ∎

<u>Lemma 4.18</u> For $D \geq 7$ and $K \geq 8$, let $S^*_{i,j}$ be the maximal compact tuple of an edge-maximal vertex sequence S. Then $i \leq 3$ and $j \geq D-3$. Further, if $i = 3$ (respectively, $j = D-3$), then n_2 (respectively, n_{D-2}) contains a removable vertex.

<u>Proof</u> It suffices to prove the result for i. Suppose then that $i \geq 4$, and denote by (x,y,z) the initial triple (n_i, n_{i+1}, n_{i+2}) of $S^*_{i,j}$. Since $S^*_{i,j}$ is maximal compact, it follows that $n_{i-1} > z$ and further, by Lemma 4.17, that $n_{i-2} + n_{i-1} > y+z$. Therefore the transformation

$$\tau : (n_{i-3}, n_{i-2}, n_{i-1}, x) \longrightarrow (n_{i-3}, n_{i-2}+n_{i-1}-z, z, x)$$

is feasible and by Lemma 2.7 increases the edge count by $(n_{i-3}-x)(n_{i-1}-z)$. Since $S^*_{i,j}$ is maximal compact, this quantity must be negative, so that $n_{i-3} < x$. Then since by Lemma 4.7, $n_1 = K > x$, it follows that $i \neq 4$. But for $i \geq 5$, it must be true that

$$n_j < n_{j+3},$$

for every $2 \le j \le i-3$, since otherwise by Lemma 4.15 a triple $(n_{j+1}, n_{j+2}, n_{j+3})$ would be minimal internal, in contradiction to the regularity of S. In particular, this means that $n_2 \le n_5$, hence by Lemma 4.16 that (n_2, n_3, n_4) is minimal internal, again in contradiction to the regularity of S. Hence $i \le 3$.

Finally, observe that if $i = 3$, then since $n_2 > n_5$, n_2 must contain a removable vertex. ∎

It is clear from Lemma 4.18 that in an edge-maximal vertex sequence, every internal triple, with the possible exception of (n_2, n_3, n_4) or $(n_{D-4}, n_{D-3}, n_{D-2})$ (but not both), is minimal internal. To complete the proof of Theorem 2.5, then, one further result is required:

<u>Lemma 4.19</u> For $D \ge 7$, $K \ge 8$, every minimal internal triple of an edge-maximal vertex sequence is lean.

<u>Proof</u> Suppose there exists a minimal internal triple $\beta = (x, y, z)$ whose order $t > K+1$. Then for $D \ge 8$ there must by Lemma 4.18 exist a 4-tuple (x, y, z, x) or (z, x, y, z). In either case, by Lemma 4.12, β is lean. Suppose then that $D = 7$ and observe that, by Lemmas 4.10, 4.11, and 4.1 exactly one of the terms x, z

contains a removable vertex. Suppose without loss of generality that x contains a removable vertex. Then

$$S = (1, K, x, y, z, n_5, K, 1) \text{ or } (1, K, n_2, x, y, z, K, 1).$$

In the first case, since β is minimal internal, $n_5 \geq x$. Suppose then that $n_5 > x$ and recall that by Lemma 14, $xz \geq K$. But x and n_5 must therefore both contain removable vertices, in contradiction to Lemma 4.1. Hence $n_5 = x$, so that by Lemma 4.12, β is lean. In the second case, since x contains a removable vertex, (n_2, x, y) cannot by Lemma 4.10 be minimal internal; hence $n_2 > z$ also has a removable vertex by Lemma 4.14, a situation which by Lemma 2.7 can arise only if $y = K$; but then y also contains a removable vertex, in contradiction to Lemma 4.1. ∎

For $D \geq 7$, Theorem 2.5 is an immediate consequence of Lemmas 4.18 and 4.19, which imply moreover that the (D-4)-tuple $(n_2, \ldots, n_{D-3})$ is 3-recurring and that only n_{D-2} can possibly contain a removable vertex; as discussed earlier, Lemma 4.8 implies Theorem 2.5 for D = 6 and, together with Lemma 4.10, establishes in this case also that only n_{D-2} can contain a removable vertex. Thus the main result of this section corresponds very closely to Theorem 2.1 for the class $\mathscr{C}_v$, a result which was very easily proved; it is curious that the proof should be so much more difficult for graphs of $\mathscr{C}_e$.

Theorem 2.5 For $D \geq 6$ and $K \geq 8$, every edge-maximal graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence in which every internal triple except possibly $(n_{D-4}, n_{D-3}, n_{D-2})$ is lean. ∎

A final remark: it seems likely that a very similar, but rather more complicated, development can be carried out for graphs of $\mathscr{C}_e(n,*,\delta,D,K)$, yielding a result closely analogous to Theorem 2.5. The main complications would derive from the fact that conditions (a) and (c) of Lemma 2.12 are expressed in terms of δ, while condition (b) is expressed in terms of K.

4.2 Structure of Edge-Maximal Graphs over $\mathscr{C}_e(n,*,*,D,K)$

This section completes the analysis of edge-maximal K-edge-connected D-critical graphs ($D \geq 6$) by establishing a collection of results (Theorems 4.1 and 2.6-2.10) which specify the structure of these graphs. The first of these results, Theorem 4.1, is a counterpart for $k \leq 7$ of Theorem 2.5, which was proved in Section 4.1; the remainder explore the further consequences of Theorem 2.5, in order to specify the precise structure of edge-maximal graphs.

In order to formulate a precise statement of the characterization theorem for $K \leq 7$, further terminology will be convenient. Recall first from Section 2.3 the definitions of the terms <u>lean</u> and <u>fat</u> for doubles, and the introduction of the quantities α, α_1, and α_2. Then a double (x,y) will be said to be smooth if $|x-y| \le 1$; otherwise, rough. Correspondingly, a vertex sequence will be said to be smooth if every internal double is smooth, and regular if every double of one of the internal (D-4)-tuples $(n_2,\ldots,n_{D-3})$, $(n_3,\ldots,n_{D-2})$ is lean and smooth, and if each terminal double is of order K+1. Table 4.1 displays the values of α, α_1, and α_2 for $K \le 7$. Observe that with the exception of K = 3 and K = 7, the smooth doubles (α_1,α_2) and (α_2,α_1) represent the only possible lean doubles; however, for K = 3 or 7, the lean smooth doubles (2,2) or (3,3), respectively, also occur, and the doubles (α_1,α_2) or (α_2,α_1) are rough.

Values of α, α_1, α_2 ($K \le 7$)

K	α	α_1	α_2
1	2	1	1
2	3	1	2
3	4	1	3
4	4	2	2
5	5	2	3
6	5	2	3
7	6	2	4

Table 4.1

As for K-connected graphs (Section 2.2), the statement of a characterization theorem for $K \le 7$ is facilitated by the definition of certain sets of tuples. Accordingly, for integers $r \ge 1$ and $0 \le s \le 1$, define

$\mathscr{S}(0,r,s)$: the set of all (2r+s+2)-tuples $S = (n_1, \ldots, n_{2r+s+2})$ which satisfy the following conditions:

* S contains r+s+2 terms 2 and r terms 1;
* $n_1 = n_{r+s+2} = 2$;
* S contains no double of order less than 3.

This set corresponds to arrangements for K = 2, where as indicated in Table 4.1, $\alpha_1 = 1$ and $\alpha_2 = 2$. It is not difficult to see that every tuple of $\mathscr{S}(0,r,0)$ contains exactly one fat double (2,2), while $\mathscr{S}(0,r,1)$ contains at least two tuples which contain the triple (2,2,2). Thus for $t \geq 0$ the following definitions are justified:

$\mathscr{S}(t,r,0)$: the set of all tuples formed from elements of $\mathscr{S}(0,r,0)$, by adding t vertices in all possible ways to the fat double;

$\mathscr{S}(t,r,1)$: the set of all tuples formed from elements of the subset of $\mathscr{S}(0,r,1)$ which contain the triple $(x,y,z) = (2,2,2)$, by adding t vertices to y.

Examples of these sets are as follows:

$\mathscr{S}(0,1,0) = \{(2,1,2,2),(2,2,1,2)\}$;

$\mathscr{S}(2,1,0) = \{(2,1,4,2),(2,1,3,3),(2,1,2,4),(4,2,1,2),(3,3,1,2),(2,4,1,2)\}$;

$\mathscr{S}(0,1,1) = \{(2,1,2,2,2),(2,2,1,2,2),(2,2,2,1,2)\}$;

$\mathscr{S}(3,1,1) = \{(2,1,2,5,2),(2,5,2,1,2)\}$.

The characterization theorem for $K \leq 7$ may now be stated:

<u>Theorem 4.1</u> For $D \geq 6$ and $K \leq 7$, suppose S is a vertex sequence of a graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then G is edge-maximal if and only if one of the following conditions is satisfied:

(a) for $K = 1$, G is edge-maximal over $\mathscr{C}_v(n,*,*,D,K)$;

(b) for $K = 2$, $S = (1,S',1)$, where $S' \in \mathscr{S}(n-(3r+2s+4),r,s)$, $r = \lfloor (D-3)/2 \rfloor$, and $s = (D-3) \bmod 2$;

(c) for $K = 4$ or for $K \geq 3$ amd D odd, S is regular;

(d) for $K = 3, 5, 6$, or 7 and D even, $S = (1,K,t,(x,\alpha-x)^r,K,1)$ or $(1,K,(x,\alpha-x)^r,t,K,1)$, where $r = \lfloor (D-3)/2 \rfloor$, $t = n-2(K+1)-r\alpha \geq \alpha_1$, and x is an integer chosen in the range $[\alpha_1,\alpha_2]$ so as to minimize

$$(r-1)x^2 + [t-K-(r-1)\alpha]x. \quad \blacksquare$$

Conditions (a) and (b) of Theorem 4.1 relate to special cases which arise for K = 1 and 2, respectively; while conditions (c) and (d) express precisely the requirement that excess vertices are added either to n_2 or to n_{D-2}. A special case of Theorem 4.1, stated in Caccetta & Smyth (1987a), asserts that a vertex-minimal graph is edge-maximal if its vertex sequence S is smooth or if, for even D, its recurring double is (α_2,α_1).

The proof of Theorem 4.1, while not as lengthy as the proof of Theorem 2.5, requires consideration of numerous detailed special cases and, from a mathematical point of view, does not add significantly to the methodology developed to deal with the case $K \geq 8$. What will be given here, therefore, is an outline of the proof, which highlights the main steps and provides some insight into the origins of the conditions (a)-(d). As in the case of Theorem 2.5, the proof of Theorem 4.1 begins with a series of lemmas or propositions which provide information about a vertex sequence $S = (1,n_1,n_2,\ldots,n_{D-1},1)$ of an edge-maximal graph of $\mathscr{C}_e(n,*,*,D,K)$, $D \geq 6$, $K \leq 7$.

<u>Proposition 4.1</u> At most two adjacent terms of S contain removable vertices.

{This is Lemma 4.1, already proved in Section 4.1.}

<u>Proposition 4.2</u> S contains no non-terminal term less than α_1.

{This result separates the range $1 \leq K \leq 7$ into subranges: $1 \leq K \leq 3$, for which $\alpha_1 = 1$; and $4 \leq K \leq 7$, for which $\alpha_1 = 2$.}

<u>Proposition 4.3</u> For $K = 1$, S is a vertex sequence of an edge-maximal graph over $\mathscr{C}_v(n,*,*,D,1)$.

{Because $\mathscr{C}_v(n,*,*,D,1) = \mathscr{C}_e(n,*,*,D,1)$. See Theorem 3.2, especially condition (d).}

<u>Proposition 4.4</u> For $K \geq 3$, or for $K = 2$ and D even, $n_1 = n_{D-1} = K$.

{For $K \geq 4$, this result is a consequence of the fact that $K > \alpha_2$, while for $K = 3$ the result follows from the fact that the smooth lean double (2,2) generally yields higher edge count than (3,1) or (1,3). For $K = 2$, the result follows from the form of the set of tuples $\mathscr{S}(t,r,s)$ specified in Theorem 4.1, since the t excess vertices are added into the middle term of the triple (2,2,2).}

<u>Proposition 4.5</u> For $K \geq 3$, or for $K = 2$ and D even, at most one term of S, n_2 or n_{D-2}, contains a removable vertex.

{Equivalent to Theorem 2.5.}

<u>Proposition 4.6</u> For $K \geq 3$, or for $K = 2$ and D even, the size m^* of S is given by

$$m^*(x) = 2\binom{K+1}{2} + r\binom{\alpha}{2} + \binom{t}{2} + K(x+t) + \left[(r-1)x+t\right](\alpha-x),$$

where $r = \lfloor (D-3)/2 \rfloor$, $t = n - 2(K+1) - r\alpha$, and $(x, \alpha-x)$ is the recurring double of S.

{By Propositions 4.4 and 4.5, it may without loss of generality be supposed that S takes the form $(1, K, (x, \alpha-x)^r, t, K, 1)$. Since for each choice of K, the only possible choices of x range between α_1 and α_2, the resulting expression for $m^*(x)$ leads directly to the function to be minimized in condition (d) of Theorem 4.1.}

<u>Proposition 4.7</u> For K = 4, or for K ≥ 3 and D odd, S is regular.

{Rough doubles occur only for K = 3 and 7; then (3,1) and (1,3) (respectively, (4,2) and (2,4)) yield lower edge counts than (2,2) (respectively, (3,3)) when D is odd.}

Turning now to the case K ≥ 8, recall from the discussion in Section 2.3 that when D mod 3 ≠ 0, every vertex-minimal vertex sequence is edge-maximal (Theorems 2.6 and 2.7(a)). Thus to complete the consideration of vertex-minimal graphs, condition (b) of Theorem 2.7 remains to be established. (Recall the definitions of the quantities α, α_1, α_2, and K_α given in Section 2.3.)

<u>Theorem 2.7</u> For $D \geq 6$ and $K \geq 8$, a vertex-minimal graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-maximal if and only if one of the following conditions is satisfied:

(a) $D \bmod 3 \neq 0$;

(b) G has a vertex sequence (2.19) where

$$(x,y,z) = \begin{cases} (\alpha_2, \alpha_1, K_\alpha), & \text{if } K \neq 16, 19, 20, 22\text{-}24, 26, 27, 36; \\ (\alpha_2+3, \alpha_1-2, K_\alpha-1), & \text{if } K = 36, 49 \\ (\alpha_2+2, \alpha_1-1, K_\alpha-1), & \text{otherwise.} \end{cases}$$

<u>Proof</u> Setting a = 0 in (2.21), define

$$g(x,z) = f(x,y,z;0) = K(x+z) - xz.$$

The values of x, y, and z which maximize g(x,z) will be determined.

Since any vertex sequence may be reversed and still retain its properties, it may without loss of generality be supposed that $x \leq z$. Observe that

(i) $g(x-1,z+1) = g(x,z) + (z-x+1) > g(x,z)$;

(ii) $g(x-1,z) = g(x,z) - (K-x) < g(x,z)$.

From (i) it follows that $g(x,z)$ takes its maximum value when z is a maximum, and from (ii) that for fixed z, x should also be a maximum. Further, since by Lemma 2.12(b), $z \le K_\alpha$, the selection of y then necessarily fixes z and x: z should be as large as possible subject to the constraint that $x+y \ge \alpha$. Observe that $y \ge 3$, since otherwise (x,y,z) could not be lean. Hence

$$3 \le y \le \alpha_1,\ \alpha_2 = \alpha-\alpha_1 \le x \le z \le K_\alpha$$

subject to the condition

$$(\alpha_1-1)(\alpha_2+1) < K. \qquad \ldots(4.11)$$

The problem therefore reduces to determining the circumstances in which there may exist positive integers a and b such that

$$d = g(\alpha_2+a+b, K_\alpha-b) - g(\alpha_2, K_\alpha) \ge 0;$$

that is, in which the edge count does not decrease when the choice $y = \alpha_1-a$ is made. Observe that since $y \ge 3$, there can be no increase for values of K such that $\alpha_1 = 3$; in particular, for no $K \le 15$. After some algebra, it turns out that

$$d = b^2 - (K_\alpha-\alpha_2-a)b + a(\alpha-1),$$

a quadratic in b which is monotonically decreasing in b provided that

$$b \le (K_\alpha - \alpha_2 - a)/2. \qquad \ldots(4.12)$$

(Here the maximum value of b just corresponds to the case in which the two arguments of g are as near as possible equal.) Since however $x \le z$, it may further be supposed that

$$K_\alpha - b \ge \alpha_2 + a + b,$$

from which (4.12) follows. Therefore, since the objective is to maximize d, it may be supposed that $b = 1$, and the condition that the edge count decreases then becomes

$$d = (a+1)\alpha + \alpha_2 - K \ge 0. \qquad \ldots(4.13)$$

Equations (4.11) and (4.13), together with

$$(\alpha_1 - a)(\alpha_2 + a + 1) \ge K, \qquad \ldots(4.14)$$

are then the basic constraints which must be satisfied by a if the edge count is to increase.

From (4.13) the condition that $d < 0$ may be re-expressed as a quadratic inequality in K:

$$4K(a+2)^2 < (K+\alpha_1)^2.$$

By inspection of this inequality, it is clear that the edge count can possibly increase only if

$$K \geq 4(a+2)^2 - 2\alpha_1. \qquad \ldots(4.15)$$

Rewriting (4.14) taking into account (4.11), observe next that

$$-a\alpha_2 + (a+1)\alpha_1 - a(a+1) \geq K-\alpha_1\alpha_2 \geq \alpha_1-\alpha_2,$$

from which it follows after some algebra that for a > 1,

$$\alpha_2 \leq a[\alpha_1-(a+1)] \;/\; (a-1),$$

or, since $\alpha_1 \leq \alpha_2$,

$$\alpha_1 \geq a(a+1). \qquad \ldots(4.16)$$

(The same inequality holds also when a = 1.) Substituting (4.16) into (4.15) leads to the conclusion that

$$K \leq 2a^2 + 14a + 16. \qquad \ldots(4.17)$$

On the other hand, since by (4.16) $\alpha \geq 2a(a+1)$, it follows that $2\sqrt{K} + 1 \geq 2a(a+1)$; in other words, that

$$K \geq a^2(a+1)^2 - a(a+1) + 1/4. \qquad \ldots(4.18)$$

After a little manipulation, equations (4.17) and (4.18) together imply that

$$a^4+2a^3 < 2a^2+15a+16,$$

which is true only if $a \leq 2$. From (4.17) it follows that the only values of K for which d may possibly not decrease are

$$a = 1,\ K \leq 32; \quad a = 2,\ K \leq 52.$$

For a = 1, Table 4.1 shows the result of direct calculation of d for $16 \leq K \leq 32$ combined with evaluation of the condition (4.14). Observe that d is strictly greater than zero, while at the same time (4.14) is satisfied in exactly those cases specified in the theorem. Observe further that for every value of K in Table 4.2,

$$(\alpha_1-2)(\alpha_2+3) < K, \qquad \ldots(4.19)$$

Determination of feasible values of K (a=1)

K	α_1	α_2	α	d	$(\alpha_1-1)(\alpha_2) \geq K$?
16	4	4	8	4	yes
17	3	6	9	7	no
18	3	6	9	6	no
19	4	5	9	4	yes
20	4	5	9	3	yes
21	3	7	10	6	no
22	4	6	10	4	yes
23	4	6	10	3	yes
24	4	6	10	2	yes
25	5	5	10	0	no
26	4	7	11	3	yes
27	4	7	11	2	yes
28	4	7	11	1	no
29	5	6	11	-1	-
30	5	6	11	-2	-
31	4	8	12	1	no
32	4	8	12	0	no

Table 4.2

leading to a similar calculation for a = 2 to cover the range $33 \leq K \leq 52$. This calculation reveals that only in the cases K = 36 and K = 49 are (4.13) and (4.19) both satisfied (d = 6 and 0, respectively); these are the cases specified in (4.11). It has therefore been shown that, for D mod 3 = 0, in order for a vertex-minimal vertex sequence to be edge-maximal, condition (b) must be satisfied. Conversely, it follows also from the above argument that whenever (b) is satisfied, the sequence is edge-maximal. This completes the proof. ■

To complete the characterization of edge-maximal vertex sequences it is necessary now to consider graphs of $\mathcal{C}_e$ that are not vertex-minimal; that is, such that the excess $a > 0$. By Theorem 2.6 it is necessary only to consider cases in which $D \bmod 3 \neq 2$. The first result deals with $D \bmod 3 = 0$ and $1 \le a \le \alpha-1$.

<u>Theorem 2.8</u> For $D \ge 6$, $D \bmod 3 = 0$, and $K \ge 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $a \le \alpha-1$. Suppose further that integers a^* and $a^‡$ are given, where $a^* = a^‡ = 0$ except as shown in the following table:

K	a^*	$a^‡$
16	2	2
19	2	2
20	1	2
22	2	2
23	1	2
24	1	1
26	1	2
27	1	1
36	2	2

Then G is edge-maximal if and only if the vertex sequence (2.19) satisfies one of the following conditions:

(a) $0 \le a \le a^*$ and (x,y,z) is specified by Theorem 2.7(b);

(b) $a^‡ \le a < \alpha-1$ and $(x,y,z) = (\alpha_2,\alpha_1,K_\alpha)$;

(c) $a = \alpha-1$ and $z = K_\alpha$.

Proof As in the proof of Theorem 2.7, it may be supposed without loss of generality that $x \le z \le K_\alpha$. It is convenient to consider transformations

$$\tau : (x,y,z) \longrightarrow (x',y',z')$$

of the edge-maximal vertex sequence (2.19) (see also the statement of Theorem 2.9). Observe first from (2.21) that the choice $x' = x-1$, $z' = z+1$ results in a change to the edge count represented by

$$\begin{aligned} \Delta f &= f(x-1,y,z+1;a) - f(x,y,z;,a) \\ &= z + a - x - 1, \qquad \ldots(4.20) \end{aligned}$$

so that for $a \ge 1$, $\Delta f \ge 0$. Thus it may always be assumed that x is the minimum value consistent with K-edge-connectivity.

Next consider the transformation τ with $(y'-y) + (x'-x) \le 0$. Since in general

$$\Delta f = K\Big[(x'-x)+(z'-z)\Big]-(x'-x)z'-(z'-z)x+a\Big[(y'-y)+(x'-x)\Big],$$

it is clear that if this transformation does not increase the edge count when $a = 0$, it cannot increase the edge count when $a > 0$. Now for $a = 0$ Theorem 2.7 implies that, starting with $(x,y,z) = (\alpha_2,\alpha_1,K_\alpha)$, the only transformations τ which increase the edge count are of the form

$$\tau : (\alpha_2, \alpha_1, K_\alpha) \longrightarrow (\alpha_2+i-1, \alpha_1-i, K_\alpha-1), \qquad \ldots(4.21)$$

where $1 \le i \le 2$. For $a \ge 0$, this transformation yields

$$\Delta f = K - \alpha_2 - (i+1)(\alpha-a),$$

which for sufficiently large a becomes positive. Now corresponding to the values of K specified in condition (b) of Theorem 2.7, the largest value a^* for which $\Delta f \le 0$ may be calculated from (4.21), as well as the changeover value $a^\ddagger$ at which Δf first becomes positive. These values turn out to be the ones tabulated in the statement of the theorem, so that condition (a) is proved and condition (b) partly proved.

To complete the proof of (b), it is now demonstrated that for $a < K-z$, the only transformations which can possibly increase the edge count are those which increase z, an impossibility since $z = K_\alpha$. Suppose first that $z' = z$. Since $y = \alpha_1$, it follows that $y'-y = x-x' > 0$ may without loss of generality be assumed, so that, for $a < K-z$,

$$\Delta f = (K-z-a)(x'-x) < 0. \qquad \ldots(4.22)$$

Suppose then that $z' < z$ and $y' \geq y$, so that $x' < x < z$ and

$$\Delta f = (z+a-K)(y'-y) + (z+a-x')(z'-z), \qquad \ldots(4.23)$$

an expression which is strictly less than zero for every transformation such that $a \leq K-z$. Hence $\Delta f < 0$ for every transformation in which $z' \leq z$ and $a < K-z = \alpha-1$, thus establishing condition (b).

To prove (c), observe that in the special case $a = \alpha-1$, the above argument implies that $z = K_\alpha$. Further, setting $a = K-z$ in (4.22), observe that $\Delta f = 0$ for any transformation satisfying $z' = z = K_\alpha$. ∎

Now suppose that $a \geq \alpha$. Recall from (4.20) that for fixed y, x takes the minimum value and z the maximum value consistent with K-edge-connectivity; similarly, (4.22) implies that for $a > K-z$ and fixed z, x and y take the minimum and maximum feasible values, respectively. Therefore, since $z \leq z \leq K_\alpha$, it must be true that $x \leq \alpha_1$. Since for $a = \alpha-1$, $z = K_\alpha$, it follows that for $a > \alpha-1$, the only transformations τ which need to be considered are those satisfying

$$x' < x,\ y' > y,\ z' < z. \qquad \ldots(4.24)$$

Moreover, since for $x' = x$, $z'-z = y-y'$ and

$$\Delta f = (K-x)(z'-z),$$

it follows that for fixed x, y and z take minimum and maximum feasible values, respectively. Hence it may be supposed that

$$y' = \lceil K/x' \rceil, \qquad \ldots(4.25)$$

and indeed that $y = \lceil K/x \rceil$ for every edge-maximal graph. Then (4.23) represents the increase in edge count resulting from a transformation τ; this quantity exceeds zero if and only if

$$z+a > \left[K(y'-y)+x'(z'-z)\right]/(x-x'). \qquad \ldots(4.26)$$

Observe now that relations (4.24)-(4.26) are just conditions (a)-(c) of Theorem 2.9, which has therefore been proved:

<u>Theorem 2.9</u> For $D \geq 6$, $D \bmod 3 = 0$, and $K \geq 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $a \geq \alpha$. Then G is edge-maximal if and only if there exists no feasible transformation

$$\tau : (x,y,z) \longrightarrow (x',y',z')$$

of (2.19) satisfying all of the following conditions:

(a) $x' < x,\ y' > y,\ z' < z$;

(b) $x' = \lceil K/x' \rceil$;

(c) $x+a > [K(y'-y)+x'(z'-z)]/(x-x')$. ■

In order to derive a more computationally useful form of this result, it is convenient to adopt the notation

$$i = x-x',\ \delta(i) = y'-y,\ \delta(i)-i = z-z', \qquad \ldots(4.27)$$

all quantities which by (4.24) should be positive integers in order that transformation τ might possibly increase the edge count. These quantities represent displacements from an initial recurring triple $(x,y,z) = (\alpha_1,\alpha_2,K_\alpha)$. (4.25) then becomes

$$\delta(i) = \lceil K/(x-i) \rceil - y, \qquad \ldots(4.28)$$

and, setting

$$b(i) = \delta(i)(K-x+i)/i + (x-i), \qquad \ldots(4.29)$$

(4.26) becomes

$$z+a < b(i). \qquad \ldots(4.30)$$

Note that (4.28) holds also for $i = 0$. It may therefore be imagined that i successively takes the values $1,2,\ldots,x-t$, where by Lemma 4.4, $t = 3$ for $D \geq 7$, and otherwise $t = 1$. Since the corresponding sequence of values $\delta(i)$ is monotone increasing in i, it is possible to write

$$\delta(i) = \sum_{0\le j\le i} v_j, \qquad \ldots(4.31)$$

where by (4.28) $v_0 = 0$, and by the definition of α_1, $v_1 \ge 2$, while for $j \ge 1$, $v_j \ge 1$. In fact, it follows from (4.28) that for $x-i \le (\sqrt{1+2K}-1)/2$, $v_i \ge 2$. Observe also that for any value of i such that $v_i \ge 2$ and, for some $h \ge 1$, $v_{i+1} = v_{i+2} = \ldots = v_{i+h} = 1$, no transformation τ for which $x' = x-i$ can yield maximum edge count, since then $\delta(i+h) = \delta(i)+h$, and so the further transformation

$$\tau' : (x-i, y+\delta(i), z+i-\delta(i)) \longrightarrow (x-i-h, y+\delta(i)+h, z+i-\delta(i))$$

is feasible, leaves $z' = z+i-\delta(i)$ unchanged, and hence, by (4.22), increases the edge count.

Imagine therefore computing a subsequence J of the sequence $I_{x-t} = \{0,1,\ldots,x-t\}$ which has the property that for every $i \in J$, the transformation τ specified by (4.27) and (4.28) may possibly be edge-maximal. J then consists of every integer $i \in I_{x-t}$ such that $v_{i+1} \ne 1$, arranged in ascending order. Let $J = \{j_0=0, j_1, \ldots, j_p\}$, and observe that for every $0 \le h \le p-1$,

$$\delta(j_{h+1})/j_{h+1} \ge \delta(j_h)/j_h, \qquad \ldots(4.32)$$

so that from (4.29), $b(j_{h+1}) \ge b(j_h)$. Then for $0 \le h \le p$ and any integer $A \ge 0$, define the triple

$$T_h(A) = (\alpha_1 - j_h, y+\delta(j_h), z+j_h-\delta(j_h)+A), \qquad \ldots(4.33)$$

and let

$$S_h = (1, K, T_h(0)^{r-1}, T_h(a), K, 1) \qquad \ldots(4.34)$$

denote the corresponding regular vertex sequence, where as before $r = \lfloor D/3 \rfloor - 1$ and $a \ge \alpha$. (4.32) can be used to determine whether the size of S_h exceeds the size of S_0. More generally, after some calculation, it can be established that for $1 \le h \le p$, the size of S_h exceeds the size of S_{h-1} if and only if

$$a > a^*(h)$$

where

$$a^*(h) = (K-\alpha_1+j_h)\left[\delta(j_h)-\delta(j_{h-1})\right]/(j_h-j_{h-1}) + (\alpha_1-j_h) - \left[K_\alpha+j_{h-1}-\delta(j_{h-1})\right]. \qquad \ldots(4.35)$$

We have proved

<u>Theorem 4.2</u> For $D \geq 6$, $D \bmod 3 = 0$, $K \geq 8$, and $a \geq \alpha$, let S be a vertex sequence (2.19) of a graph $G \in \mathscr{C}_e(n,*,*,D,K)$. Then G is edge-maximal if and only if for some integer $h \geq 0$,

$$S = S_h \text{ and } a^*(h) \leq a \leq a^*(h+1),$$

where S_h is defined by (4.33) and (4.34), and $a^*(h)$ is defined by (4.35). ■

Note that if $a^*(h)$ is an integer, then both S_{h-1} and S_h are edge-maximal when $a = a^*(h)$. The recognition (or construction) of an edge-maximal graph G thus reduces essentially to the determination of the sequence J and the corresponding calculation of the functions $a^*(h)$, $j_h \in J$.

This section concludes with a proof of the characterization theorem for D mod 3 = 1. With reference to the vertex sequence (2.19), observe that in this case it suffices to consider $n^* = n_{D-2} = x+a \geq K_\alpha$, since for smaller values of n^* the result is covered by Theorem 2.7(a) for vertex-minimal graphs.

<u>Theorem 2.10</u> For $D \geq 6$, $D \bmod 3 = 1$, and $K \geq 8$, suppose that a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ has a vertex sequence (2.19) in which $n^* > K_\alpha$. Let j^* and k^* be the values of j and k which maximize

$$(i+j)\left\{a-i\left[K-\alpha_2)/(i+j)\ -\ 1\right]\right\}$$

over $0 \le j \le \alpha_1-3$ and $1 \le K \le 2$, where

$$i = i(j,k) = \lceil K/(\alpha_1-j)\rceil-\overline{\alpha_2}-j,\ \text{for } k = 1;$$
$$= K_\alpha - \lceil K/(\alpha_1-j)\rceil,\ \text{for } k = 2.$$

Let $i^* = i(j^*,k^*)$. Then G is edge-maximal if and only if

$$(x,y,z) = (K_\alpha-i^*,\alpha_1-j^*,\alpha_2+i^*+j^*).$$

Further, there exist integers $a_* \le \alpha-1$ and $a^* \ge \alpha-1$ such that

(a) for $a \le a_*$, $(x,y,z) = (K_\alpha,\alpha_1,\alpha_2)$;

(b) for $a \ge a^*$, $(x,y,z) = (\alpha_2,\alpha_1,K_\alpha)$. ∎

<u>Proof</u> Recall from (2.22) that it is required to determine the maximum of the function

$$f(x,y,z;a) = K(2x+a) + az$$
$$= Kx + az + n^*K.$$

Since n^* is constant, this reduces to the problem of maximizing $Kx + az$, from which it may be concluded that y is always as small as possible. Thus $y \le \alpha_1$, and in fact, by Lemma 2.12(b),

$$y = \max\left\{\lceil K/x \rceil, \lceil K/z \rceil\right\}. \qquad \ldots(4.36)$$

Since $n^* = a+x > K_\alpha$, it follows then that, in order to maximize the edge count, only transformations of the form

$$\tau : (K_\alpha, \alpha_1, \alpha_2) \longrightarrow (K_\alpha - i, \alpha_1 - j, \alpha_2 + i + j), \qquad \ldots(4.37)$$

need be considered, where by (4.36),

$$i = \lceil K/(\alpha_1 - j) \rceil - \alpha_2 - j \text{ or } K_\alpha - \lceil K/(\alpha_1 - j) \rceil,$$

and by Lemma 4.4, $0 \le j \le \alpha_1 - 3$. Observe that the change in the edge count produced by (4.37) is

$$m(j,k) = (a+i)(\alpha_2 + i + j) - a\alpha_2 - iK,$$

which may be written in the form

$$m(j,k) = (i+j)\left[a - \hat{a}(j,k)\right], \qquad \ldots(4.38)$$

where

$$\hat{a}(j,k) = i\left[(K-\alpha_2)/(i+j) - 1\right]. \qquad \ldots(4.39)$$

Since m(0,1) = 0, it follows that

$$\max_{j,k} \{\hat{a}(j,k)\} \geq 0,$$

which establishes the main part of the theorem.

To establish condition (a), let $\hat{j}$ and $\hat{k}$ be the values of j and k, respectively, which minimize (4.39). Observe that for j = 0,

$$\hat{a}(0,k) = K - \alpha_2 - i,$$

a quantity which is minimized by the choice k = 2 $(i = K_\alpha - \alpha_2)$ for which $\hat{a}(0,2) = \alpha - 1$. Then set $a_* = \hat{a}(\hat{j},\hat{k}) \leq \alpha - 1$, and observe that for $a \leq a_*$, $(x,y,z) = (K_\alpha, \alpha_1, \alpha_2)$.

To establish condition (b), observe that i+j attains its maximum value $K_\alpha - \alpha_2$ if and only if j = 0 and k = 2, hence that for sufficiently large $a \geq a^* \geq \alpha - 1$, m(0,2) is a maximum over all m(j,k), so that $(x,y,z) = (\alpha_2, \alpha_1, K_\alpha)$. This completes the proof. ∎

In connection with this result, it is tempting to suppose that the functions m(j,k) or $\hat{a}(j,k)$ may be monotone in j. However this turns out not in general to be true, as the following table, based on the example of Table 2.5, indicates;

Values of $\hat{a}(j,k)$ for $K = 70$ ($\alpha = 17$, $\alpha_1 = 7$, $\alpha_2 = 10$, $K_\alpha = 54$)

j	k	(x′,y′,z′)	$\hat{a}(j,k)$
0	1	(54,7,10)	-
1	1	(53,6,12)	29
2	1	(52,5,14)	28
3	1	(49,4,18)	32.5
4	1	(44,3,24)	33.8
0	2	(10,7,54)	16
1	2	(12,6,53)	10.7
2	2	(14,5,52)	11.4
3	2	(18,4,49)	13.8
4	2	(24,3,44)	17.6

Table 4.3

4.3 Edge-Minimal Graphs over $\mathscr{C}_e(n,*,*,D,K)$

In this section two results are presented which characterize edge-minimal graphs of $\mathscr{C}_e$ for values of n which are large with respect to the product DK. Examples are then given which illustrate the difficulties involved in characterizing edge-minimal graphs for smaller values of n.

<u>Theorem 2.11</u> For even $D \geq 4$ and $n \geq D(K+1)/2 + 1$, a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-minimal if and only if G is edge-minimal over $\mathscr{C}_v(n,*,*,D,1)$. ∎

<u>Proof</u> Observe that for any positive integer t, $n \geq D(t+1) + 1$, $K = 1$, and even D, the edge-minimal graphs of $\mathscr{C}_v(n,*,*,D,1)$ given in Theorems 3.1 and 3.2 are all t-edge-connected. The result then follows from the fact that $\mathscr{C}_e(n,*,*,D,K) \subseteq \mathscr{C}_v(n,*,*,D,1)$. ∎

On the other hand, for $n \geq (D-1)t + 2$, $K = 1$, and odd $D \geq 5$, there exists, again by Theorems 3.1 and 3.2, an edge-minimal vertex sequence for $G \in \mathscr{C}_v(n,*,*,D,1)$ of the form

$$(1,(2t-1,1)^{(D-5)/2},t,t,1,2t-1,1).$$

G is therefore t-edge-connected. Moreover, every t-edge-connected edge-minimal graph of $\mathscr{C}_v(n,*,*,D,1)$ has a vertex sequence of this form. Hence:

<u>Theorem 2.12</u> For odd $D \geq 5$ and $n \geq (D-1)K + 2$, a graph $G \in \mathscr{C}_e(n,*,*,D,K)$ is edge-minimal if and only if it has a vertex sequence which satisfies all of the following conditions:

(a) $n_j = 1$, for $j = D-1$ and $j = 2,4,\ldots,D-5$;

(b) $\min\{n_{D-4},n_{D-5}\} \geq K$;

(c) for every $1 \leq j \leq (D-5)/2$,

  (i) $n_{2j} \geq 2K-1$;

  (ii) for every $1 \leq j' \leq (D-5)/2$,

$$|n_{2j-1}-n_{2j'-1}| \leq 1;$$

  (iii) $|n_{2j-1}-n_{D-1}| \leq 1$;

  (iv) $|n_{2j-1}-n_{D-4}-n_{D-3}| \leq 1$. ∎

CHAPTER 5

UPPER BOUNDS ON THE DIAMETER OF GRAPHS OF $\mathcal{G}_V(n,m,\delta,*,K)$

In this chapter the methodology of Chapters 3 and 4 is applied first to graphs of $\mathcal{G}_V(n,*,\delta,D,K)$ in order to compute $m^* = m^*(n,\delta,D,K)$, the size of an edge-maximal graph of $\mathcal{G}_V(n,*,\delta,D,K)$. This result is then used to determine an upper bound on the diameter of any graph of $\mathcal{G}_V(n,m,\delta,*,K)$, thus extending a result of Klee & Quaife (1976), who effectively determine an upper bound on the diameter of graphs of $\mathcal{G}_V(n,*,\delta,*,K)$ (see Theorem 2.17). The references for this chapter are Caccetta & Smyth (1989b, 1989c).

5.1 Edge-Maximal Graphs over $\mathcal{G}_V(n,*,\delta,D,K)$

Table 2.5 displays vertex sequences of the edge-maximal graphs of $\mathcal{G}_V = \mathcal{G}_V(n,*,\delta,D,K)$, $D \le 5$. Throughout this section the term <u>edge-maximal</u> will always be used with respect to graphs of $\mathcal{G}_V$, and hence, by Lemma 2.4, will always specify a graph of $\mathcal{G}_V(n,*,\delta,D,K)$. For $D \ge 6$, recall from Section 2.1 the definition of an <u>h-recurring</u> vertex sequence. Further, the term <u>removable</u> of Section 4.1 is here redefined to apply to a vertex u of a graph $G \in \mathcal{G}_V(n,*,\delta,D,K)$ which has the property that $G-\{u\} \in \mathcal{G}_V(n-1,*,\delta,D,K)$. Then two preliminary results may be stated and proved:

<u>Lemma 5.1</u> There exists an edge-maximal vertex sequence in which every subsequence of adjacent disjoint lean triples is 3-recurring.

<u>Proof</u> Consider a subsequence

$$\beta = (u, x, y, z, x', y', z', v),$$

where (x,y,z) and (x',y',z') are lean triples. Recall that by Lemma 2.5,

$$u \geq z \geq z', \quad v \geq x' \geq x.$$

It is then easy to verify directly, using Lemma 2.6, that the size of each of

$$(u, (x,y,z)^2, v) \quad \text{and} \quad (u, (x',y',z')^2, v)$$

is at least the size of β. ∎

<u>Lemma 5.2</u> Let $S = S_D(u)$ be a vertex sequence of an edge-maximal graph $G \in \mathcal{G}_V(n,*,\delta,D,K)$. If any two terms n_i and n_j of S contain a removable vertex of G, then $|i-j| \leq 1$.

<u>Proof</u> See Lemma 4.1. ∎

A series of lemmas is now proved which allow the specification, for given $\mathcal{G}_V(n,*,\delta,D,K)$, of an edge-maximal vertex sequence. First, however, another definition: a lean triple (n_i,n_{i+1},n_{i+2}) is said to be <u>isolated</u> if $(n_{i+1},n_{i+2},n_{i+3})$ is fat and there exists a lean triple (n_j,n_{j+1},n_{j+2}), $j > i + 1$.

<u>Lemma 5.3</u> There exists an edge-maximal vertex sequence in which no lean triple is isolated.

<u>Proof</u> Suppose that, corresponding to $G \in \mathcal{G}_V$ and for some $2 \le i \le j-4 \le D-8$, there exists an edge-maximal vertex sequence S_D containing two lean triples (n_i,n_{i+1},n_{i+2}) and (n_j,n_{j+1},n_{j+2}). Consider the vertex sequence

$$S' = (\ldots,n_{i-1},(n_i,n_{i+1},n_{i+2})^2,n_{i+3},\ldots,n_{j-1},n_{j+3},\ldots),$$

where the unspecified terms before n_{i-1}, between n_{i+3} and n_{j-1}, and after n_{j+3} are identical with those in S. Observe that the orders of S and S′ are equal and further, by Lemma 2.5, that $n_{j-1} \ge n_{j+2}$, $n_{j+3} \ge n_j$. Suppose now that edges are introduced so that the graph G′ giving rise to S′ satisfies Lemma 2.1(b). Recall from the proof of Lemma 2.3 that this can always be done so that the minimum degree of G′ is δ. Then $G' \in \mathcal{G}$. However, using Lemma 2.6 it follows that the size of G′ exceeds that of G by

$$(n_{j-1}-n_{j+2})(n_{j+3}-n_j) \geq 0.$$

It has been shown that the edge count is at least as great as the maximum if disjoint lean triples are made adjacent. ∎

<u>Lemma 5.4</u> For $D \geq 6$, there exists an edge-maximal vertex sequence in which at most one of the triples (n_2, n_3, n_4), $(n_{D-4}, n_{D-3}, n_{D-2})$ is fat.

<u>Proof</u> For $D = 6$ the result is trivially true. For $D > 6$ observe that if both of the specified triples are fat and $\min\{n_2, n_{D-2}\} > K$, then n_2 and n_{D-2} must both contain a removable vertex, in contradiction to Lemma 5.2. It may be supposed therefore that $n_2 = K$ and that (n_2, n_3, n_4) is fat. Observe that, since $n_5 \geq K$, one may by Lemma 2.7 choose $n_3 = K$, yielding

$$(n_2, n_3, n_4) = (K, K, K'+a)$$

for some integer $a \geq 1$, where $K' = M - 2K$. Then L_4 contains a removable vertex, and it follows from Lemma 5.2 that no level other than L_5 can possibly contain a removable vertex. Thus for $D > 7$ it cannot be true that $(n_{D-4}, n_{D-3}, n_{D-2})$ is fat and $n_{D-2} > K$. For $D = 7$ observe that by Lemma 2.7 the transformation

$$(K, K, K'+a, n_5) \longrightarrow (K, K, K', n_5+a)$$

does not decrease the edge count, while leaving (n_2, n_3, n_4) lean.

The only case left to consider is $D \geq 8$, $n_{D-2} = K$, and $(n_{D-4}, n_{D-3}, n_{D-2})$ fat. Using the argument of the previous paragraph, it may be concluded that

$$(n_{D-4}, n_{D-3}, n_{D-2}) = (K'+b, K, K)$$

for some $b \geq 1$, hence that L_{D-4} contains a removable vertex. This is possible only if $D = 8$ or 9.

For $D = 8$ observe that the transformation

$$(K, K, K'+a, K, K) \longrightarrow (K, K, K', K+a, K)$$

does not decrease the edge count and leaves (n_2, n_3, n_4) lean, while similarly for $D = 9$ the transformation

$$(K, K, K'+a, K'+b, K, K) \longrightarrow (K, K, K', K'+a+b, K, K)$$

does not decrease the edge count. Then for $D = 8, 9$ the triple (n_2, n_3, n_4) may without loss of generality be supposed to be lean. Since all other cases have been excluded, the result is proved. ∎

Lemma 5.5 For $D \geq 7$, there exists an edge-maximal vertex sequence S_D and an integer $k \geq 3$ such that

(a) the k-tuple $(n_2, \ldots, n_{k+1})$ is 3-recurring;

(b) every triple (if any) in the (D-k-1)-tuple $(n_k, \ldots, n_{D-2})$ is fat.

Proof By Lemma 5.4 it may supposed that (n_2, n_3, n_4) is lean. By Lemmas 5.1 and 5.3 it may further be supposed that there exists a 3-recurring j-tuple $(n_2, \ldots, n_{j+1})$, $j \geq 3$, such that for $j' > j + 1$ there exists no lean triple $(n_{j'}, n_{j'+1}, n_{j'+2})$. Then it suffices to consider the triples $T_j = (n_j, n_{j+1}, n_{j+2})$ and $T_{j+1} = (n_{j+1}, n_{j+2}, n_{j+3})$.

If T_j and T_{j+1} are both fat, the proof is complete and $k = j$. If T_j is lean and T_{j+1} is fat, the result is true for $k = j + 1$. If T_j and T_{j+1} are both lean, the result is true for $k = j + 2$. Finally, if T_j is fat and T_{j+1} is lean, observe that for $j > 3$, $T_{j-2} = (n_{j-2}, n_{j-1}, n_j)$ and T_{j+1} are adjacent but not 3-recurring, contrary to Lemma 5.1; on the other hand, for $j = 3$, observe that

$$n_5 + n_6 = n_2 + n_3, \qquad n_3 > n_6, \; n_5 > n_2;$$

and it is then not difficult to see, using Lemma 2.6, that the edge count is increased by the substitution

$$n_3 \longrightarrow n_6; \qquad n_2 \longrightarrow n_5.$$

Hence this last case cannot arise in an edge-maximal vertex sequence. ∎

Lemma 5.6 For $D \geq 6$, there exists an edge-maximal vertex sequence in which no term other than n_{D-2} gives rise to a removable vertex.

Proof Observe first that for $D = 6$ we may by virtue of Lemma 5.2 suppose that at most L_3 and L_4 contain removable vertices.

For $D \geq 7$, Lemma 5.5 implies that there exists $k \geq 3$ such that every

$$T_j = (n_j, n_{j+1}, n_{j+2}), \qquad j \geq k,$$

is fat, and every T_j, $j < k$, is lean. If $k = D-3$, there is no removable vertex and the theorem is true. Suppose then that $k < D-3$. It follows that at least one of the following statements is true:

(a) L_{k+2} contains a removable vertex;

(b) $n_{k+2} = 1$.

Statement (b) is false because $n_{k+2} > n_{k-1}$. Then statement (a) is true. Since no lean triple can give rise to a removable vertex, it follows moreover that the lemma is true for $k+2 \geq D-2$. Hence suppose that $k \leq D-5$.

Recall from Lemma 5.4 that for the fat triple $T_{D-4} = (n_{D-4}, n_{D-3}, n_{D-2})$ two possibilities arise:

(a) $n_{D-2} > 1$

This is the case which arises also when D = 6. Here L_{D-2} contains a removable vertex, so that by Lemma 5.2, k = D-4 or D-5. In the former case the proof is complete; the latter case is impossible, since removable vertices can be transferred from L_{D-5} to L_{D-4} until T_{D-3} is a lean triple; this transfer must increase the edge count.

(b) $n_{D-2} = 1$

In this case, as in the proof of Lemma 5.4, Lemma 2.7 allows the supposition that $T_{D-4} = (K'+a, K, K)$, $a \geq 1$, where as before $K' = M-2K$, so that L_{D-4} contains a removable vertex. Then by Lemma 5.2, k = D-6 or D-7, both of which are impossible by virtue of transformations similar to those used in Lemma 5.4. ■

From these lemmas follows the first main result of this section:

Theorem 2.13 For $D \geq 6$, there exists an edge-maximal graph of $\mathcal{G}_V(n, *, \delta, D, K)$ with a vertex sequence in which every internal triple except possibly $(n_{D-4}, n_{D-3}, n_{D-2})$ is lean. ■

This is essentially the same result found in Section 4.1 (Theorem 2.5) to hold for K-edge-connected graphs, and hence gives rise to similar expressions (see equations (2.24)-(2.28) of Section 2.4). In order to analyze these expressions, the three values of D mod 3 need to be considered.

Observe first that for D mod 3 = 2, the maximization problem disappears, since $x + y + a = n^*$, a constant (see also Theorem 2.8). In this case, then,

$$f^* = \delta n^* . \qquad ...(5.1)$$

Observe also that for $\delta + 1 \le 3K$, the only possibility is that $M_v = 3K$, $x = y = z = K$, so that, for D mod 3 = 0 ($n^* = a$),

$$f^* = (\delta+2K)n^* + (2\delta-K)K; \qquad ...(5.2)$$

while for D mod 3 = 1 ($n^* = K + a$),

$$f^* = 2\delta n^* - (\delta-K)(n^*-K). \qquad ...(5.3)$$

It may be supposed therefore that $M = \delta+1 > 3K$, so that $x + y + z = \delta+1$ and $n^* = n - (r+3)(\delta+1)$. Two cases need to be considered:

D mod 3 = 0

In this case $n^* = a \geq 0$. Setting $y = \delta+1-x-z$, $K' = \delta+1-2K$, it is found after some manipulation that

$$f = (\delta+K'+2K)n^* + (\delta-n^*)x + (\delta-x)z. \qquad \ldots(5.4)$$

Suppose now that x takes the value required in order to maximize (5.4). Since $\delta-x > 0$, it follows that z must be as large as possible; that is, $y = K$, $z = K'+K-x$. By substituting in (5.4), one is led to consider

$$g(x) = x^2 - (n^*+K'+K)x + (\delta+K'+2K)n^* + (K'+K)\delta,$$

to be maximized by choice of x in the range $[K,K']$. (In order to determine this value, it is convenient temporarily to treat g as a continuous function of x.) Since the derivative

$$\frac{dg}{dx} = 2x - (n^*+K'+K)$$

is negative at $x = K$, and since $g(K) > g(K')$, it must be true that setting $x = K$ maximizes g. Hence, substituting $x = K$, $z = K'$ in (5.4), observe that

$$f^* = (\delta+K'+K)n^* + (K'\delta+\delta K-KK'), \qquad \ldots(5.5)$$

an expression which reduces to (5.2) when $K' = K$ ($M = 3K$).

<u>D mod 3 = 1</u>

In this case $x = K + \alpha$, for some $0 \le \alpha \le a$, and $n^* = K + a \ge K$. Then

$$f = (n^*+x)\delta + (n^*-x)z. \qquad \ldots(5.6)$$

Suppose that x takes the value required to maximize (5.6). Since $n^*-x \ge 0$, it follows that z must be as large as possible; that is, $y = k$, $z = \delta+1-K-x$. Then, substituting in (5.6), consider

$$g(x) = x^2 - (n^*+1-K)x + (2\delta+1-K)n^*,$$

to be maximized by choice of $x \in [K,K']$. This function is convex downward and assumes its minimum value at

$$x_0 = (a+1)/2 > 0.$$

Three subcases may be distinguished:

(a) <u>$0 \le a \le 2K-1$</u>

In this case $x_0 \le K$, and $g(x)$ therefore achieves its maximum value either at $x = K+a$ (when $a < K'-K$) or at $x = K'$ (otherwise).

(b) <u>$2K \le a \le K'-K-2$</u>

In this case $K < x_0 < (K'-K)/2$, and since therefore $(K+a) - x_0 > x_0-K$, it follows that $g(x)$ achieves its maximum value at $x = K+a$.

(c) <u>$a \geq 2K$ and $a \geq K'-K-1$</u>

Here $x_0 \geq (K'-K)/2$, so that $g(x)$ achieves its maximum value at $x = K$.

Putting together these results, and making the appropriate substitutions for x and z in (5.6), one finds that

$$f^* = f(n^*,K,K'-a;0) = 2\delta n^*, \qquad \ldots(5.7)$$

for $a \leq 2K-1$ and $a \leq K'-K$,

for $2K \leq a \leq K'-K-2$;

$$f^* = f(K',K,K;n^*-K') = 2\delta n^*-(\delta-K)(n^*-K') \qquad \ldots(5.8)$$

for $a \leq 2K-1$ and $a > K'-K$;

$$f^* = f(K,K,K';a) = 2\delta n^* - (\delta-K')(n^*-K), \qquad \ldots(5.9)$$

for $a \geq 2K$ and $a \geq K'-K-1$.

Note that for $K' = K$, (5.8) and (5.9) reduce to (5.3).

These results may be formally expressed as follows:

<u>Theorem 2.14</u> For $D \geq 5$, the size m^* of an edge-maximal graph of $\mathcal{G}_V(n,*,\delta,D,K)$ is given by (2.25), where

(a) for $D \bmod 3 = 0$,

$$f^* = (\delta+K+K')n^* + (K'\delta+\delta K-KK');$$

(b) for D mod 3 = 1,

(i) $f^* = 2\delta n^*$,

for $a \le 2K-1$ and $a \le K'-K$, <u>or</u>

for $2K \le a \le K'-K-2$;

(ii) $f^* = 2\delta n^* - (\delta-K)(n^*-K')$,

for $a \le 2K-1$ and $a > K'-K$;

(iii) $f^* = 2\delta n^* - (\delta-K')(n^*-K)$,

for $a \ge 2K$ and $a \ge K'-K-1$;

(c) for D mod 3 = 2, $f^* = \delta n^*$;

and $K' = \delta+1-2K$. ■

<u>Theorem 2.15</u> For fixed n, δ, and K, the function $m^*(n,\delta,D,K)$ specified by (2.25) is monotone decreasing in D.

<u>Proof</u> For $D \le 5$ inspection of Table 2.6 easily establishes monotonicity. For $D \ge 6$ observe first that $r = d-1$ for any three consecutive values D = 3d, 3d+1, 3d+2; then it is tedious but not difficult to verify that m^* is monotone decreasing for constant r. When r increases by one, the following computation results, corresponding to D mod 3 = 2:

$$m^*(n,\delta,D,K) - m^*(n,\delta,D+1,K)$$

$$= \left\{ r\binom{M_V}{2} + 2\binom{\delta+1}{2} + \binom{n^*}{2} + \delta n^* \right\}$$

$$- \left\{ (r+1)\binom{M_v}{2} + 2\binom{\delta+1}{2} + \binom{n^*-M_v}{2} \right.$$

$$\left. + (\delta+K'+K)(n^*-M_v) + (K'\delta+\delta K-KK') \right\}$$

$$= \binom{M_v}{2} + K(n^*+\delta-2K)$$

$$> 0.$$

Thus monotonicity is established for $D \geq 6$. To establish that $m^*(n,\delta,6,K) < m^*(n,\delta,5,K)$, it can easily be verified that the value of $m^*(n,\delta,5,K)$ given in Table 2.6 equals the value of $m^*(n,\delta,5,K)$ specified by Theorem 2.14 ($r = 0$). Then the preceding calculation is valid also for $D = 5$. This completes the proof. ■

5.2 Maximum Diameter of Graphs of $\mathcal{G}_v(n,m,\delta,*,K)$

It is supposed throughout this section that a graph $G \in \mathcal{G}_v(n,m,\delta,*,K)$ is given, the results of Section 5.2 are used to determine a sharp upper bound D^* on the diameter of G.

Observe first from Table 2.6 that for $m \geq \binom{n-1}{2}$, $1 \leq D^* \leq 3$, and D^* can immediately be determine by comparing m with $\binom{n}{2}-1$. For $m < \binom{n-1}{2}$, the monotonicity of m^* implies that $D^* > 3$, hence Table 2.6 and the monotonicity property may be used again to determine whether $D^* = 4$ or 5. For $D^* \geq 6$, it becomes convenient to consider the new function

$$G(n,\gamma,\bar{r},K) = 2[m^*(n,\delta,D,K)-m], \qquad \ldots(5.10)$$

where $\bar{r} = M_V\gamma$ and $\gamma = \delta+1$, so that $n^* = n-2\gamma-\bar{r}$. After some manipulation, (5.10) becomes

$$\bar{r}^2 - (2n-4\gamma-M_V)\bar{r} + [n(n-4\gamma-1) + 6\gamma^2 - 2m + 2f^*], \qquad \ldots(5.11)$$

a quadratic expression in $\bar{r}$. Substitution for f^* from Theorem 2.14 yields five functions F_j, $1 \le j \le 5$, as in (2.29):

$$F_j(\bar{r}) = \bar{r}^2 - [b_j+2(n-1)]\bar{r} + [c_j+n(n-3)-m],$$

where the coefficients b_j and c_j are specified by Table 2.7.

Then, as described in Section 2.4, the condition

$$F_j(\bar{r}) = 0, \ 1 \le j \le 5,$$

yields the main result of this section:

Theorem 2.16 For a graph of $\mathcal{G}_V(n,m,\delta,*,K)$, where $m < m^*(n,\delta,5,K)$, the maximum diameter

$$D^* = D^*(n,m,\delta,K) = \min(D_1^*,D_j^*,,D_5^*),$$

where $j' = 2, 3$, or 4 according as

$$a = n - (\lfloor D/3 \rfloor - 1)M_v - 2\gamma - K,$$

satisfies Theorem 2.14(b) (i), (ii), or (iii), respectively; and the D^*_j, $1 \le j \le 5$, are specified by (2.29), (2.30), and Table 2.7. ∎

Some of the implications of this theorem are discussed in Section 2.4, where also Theorem 2.17, due to Klee and Quaife (1976), is quoted, specifying an upper bound on the diameter of graphs of $\mathcal{G}_v(n,*,\delta,*,K)$. Computer experiments confirm the natural expectation that Theorem 2.16 yields a much sharper upper bound in most cases than Theorem 2.17 does, but the immediate practical utility of Theorem 2.16 in a graph diameter algorithm is nevertheless not clear. From an algorithmic point of view, a reasonably good estimate of the diameter D of a given graph $G \in \mathcal{G}_c(n,m,\delta,*,*)$ can be obtained in O(m) time (the time required to read G into main memory) by considering the vertex sequence $S_d(u)$ corresponding to a vertex u of maximum degree Δ; then $D \le 2d$ and "usually" D is not much greater than d. This observation is the basis of a well-known algorithm due to Cuthill and McKee (1969), later refined into a standard O(m) "pseudo-diameter" algorithm by Gibbs, Poole, and Stockmeyer (1976). Related strategies for determining or estimating the diameter have been proposed by Smyth and Benzi (1974) and Smyth (1985), but all such methods, like the classical algorithms of

Dijkstra (1959) and Floyd (1962), either require $O(n^3)$ time in the worst case or fail to guarantee the exact determination of D. Thus Theorem 2.16 might be useful in special circumstances in a graph diameter algorithm, but its main importance would appear to reside in the light it might shed on the general relationship among the five parameters n, m, δ, D, and K.

GLOSSARY OF TERMS & SYMBOLS

The purpose of this glossary is to provide a single reference point for terminology and notation as a service to the reader who may have forgotten a previous definition. A word of warning: the definitions given here are designed to *remind*, and are therefore not always expressed with full mathematical rigour.

a : The number of excess vertices in a vertex sequence of a graph of $\mathscr{C}_c$.

α : $\lceil 2\sqrt{K} \rceil$.

α_1 : The least integer such that $\alpha_1(\alpha-\alpha_1) \geq K$.

α_2 : $\alpha-\alpha_1$.

b : The sum of the excess terms δ_1 and δ_{D-1} of a vertex sequence of a graph of $\mathscr{C}_v$.

c : A subscript denoting connectivity (c = v) or edge-connectivity (c = e).

$\mathscr{C}_c$: The class $\mathscr{C}(n,m,\delta,D,K)$ of diameter-critical graphs corresponding to parameters n, m, δ, D, and K, where the graphs are K-connected (respectively, K-edge-connected) according as c = v or e.

clump : One of the $\lceil k/2 \rceil$ elements of a partition of a k-tuple.

compact : An internal tuple T of a vertex sequence is compact if every triple of T is minimal internal.

complete : (1) A double (x,y) is complete if its size is $\binom{x+y}{2}$.

(2) A k-tuple T is complete if every double of T is complete.

connectivity : The least number of vertices which can be removed from a graph and leave it disconnected.

critical : See lower (upper) $\mathscr{P}$-critical ($\mathscr{P}$-edge-critical).

δ : The minimum degree over all vertices of a graph.

δ_j : The excess over K in the j-th term of a vertex sequence of a graph of $\mathscr{C}_v$, $1 \le j \le D-1$.

Δ : The maximum degree over all vertices of a graph.

D : The diameter of a graph.

D-critical : Diameter-critical of diameter D.

diameter : The maximum distance between any two vertices of a graph.

diameter-critical : A graph is diameter-critical if the addition of any edge changes (decreases) the diameter.

distance : The number of edges on a shortest path joining two given vertices.

double : Two adjacent terms of a vertex sequence.

edge-connectivity : The least number of edges which can be removed from a graph and leave it disconnected.

edge-minimal : A graph G is edge-minimal over a given class $\mathcal{F}$ of graphs if no other graph of $\mathcal{F}$ has size less than the size of G. Applied also to vertex sequences of G.

edge-maximal : A graph G is edge-maximal over a given class $\mathcal{F}$ of graphs if no other graph of $\mathcal{F}$ has size greater than the size of G. Applied also to vertex sequences of G.

exceptional : A vertex in level $L_j(u)$ is exceptional it its degree is less than $n_{j-1} + n_j + n_{j+1} - 1$.

excess : $n-n_*$.

fat : Not lean.

feasible : (1) A vertex sequence is feasible if it satisfies Lemma 2.12.

(2) A transformation is feasible if it transforms a feasible vertex sequence into another feasible vertex sequence.

G : A graph.

$\mathcal{G}_c$: The class $\mathcal{G}_c(n,m,\delta,D,K)$ of graphs corresponding to parameters n, m, δ, D, and K, where the graphs are K-connected (respectively, K-edge-connected) according as c = v or e.

graph : Finite, non-empty, connected, simple, undirected.

$\hbar$-recurring : (1) A k-tuple T is $\hbar$-recurring if every $\hbar$-th term of T is the same.

(2) A vertex sequence $S_d(u)$ is $\hbar$-recurring if it contains an internal $\hbar$-recurring (d-3)-tuple.

i : The size of a complete (D-1)-tuple $(\delta_1, \delta_2, \ldots, \delta_{D-1})$ of the excess vertices of a graph of $\mathscr{C}_v$.

internal : A k-tuple $(n_j, n_{j+1}, \ldots, n_{j+k-1})$ of $S_d(u)$ is internal if $1 < j < d-k$.

isolated : A lean triple (n_j, n_{j+1}, n_{j+2}) of a vertex sequence of a graph of $\mathscr{C}_v$ is isolated if $(n_{j+1}, n_{j+2}, n_{j+3})$ is fat and there exists a lean triple $(n_{j'}, n_{j'+1}, n_{j'+2})$, $j' > j+1$.

K : The (edge-) connectivity of a graph.

K_α : $K+1-\alpha$.

K-connected : A graph is K-connected if its connectivity $K' \le K$.

K-edge-connected : A graph is K-edge-connected if its edge-connectivity $K' \le K$.

k-tuple : (1) A subsequence $(n_j, n_{j+1}, \ldots, n_{j+k-1})$ of $S_d(u)$.

(2) A sequence $(\delta_1, \delta_2, \ldots, \delta_k)$ of non-negative integers.

$L_j(u)$: The j^{th} level of $S_d(u)$.

lean : A k-tuple of $S_d(u)$ is lean if its order is the least possible value consistent with the parameters δ and K. In particular,

(1) for a graph of $\mathscr{C}_v$, a lean k-tuple has order kK;

(2) for a graph of $\mathscr{C}_e$, a lean double has order α;

(3) for a graph of $\mathscr{C}_c$, a lean triple has order M_c.

length : The index of the last non-zero term in a vertex sequence (the last non-empty set in a rooted level structure).

level : The set of all vertices located at the same distance from a specified vertex u.

level structure : See rooted level structure.

lower $\mathcal{P}$-critical : A graph is lower $\mathcal{P}$-critical if the removal of any vertex changes the value of property $\mathcal{P}$.

lower $\mathcal{P}$-edge-critical : A graph is lower $\mathcal{P}$-edge-critical if the removal of any edge changes the value of property $\mathcal{P}$.

m : The size of a graph.

M_c : The minimum order of a triple in a vertex sequence of a graph $G \in \mathcal{G}_c(n,*,\delta,*,K)$:

$$M_v = \max\{\delta+1, 3K\}, \quad M_e = \delta+1.$$

maximal : A triple T of a vertex sequence S is maximal if no triple of S has order greater than that of T.

maximal compact : An internal tuple $T = (n_j,\ldots,n_{j'})$ of a vertex sequence is maximal compact if T is compact and if neither of the triples (n_{j-1},n_j,n_{j+1}), $(n_{j'-1},n_{j'},n_{j'+1})$ is minimal internal.

maximal internal : An internal triple T of a vertex sequence S is maximal internal if no internal triple of S has order greater than that of T.

minimal : A triple T of a vertex sequence S is minimal if no triple of S has order less than that of T.

minimal internal : An internal triple T of a vertex sequence S is minimal internal if no internal triple of S has order less than that of T.

mod : For integers $x \geq 0$, $y \geq 1$, $x \bmod y = x - y\lfloor x/y \rfloor$ is the remainder after division of x by y.

n : The order of a graph.

n_* : The least order of a graph of $\mathscr{C}_c(*,*,\delta,D,K)$; hence, for $c = v$, $n_* = (D-3)K + 2(\delta+1)$.

n_j : The j^{th} term in a vertex sequence.

non-trivial : A double (x,y) is non-trivial if $x > 0$ and $y > 0$.

order : (1) The order of a graph is the number of vertices in it.

(2) The order of a k-tuple is the sum of its terms.

partition : A partition of a partitionable k-tuple is the set of $\lfloor k/2 \rfloor$ doubles $(\delta_{2j-1},\delta_{2j})$, $1 \leq j \leq \lfloor k/2 \rfloor$, together with the single term δ_k when k is odd.

partitionable : A k-tuple is partitionable if for every j satisfying $1 \le j \le \lceil k/2 \rceil - 1$, δ_{2j} and δ_{2j+1} are not both non-zero.

peripheral : A vertex is peripheral if it is distance D from some other vertex.

regular : (1) For $K \le 7$, a vertex sequence of a graph of $\mathcal{G}_e(n,*,*,D,K)$ is regular if every double of one of the internal (D-4)-tuples $(n_2, n_3, \ldots, n_{D-3})$, $(n_3, n_4, \ldots, n_{D-2})$ is lean and smooth, and if each terminal double is of order K+1.

(2) For $K \ge 8$, a vertex sequence of a graph of $\mathcal{G}_e(n,*,*,D,K)$ is regular if it contains exactly one maximal compact k-tuple.

removable : A vertex u of a graph $G \in \mathcal{G}_c(n,*,\delta,D,K)$ is removable if $G-\{u\} \in \mathcal{G}_c(n-1,*,\delta,D,K)$.

return(X) : A pseudocode feature which assigns the value of X to the function F currently being executed and then exits from F; recommended for inclusion in future computer languages.

rooted level structure : Corresponding to a given vertex u of a graph G, a rooted level structure is an arrangement of the vertices of G into subsets $L_j(u)$, $j = 0, 1, \ldots,$ consisting of the vertices distance exactly j from u.

rough : Not smooth.

S : A vertex sequence.

$S_d(u)$: A vertex sequence of length d corresponding to a given vertex u.

$S_D(u)$: A vertex sequence of length D corresponding to a peripheral vertex u.

$\mathcal{S}$: A set of vertex sequences.

$\sigma(a,k)$: The least size of a complete k-tuple of order a.

size : (1) The size of a graph is the number of edges in it.

(2) A vertex sequence is smooth if every internal double is smooth.

τ : A transformation.

term : A 1-tuple of a vertex sequnce.

terminal : A k-tuple $(n_i, n_{i+1}, \ldots, n_{i+k-1})$ of $S_d(u)$ is terminal if i = 0 or d-k+1.

transformation : An operation which carries a k-tuple into another k-tuple of the same order.

triple : Three adjacent terms of a vertex sequence.

tuple : See k-tuple

u : A vertex.

upper $\mathcal{P}$-edge-critical : A graph is upper $\mathcal{P}$-edge-critical if the addition of any edge changes the value of property $\mathcal{P}$.

v : A vertex.

vertex-minimal : A graph $G \in \mathcal{C}_e$ is vertex-minimal if it has a vertex sequence $S_D(u)$ satisfying one of the following conditions:

(for $K \leq 7$) Every internal double is lean.

(for $K \geq 8$) Every internal triple is lean.

Applied also to vertex sequences.

vertex sequence : Corresponding to a given vertex u of a graph G, a vertex sequence $S_d(u) = (n_0, n_1, \ldots, n\ )$, where for every $0 \le j \le D$, $n_j = |L_j(u)|$ is the number of vertices distance j from u.

(x,y,z) : The recurring lean triple in a vertex sequence.

REFERENCES

Amar, D., Fournier, I. & Germa, A. (1983). Ordre minimum d'un graphe simple de diamètre, degré minimum et connexité donnés. Annals of Discrete Mathematics, pp60-69.

Arany, I., Smyth, W.F. & Szóda, L. (1971). An improved algorithm for reducing the bandwidth of sparse symmetric matrices. Proceedings of the Congress of the International Federation for Information Processing, pp1246-1250.

Bermond, J.-C. & Bollobás, B. (1981). The diameter of graphs — a survey. Congressus Numerantium, 32 pp3-27.

Bhattacharya, D. (1985). The minimum order of n-connected n-regular graphs with specified diameters. Institute of Electrical & Electronic Engineers Transactions on Circuits & Systems, 32(4) pp407-409.

Bloom, G.S., Kennedy, J.W. & Quintas, L.V. (1987). A characterization of graphs of diameter two. American Mathematical Monthly, 94 pp37-38.

Boesch, F.T., Harary, F. & Kabell, J.A. (1981). Graphs as models of communication network vulnerability: connectivity and persistence. Networks, 11(1) pp57-63.

Bollobás, B. (1981). The diameter of random graphs. Transactions of the American Mathematical Society, 267 pp41-52.

Bollobás, B. & de la Vega, W.F. (1982). The diameter of random regular graphs. Combinatorica, 2 pp125-134.

Bosák, J., Rosa, A. & Znám, S. (1968). On decompositions of complete graphs into factors with given diameters, in Theory of Graphs (Proceedings of the 1966 Tihany Colloquium), pp37-56. Academic Press, New York.

Bondy, J.A. & Hell, P. (1983). Counterexamples to theorems of Menger type for the diameter. Discrete Mathematics, 44(2) pp217-220.

Bondy, J.A. & Murty, U.S.R. (1977). Graph Theory with Applications. American Elsevier, New York.

Caccetta, L. (1984). Vulnerability of communication networks. Networks, 14 pp141-146.

Caccetta, L. (1989). Graph theory in network design and analysis, in Kulli, V.R. (ed.) Recent Studies in Graph Theory, pp29-63. Vishwa International Publications, Gulbarga, India.

Caccetta, L. & Smyth, W.F. (1987a). K-edge-connected D-critical graphs of minimum order. Congressus Numerantium, 58 pp225-232.

Caccetta, L. & Smyth, W.F. (1987b). Properties of edge-maximal K-edge-connected D-critical graphs. Journal of Combinatorial Mathematics & Combinatorial Computing, 2 pp111-131.

Caccetta, L. & Smyth, W.F. (1988a). Redistribution of vertices for maximum edge count in K-edge-connected D-critical graphs. Ars Combinatoria, 26B pp115-132.

Caccetta, L. & Smyth, W.F. (1988b). Diameter-critical graphs with a minimum number of edges. Congressus Numerantium, 61 pp143-153.

Caccetta, L. & Smyth, W.F. (1989a). A characterization of edge-maximal diameter-critical graphs. Submitted for publication.

Caccetta, L. & Smyth, W.F. (1989b). Graphs of maximum diameter. Discrete Mathematics (to appear).

Caccetta, L. & Smyth, W.F. (1989c). A sharp upper bound on the diameter of a graph of given order, size, connectivity, and minimum degree. Submitted for publication.

Caccetta, L. & Smyth, W.F. (1989d). The construction of K-connected D-critical graphs of given order and size. Submitted for publication.

Caccetta, L. & Vijayan, K. (1987). Applications of graph theory. Ars Combinatoria, 23B pp21-77.

Chartrand, G. & Lesniak, L. (1986). Graphs and Digraphs, 2nd ed. Wadsworth & Brooks, Monterey, California.

Chung, F.R.K. (1984). Diameters of communication networks, in Mathematics of Information Processing, pp1-18. American Mathematical Society Short Course Lecture Notes.

Chung, F.R.K. (1987). Diameters of graphs: old problems and new results. Congressus Numerantium, 60 pp295-317.

Chung, F.R.K. & Garey, M.R. (1984). Diameter bounds for altered graphs. Journal of Graph Theory, 8 pp511-534.

Cuthill, E. & McKee, J. (1969). Reducing the bandwidth of sparse symmetric matrices. Proceedings of the 24th National Conference of the Association for Computing Machinery, pp157-172.

Dijkstra, E.W. (1959). A note on two problems in connexion with graphs. Numerische Mathematik, 1 pp269-271.

Fan, G. (1987). On diameter 2-critical graphs. Discrete Mathematics, 67 pp235-240.

Floyd, R.W. (1962). Algorithm 97: shortest path. Communications of the Association for Computing Machinery, 5, p345.

George, J.A. & Liu, J.W. (1981). Computer Solution of Large Sparse Positive Definite Systems. Prentice Hall, Englewood Cliffs, N.J.

George, J.A. & Liu, J.W. (1989). The evolution of the minimum degree ordering algorithm. Review of the Society for Industrial & Applied Mathematics, 31(1) pp1-19.

Gibbs, N.E., Poole, W.G. & Stockmeyer, P.K. (1976). An algorithm for reducing the bandwidth and profile of a sparse matrix. Society of Industrial & Applied Mathematics Journal on Numerical Analysis, 13(2) pp236-250.

Goldsmith, D.L., Manvel, B. & Faber, V. (1981). A lower bound for the order of a graph in terms of the diameter and minimum degree. Journal of Combinatorics and Information System Sciences, 6(4) pp315-319.

Harary, F. & Robinson, R.W. (1985). The diameter of a graph and its complement. American Mathematical Monthly, 92 pp211-212.

Kane, V.G. & Mohanty, S.P. (1978). A lower bound on the number of vertices of a graph. Proceedings of the American Mathematical Society, 72(1) pp211-212.

Klee, V. (1980). Classification and enumeration of minimum (d,3,3)-graphs for odd d. Journal of Combinatorial Theory Series B, 28(2) pp184-207.

Klee, V. & Larman, D. (1981). Diameters of random graphs. Canadian Journal of Mathematics, 33 pp618-640.

Klee, V. & Quaife, H. (1976). Minimum graphs of specified diameter, connectivity and valence. Mathematics of Operations Research, 1(1) pp28-31.

Moon, J. (1965). On the diameter of a graph. Michigan Mathematical Journal, 12 pp349-351.

Myers, B.R. (1980). The Klee and Quaife minimum (d,1,3)-graphs revisited. Institute of Electrical & Electronic Engineers Transactions on Circuits & Systems, 27(3) pp214-220.

Myers, B.R. (1981). Regular separable graphs of minimum order with given diameter. Discrete Mathematics, 33(3) pp289-311.

Ore, O. (1968). Diameters in graphs. Journal of Combinatorial Theory, 5(1) pp75-81.

Parthasarathy, K.R. & Srinivasan, N. (1984). An extremal problem in geodetic graphs. Discrete Mathematics, 49(2) pp151-159.

Ringel, G. (1963). Selbskomplementäre graphen. Archiv für Mathematik, 14 pp354-358.

Sachs, H. (1962). Über selbskomplementäre graphen. Publicationes Mathematicae Debrecensis, 9 pp270-288.

Schoone, A.A., Bodlaender, H.L. & van Leeuwen, J. (1987). Diameter increase caused by edge deletion. Journal of Graph Theory, 11(3) pp409-427.

Seidman, S.B. (1983). Network structure and minimum degree. Social Networks, 5(3) pp269-287.

Smyth, W.F. (1985). Algorithms for the reduction of matrix bandwidth and profile. Journal of Computational & Applied Mathematics, 12 & 13 pp551-561.

Smyth, W.F. (1987). Sharp bounds on the diameter of a graph. Canadian Mathematical Bulletin, 30(1) pp72-74.

Smyth, W.F. & Benzi, W.M.L. (1974). An algorithm for finding the diameter of a graph. Proceedings of the Congress of the International Federation for Information Processing, pp500-503.

Straffin, P.D. (1986). Letter to the editor. American Mathematical Monthly, 93 p76.

Usami, Y. (1985). Extremal graphs of diameter at most 6 after deleting any vertex. Journal of Graph Theory, 9 pp221-234.

Wong, P.-K. (1982). Cages — a survey. Journal of Graph Theory, 6 pp1-22.