A FAMILY OF SPARSE GRAPHS OF LARGE SUM NUMBER

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ABSTRACT

Given an integer $r \geq 0$, let $G_r = (V_r, E)$ denote a graph consisting of a simple finite undirected graph $G = (V, E)$ of order n and size m together with r isolated vertices $\overline{K}_r$. Then $|V| = n$, $|V_r| = n + r$, and $|E| = m$. Let $L : V_r \to \mathrm{Z}^+$ denote a labelling of the vertices of G_r with distinct positive integers. Then G_r is said to be a *sum graph* if there exists a labelling L such that for every distinct vertex pair u and v of V_r, $(u, v) \in E$ if and only if there exists a vertex $w \in V_r$ whose label $L(w) = L(u) + L(v)$. For a given graph G, the *sum number* $\sigma = \sigma(G)$ is defined to be the least value of r for which G_r is a sum graph. Gould and Rödl have shown that there exist infinite classes $\mathcal{G}$ of graphs such that, over $G \in \mathcal{G}$, $\sigma(G) \in \Theta(n^2)$, but no such classes have been constructed. In fact, for all classes $\mathcal{G}$ for which constructions have so far been found, $\sigma(G) \in o(m)$. In this paper we describe constructions which show that for wheels W_n of (sufficiently large) order $n + 1$ and size $m = 2n$, $\sigma(W_n) = n/2 + 3$ if n is even and $n \leq \sigma(W_n) \leq n + 2$ if n is odd. Hence for wheels $\sigma(W_n) \in \Theta(m)$.

1 INTRODUCTION

Sum graphs were introduced by Harary [Har88]. In 1989 Hao [Hao89] established a lower bound on $\sigma(G)$ in terms of the degree sequence of G, and also showed that a sum graph of order n and size m exists if and only if

$$m \leq \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2}.$$

These results have recently been applied by Smyth [S92] to *unit graphs* (that is, graphs G for which $\sigma(G) = 1$). He shows that there exists a unit graph of order $n > 1$ and size m if and only if $\lceil n/2 \rceil \leq m \leq \lfloor n^2/4 \rfloor$, and provides a methodology for constructing at least one such graph for each suitable value of m. The same paper also shows how to construct graphs G of given order $n \geq 4$ and size m whose sum number $\sigma(G) \in o(m)$. Other constructions have been found for specific classes:

* Ellingham [E89] shows that every nontrivial tree is a unit graph;

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* Bergstrand et al. [BHHJKW89] show that for a complete graph K_n, $n \geq 4$, $\sigma(K_n) = 2n - 3$;
* Hartsfield and Smyth [HS92] show that for a complete bipartite graph $K_{p,q}$, $2 \leq p \leq q$, $\sigma(K_{p,q}) = \lceil (3p + q - 3)/2 \rceil$.

Thus to date constructions have focussed on sum graphs of "small" sum number: no class of graphs is known whose sum number is even close to the bound of Gould and Rödl [GR89] given in the Abstract. In this paper we consider wheels W_n of order $n + 1$ and size $m = 2n$; we show that, over all wheels $\mathcal{W}$, $\sigma(W_n) \in \Theta(m)$. Section 2 shows that, for odd n, $n \leq \sigma(W_n) \leq n+2$, and describes labellings which achieve the upper bound. Section 3 considers the case in which n is even, and presents a construction which achieves sum number $\sigma(W_n) = n/2 + 3$. Section 4 discusses some conjectures and open problems.

2 WHEELS WITH AN ODD NUMBER OF SPOKES

For every integer $n \geq 3$, a *wheel* $W_n = (V, E)$ is the graph defined by $V = \{c, a_1, a_2, \ldots, a_n\}$, $E = \{(c, a_i), (a_i, a_{i+1}) | i = 0, 1, \ldots, n-1\}$, where here and throughout this paper arithmetic on indices is interpreted modulo n. (Thus, for example, $(n-1)+1 = 0$.) The vertex c is called the *center* of the wheel, each edge (c, a_i), $i = 0, 1, \ldots, n-1$, is called a *spoke*, and the cycle $C_n = W_n - c$ is called the *rim*. We let $\mathcal{W} = \{W_3, W_4, \ldots\}$ and, to simplify notation, we suppose that the vertices V are already identified by their labels (distinct positive integers) under L.

Some general sum graph terminology will be useful. In a sum graph G_r, a vertex u is said to *label* an edge (x, y) if and only if $u = x+y$. The number of edges labelled by a given vertex u is called its *multiplicity* and is denoted by $\mu(u)$; if $\mu(u) > 0$ then u is said to be a *working vertex*. If $G_r = G + \overline{K}_r$ and G contains no working vertices, then G_r is said to be *exclusive*; otherwise G_r is said to be *inclusive*. One of the interesting results of Bergstrand et al. [BHHJKW89] is that for $n \geq 4$ every sum graph $G_r = K_n + \overline{K}_r$ is exclusive. In this section we first show that every sum graph $G_r = W_n + \overline{K}_r$ is exclusive provided n is odd.

Lemma 2.1 Suppose that $G_r = W_n + \overline{K}_r$ is a sum graph. If for some integer i satisfying $0 \leq i \leq n-1$, $c + a_i \in V$, then
(a) n is not odd;
(b) the vertices of the rim consist of $n/2$ working vertices $c + b_j$, $1 \leq j \leq n/2$, which label spokes, alternating with $n/2$ vertices b_k, $1 \leq k \leq n/2$, which do not label spokes.

Proof By hypothesis, there exists $a_j \in V$ such that $(a_j, c + a_i) \in E$ and $j \neq i$. Then $c + a_j$ and $c + a_j + a_i$ are both vertices, so that $(c + a_j, a_i) \in E$ and $c + a_j \in V$. Thus every working vertex $c + a_i$ implies the existence of another working vertex $c + a_j$ adjacent to a_i, and we see that (b) follows. Observe then that (b) cannot hold if n is odd. $\square$

This result tells us that either there are no working vertices in V which label spokes or there are exactly $n/2$ of them. We shall see in the next section that the latter case can in fact occur when n is even.

Lemma 2.2 Suppose that $G_r = W_n + \overline{K}_r$ is a sum graph. Then no edge of the rim is labelled by a vertex of V.

Proof For $n = 3$, $W_n = K_{n+1}$ and the result follows from the fact [BHHJKW89] that every labelling of the complete graph is exclusive. Then suppose that $n \geq 4$ and that, for some choice of i, $a_i + a_{i+1} \in V$. Two cases now arise: either $c = a_i + a_{i+1}$ or there exists an integer $j \notin \{i, i+1\}$ such that $a_j = a_i + a_{i+1}$. Suppose the former. Then there exists a rim vertex $a_j \neq a_{i+1}$ such that $(a_i, a_j) \in E$, and therefore there exists a vertex $a_i + a_j \in V_r$. But it must also be true that $a_i + a_{i+1} + a_j \in V_r$, from which it follows that $(a_i + a_j, a_{i+1}) \in E$, and so $a_i + a_j$ must be a rim vertex. Thus the first case implies the second.

Suppose then that $a_j = a_i + a_{i+1} \in V$. It follows that $\{c + a_i + a_{i+1}, c + a_i, c + a_{i+1}\} \subseteq V_r$, hence that $(c + a_i, a_{i+1}) \in E$, $(c + a_{i+1}, a_i) \in E$. Then the rim contains a cycle $(c + a_{i+1}, a_i, a_{i+1}, c + a_i)$ in which the vertices that label spokes do not alternate. This contradicts Lemma 2.1 and completes the proof. $\square$

These two lemmas immediately yield

Theorem 2.1 For odd n, every sum graph $G_r = W_n + \overline{K}_r$ is exclusive. $\square$

This result allows us to express the set $\overline{K}_r$ of isolated vertices as a union of two subsets: $R = \{a_i + a_{i+1} \,|\, i = 0, 1, \ldots, n-1\}$, which labels edges of the rim, and $S = \{c + a_i \,|\, i = 0, 1, \ldots, n-1\}$, which labels spokes. Since the elements of S must all be distinct, it follows that, for odd n, $\sigma(W_n) \geq n$. However, it does not follow that $\sigma(W_n) = 2n$: there are two strategies available to us to reduce r, which we now describe. We can try to choose vertex labels so that

(a) for certain nonnegative integers i and $j \notin \{i-1, i\}$,

$$c + a_i = a_j + a_{j+1}; \qquad \ldots (2.1)$$

(b) for certain nonnegative integers i and $j \notin \{i-1, i, i+1\}$,

$$a_i + a_{i+1} = a_j + a_{j+1}. \qquad \ldots (2.2)$$

We consider first strategy (b). Observe that the best possible result that can be achieved using equations (2.2) is to give alternate edges of the rim the same label; that is, to choose labels so that

$$a_0 + a_1 = a_2 + a_3 = \cdots = a_{n-3} + a_{n-2},$$

$$a_1 + a_2 = a_3 + a_4 = \cdots = a_{n-2} + a_{n-1}.$$

Thus R would reduce to $\{a_0 + a_1, a_1 + a_2, a_{n-1} + a_0\}$, yielding a labelling for which $r = n + 3$. (In fact it turns out that, for odd $n \geq 7$, the best that can be done using strategy (b) is $r = n + 4$.) We shall now consider a construction based on strategy (a) which yields $r = n + 2$.

In order to describe this construction, we first make the assumption that $c < a_i$ for every $i = 0, 1, \dots, n-1$. Then each equation (2.1) implies that both $a_i > a_j$ and $a_i > a_{j+1}$ hold, and the aggregate of such equations defines a partial order on the vertices of the rim. Let us say that a vertex a_j is *dominated* if it occurs on the right hand side of some equation (2.1). Then in order that the partial order should be consistent, there must exist at least one vertex, say a_k, which is not dominated. Call such a vertex *exceptional.* It follows that at most $n-2$ of the equations (2.1) can be simultaneously satisfied, and in that case both of the unsatisfied equations must include the exceptional vertex a_k on the right hand side:

$$c + a_i \neq a_{k-1} + a_k,\; c + a_j \neq a_k + a_{k+1},$$

for some integers i and j. Thus the only possible way to improve on a labelling induced by $n-2$ equations (2.1) would be to apply equation (2.2), setting $a_{k-1} + a_k = a_k + a_{k+1}$, an impossibility since $a_{k-1} \neq a_{k+1}$. We conclude that if $n-2$ of the equations (2.1) can be simultaneously satisfied, and if c is the least label of W_n, then the resulting labelling cannot be further improved.

To produce $n-2$ consistent equations (2.1), set $j = i-2$ for every $i = 2, 3, ..., n-1$; then

$$c + a_2 = a_0 + a_1, c + a_3 = a_1 + a_2, \dots, c + a_{n-1} = a_{n-3} + a_{n-2}. \qquad \dots (2.3)$$

These equations can easily be solved in terms of a_0, a_1 and c, yielding

$$a_i = a_0 f_{i-1} + a_1 f_i - (f_{i+1} - 1)c, \qquad \dots (2.4)$$

for $i = 2, 3, \dots, n-1$, where f_i is the ith Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_i = f_{i-2} + f_{i-1}$ for every $i > 1$). The solutions (2.4) may also be rewritten in Fibonacci form as

$$a_i = a_{i-2} + a_{i-1} - c, \qquad \dots (2.4a)$$

exhibiting the labels of the rim vertices as a monotone increasing sequence. Then the set of isolated vertices corresponding to (2.3) is

$$\overline{K}_{n+2} = S \cup R = \{c + a_i \mid i = 0, 1, \dots, n-1;\; a_{n-2} + a_{n-1};\; a_{n-1} + a_0\}, \qquad \dots (2.5)$$

where the a_i, $i = 2, 3, \dots, n-1$, satisfy (2.4). Call this construction $\mathcal{C}_1$: it remains to be seen whether a_0, a_1 and c can be chosen so that $\mathcal{C}_1$ yields a sum graph. It is clear that every edge of E is labelled by an element of (2.5); what needs to be shown is that no vertex of $G_{n+2} = W_n + \overline{K}_{n+2}$ is a sum of any pair of vertices of G_{n+2} which do not form an edge.

Theorem 2.2 For odd n and choices $c = 1$, $a_0 = 10$, $a_1 = 100$, the construction $\mathcal{C}_1$ defined by (2.5) yields a sum graph $G_{n+2} = W_n + \overline{K}_{n+2}$.

Proof It is easy to verify, using (2.4a), that for every $i = 2, 3, \dots, n-1$, the sums $a_j + a_i$, $j = 0, 1, \dots, i-1$, satisfy

$$a_i + 1 < a_j + a_i < a_{i+1} - 1;$$

and further that

$$a_{i+1} < a_i + (a_i + 1) < a_{i+2}.$$

From these inequalities, and from the monotonicity of the a_i, it follows that no vertex of V_{n+2} takes any of the values $a_j + a_i$, $a_j + (a_i + 1)$, or $2a_i + 1$, over the indices i and j specified above. Since clearly c does not give rise to any unwanted edges, the result follows. □

The construction $\mathcal{C}_1$ was based on choosing c to be the least label of V. It turns out, however, that if c is chosen to be a label of intermediate value in V, then up to $n-1$ equations (2.1) may be consistent and in at least some cases a labelling can be found for which $r = n$ or $n+1$. For $n = 3$, $\mathcal{C}_1$ yields $\sigma(W_n) = 5$ by the result of Bergstrand et al. [BHHJKW89]. But for $n = 5$ the assignment

$$c = x, a_1 = x + 5d, a_2 = x + 2d, a_3 = x + d, a_4 = x + 4d, a_5 = x$$

induces a sum graph $G_5 = W_5 + \overline{K}_5$ for suitable choices of the integers x and d; hence $\sigma(W_5) = 5$. Similarly, for $n = 9$ we have found a labelling which yields $G_{10} = W_9 + \overline{K}_{10}$ (so that $\sigma(W_9) \leq 10$), while for $n = 7$ it appears that $\mathcal{C}_1$ is optimal. We return to this subject briefly in Section 4. For now we restrict ourselves to a formal statement of the main result of this section:

Theorem 2.3 For odd n, $n \leq \sigma(W_n) \leq n + 2$. □

3 WHEELS WITH AN EVEN NUMBER OF SPOKES

We begin this section by looking again at Lemma 2.1 in the case that n is even. Suppose that we start to give the rim vertices alternating working and nonworking labels, as prescribed by the lemma. If we begin somewhere with a nonworking label b_1, then adjacent to b_1 there exists a working label $c + b_2$. Therefore, elsewhere on the rim, there exists a label b_2 adjacent to $c + b_1$. If the directed edges $(b_1, c + b_2)$ and $(c+b_1, b_2)$ are either both oriented clockwise or both oriented counterclockwise, then we say that they are *harmonious*; otherwise, we say that they are *contrary*. If all such pairs of directed edges on the rim are harmonious, we say that we have a *harmonious labelling*; conversely, if all such pairs are contrary, then we have a *contrary labelling*.

Lemma 3.1 The labelling of every inclusive sum graph $G_r = W_n + \overline{K}_r$ is either harmonious or contrary.

Proof It is easy to see that if one pair of edges is harmonious (respectively, contrary), then the edges adjacent to them must also be harmonious (respectively, contrary). □

This result tells us that there are exactly two strategies for choosing working rim labels when n is even. Further, it is easy to see that for a contrary labelling, there must exist exactly two edges, say $(b_1, c + b_1)$ and $(b_{n/2}, c + b_{n/2})$, joining nonworking vertices to their corresponding working vertices. On the other hand, for a harmonious labelling, every pair b_i and $c + b_i$ of corresponding vertices must be antipodal. It follows from these observations that, for $n \equiv 2 \pmod 4$, both harmonious and contrary labellings are possible, while for $n \equiv 0 \pmod 4$, an inclusive sum graph can only have a contrary labelling.

We consider first, therefore, for any even $n \geq 4$, a contrary labelling of the rim. Then there exists a nonworking vertex, say x, which is adjacent to its corresponding working vertex $x + c$, and thus by Lemma 2.1 there must exist an isolated vertex $2x + c \in \overline{K}_r$. Further, there exists a second vertex y adjacent to x such that $y \neq x + c$. Using Lemma 2.2, we may without loss of generality set $y = x - d + c$ for some non-zero integer d, and it follows that there must exist a second isolated vertex $2x - d + c \in \overline{K}_r$. Observe now that these two isolated vertices could possibly be used to label alternating edges of the rim. If this were done, the vertices of the rim could be expressed as follows, in terms of two disjoint paths, one clockwise in direction, the other counterclockwise, around the rim:

$$\begin{gathered}(x, x-d+c, x+d, x-2d+c, x+2d, \ldots, w)\\(x+c, x-d, x+d+c, x-2d, x+2d+c, \ldots, z)\end{gathered} \qquad \ldots(3.1)$$

where the terminating vertices w and z are given by

$$\begin{aligned} w &= x - \lfloor n/4 \rfloor d + c, && \text{for } n \equiv 0 \pmod 4,\\ &= x + \lfloor n/4 \rfloor d, && \text{for } n \equiv 2 \pmod 4; \end{aligned} \qquad \ldots(3.2)$$

$$\begin{aligned} z &= x - \lfloor n/4 \rfloor d, && \text{for } n \equiv 0 \pmod 4,\\ &= x + \lfloor n/4 \rfloor d + c, && \text{for } n \equiv 2 \pmod 4. \end{aligned} \qquad \ldots(3.3)$$

Then a third distinct isolated vertex

$$\begin{aligned} w + z &= 2x - 2\lfloor n/4 \rfloor d + c, && \text{for } n \equiv 0 \pmod 4,\\ &= 2x + 2\lfloor n/4 \rfloor d + c, && \text{for } n \equiv 2 \pmod 4, \end{aligned} \qquad \ldots(3.4)$$

must exist to label the edge (w, z). As in Section 2, denote by R the set of isolated vertices which label the rim; then

$$R = (2x + c, 2x - d + c, 2x \pm 2\lfloor n/4 \rfloor d + c), \qquad \ldots(3.5)$$

where the ambiguous sign in the third element is governed by (3.4). Similarly, the set of isolated vertices labelling spokes is

$$S = (x - id + 2c \,|\, i = 1, 2, \ldots, \lfloor n/4 \rfloor;\ x + id + 2c \,|\, i = 0, 1, \ldots, \lfloor (n-2)/4 \rfloor). \ \ldots(3.6)$$

Let $\mathcal{C}_0$ denote the construction described by (3.1)-(3.6). Clearly $\mathcal{C}_0$ yields vertices which provide labels for every edge of W_n and includes at most

$$|R| + |S| = n/2 + 3$$

isolated vertices. Observe further that by choosing (for example)

$$x = 10^n, \quad d = 2n, \quad c = 1, \qquad \ldots(3.7)$$

we can easily ensure that $\mathcal{C}_0$ yields a sum graph; that is, that for every pair of vertices $u, v \in V \cup R \cup S$ such that $(u,v) \notin E$, $u+v \notin V \cup R \cup S$. We have then

Theorem 3.1 For even $n \geq 4$, the construction $\mathcal{C}_0$ defined by (3.1)-(3.6) with labels (3.7) yields a sum graph $G_{n/2+3} = W_n + \overline{K}_{n/2+3}$. $\square$

We consider now the question of whether or not $\mathcal{C}_0$ is optimal — that is, yields a minimum number of isolated vertices. First observe that every exclusive sum graph derived from W_n must contain at least n isolated vertices. For the special case $n = 4$ we leave it as an exercise for the reader to show that $\sigma(W_4) = 5$. For $n \geq 6$ the fact that $n \geq n/2+3$ tells us that $\mathcal{C}_0$ is at least as good as any construction based on an exclusive labelling. Moreover, in view of Lemmas 2.1 and 2.2, we see that, at least for contrary labellings, the only possible strategy to reduce the number of isolated vertices below $n/2+3$ is to label the vertices in such a way that at least one isolated vertex labels both a spoke and a vertex of the rim. The following lemma shows that such a strategy is impossible, no matter whether the labelling of the rim is harmonious or contrary.

Lemma 3.2 No vertex of an inclusive sum graph $G_r = W_n + \overline{K}_r$ labels both a spoke and an edge of the rim.

Proof Suppose on the contrary that there exists a vertex t which labels both a spoke (c, y) and an edge (x_1, x_2) of the rim. Then $t = x_1 + x_2 = y + c$ and, by Lemma 2.2, t is isolated. Since G_r is inclusive, it follows from Lemma 2.1 that the rim of W_n consists of alternating working and nonworking vertices. Therefore we may without loss of generality suppose that $x_2 = x_2' + c$, where x_2' is a vertex of the rim. Observe also that if y is nonworking, then $t = y + c$ is on the rim, a contradiction. Hence $y = y' + c$ is a working vertex corresponding to a nonworking vertex y' of the rim.

From the above expression for t, it follows now that $y = y' + c = x_1 + x_2'$, from which we conclude that (x_1, x_2') is an edge joining two nonworking vertices, in contradiction to the assumption that G_r is inclusive. $\square$

Lemma 3.2 establishes the optimality of $\mathcal{C}_0$ in the case that $n \equiv 0 \pmod 4$, when no harmonious labelling exists. But for $n \equiv 2 \pmod 4$ it is easy to see that there exists no harmonious labelling with less than three distinct labels for rim edges. Thus we have established

Theorem 3.2 For even n, $\sigma(W_n) = n/2 + 3$. $\square$

4 REMARKS

The results and conjectures of the preceding two sections establish the fact that for wheels the sum number is of the same order of magnitude as the number of edges. Wheels are the first class of graphs for which this property has been proven to hold, even though the existence of such classes has been established by [GR89]. Our results are not complete, however: for odd n, we have been able only to bound the sum number between n and $n+2$. The exact computation of $\sigma(W_n)$ in this case appears to be a rather difficult problem, to which we are able to contribute only the following

Conjecture 4.1 For $n \equiv 1 \pmod 4$ and $n \geq 5$, $\sigma(W_n) = n + 1$; for $n \equiv 3 \pmod 4$, $\sigma(W_n) = n + 2$. □

Indeed, the fact that results for wheels have been rather difficult to establish, even given the generous structural information provided by Lemmas 2.1, 2.2, and 3.1, attests to the difficulty of sum number problems in general. In particular, it would be of great interest to determine a class of dense graphs for which the sum number is of the same order of magnitude as the size. Here also we make a conjecture:

Conjecture 4.2 For every integer $n \geq 2$, let O_{2n} denote the generalized octohedron — that is, K_{2n} less a 1-factor. Then over all octahedra, $\sigma(O_{2n}) \in \Theta(n^2)$. □

It turns out that every sum graph derived from O_{2n} must be exclusive; but as for wheels this structural information is not sufficient to easily give rise to optimal constructions.

Finally, we mention an important conjecture for graphs of small sum number, first stated in [H89]:

Conjecture 4.3 Any disjoint union of unit graphs is a unit graph. □

This conjecture is known to hold for trees [E89].

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