Functional Programming

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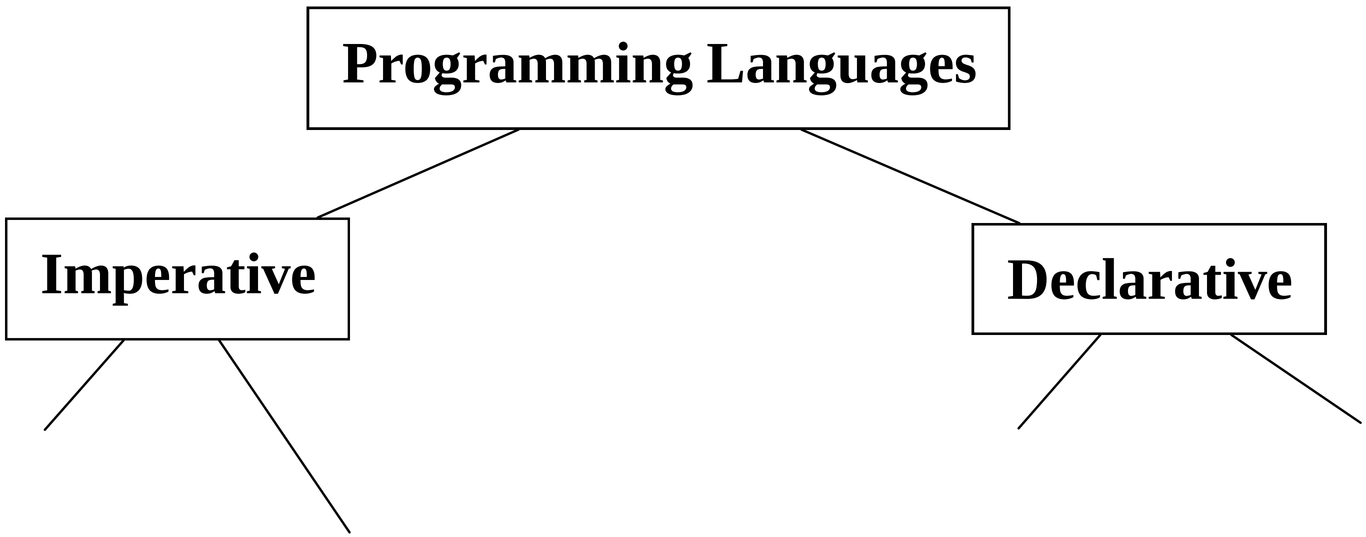
Declarative — "telling the machine what to **achieve**"



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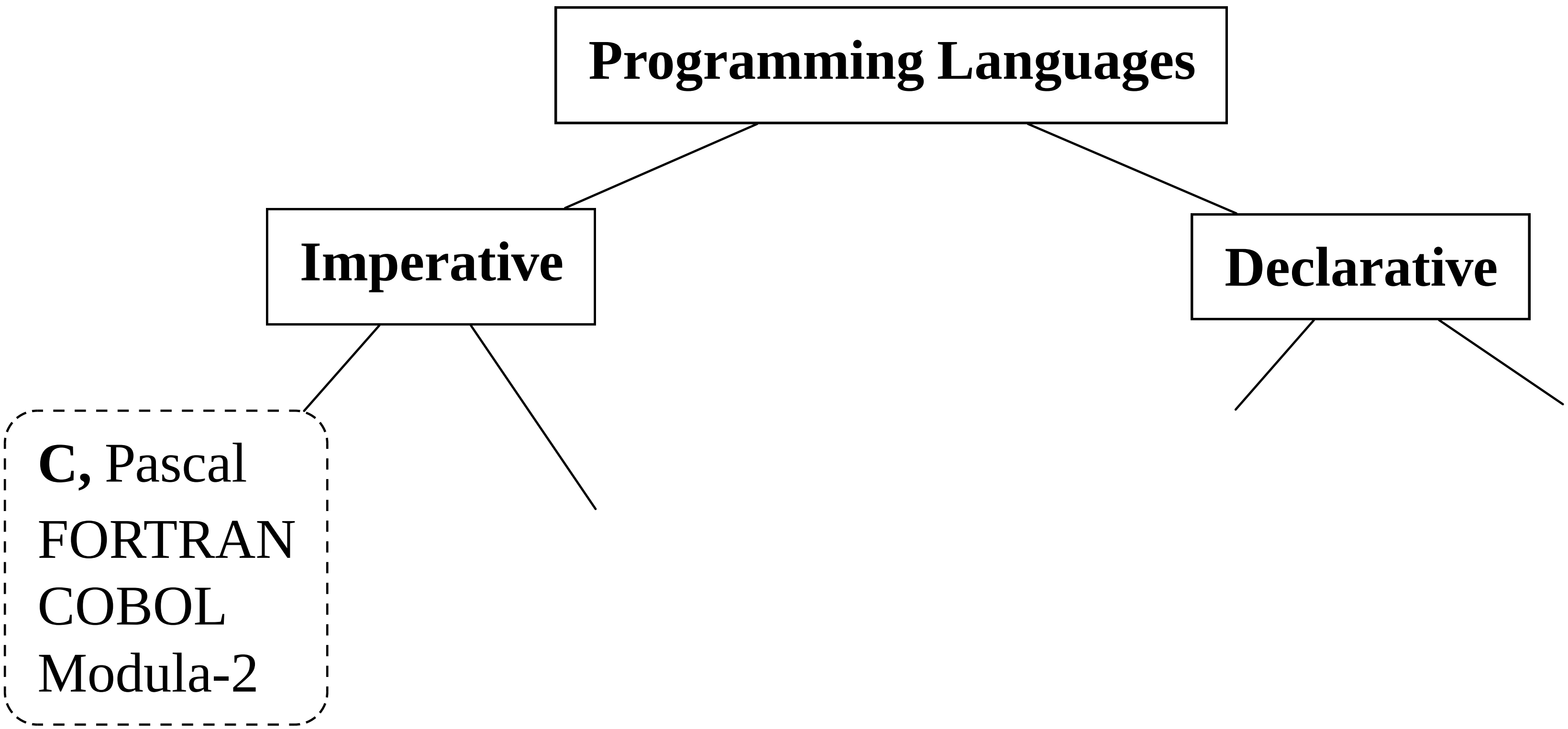

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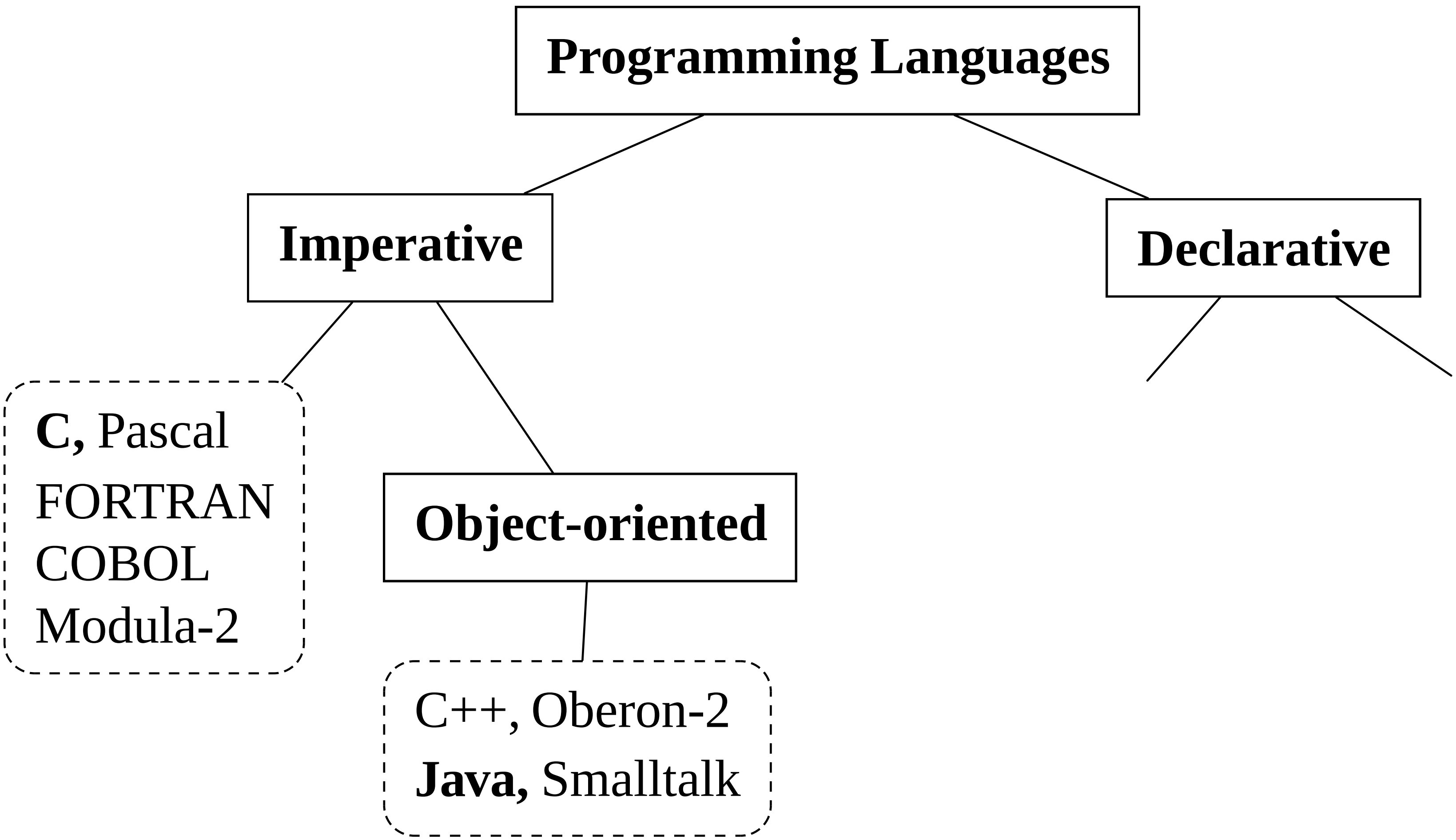

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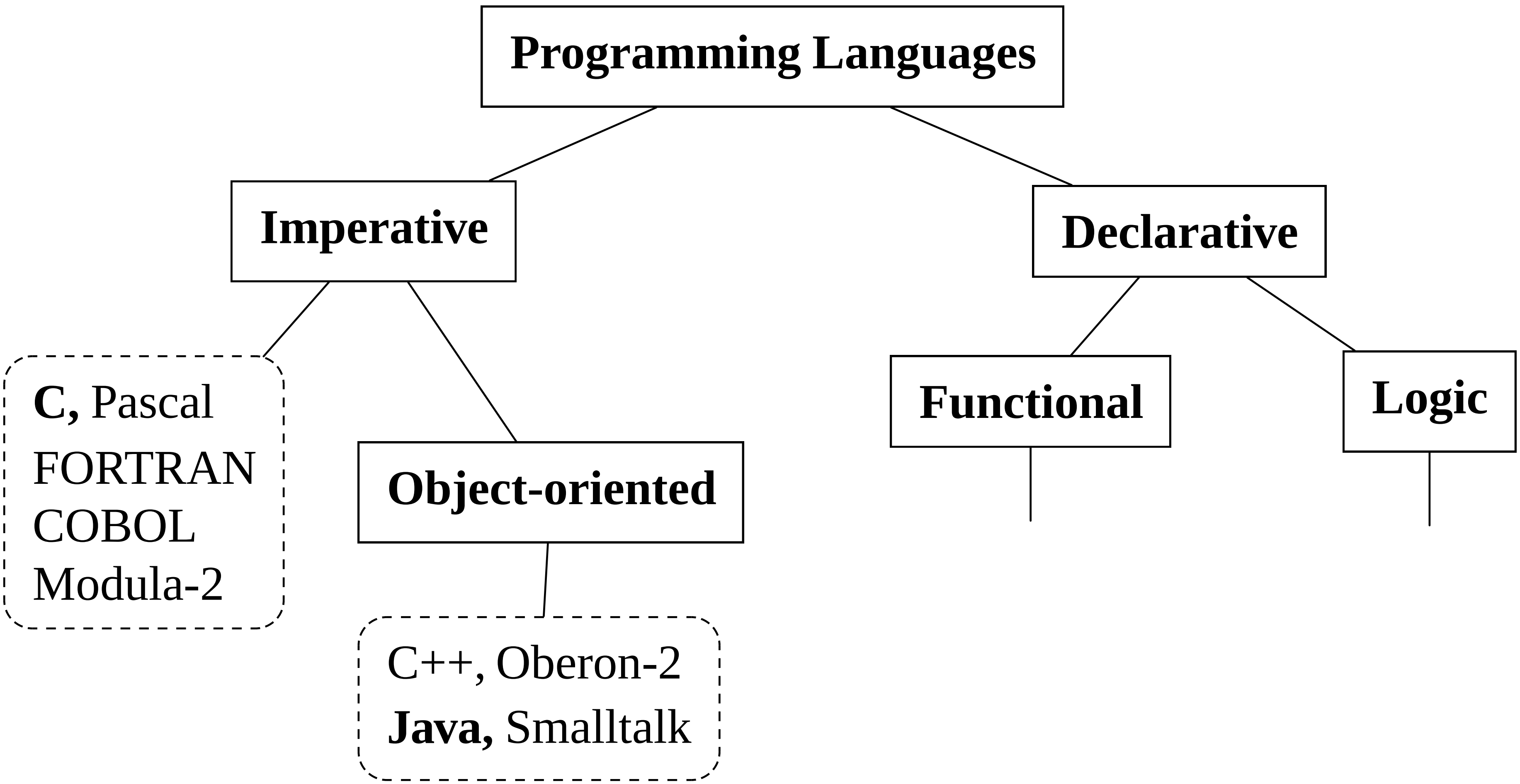

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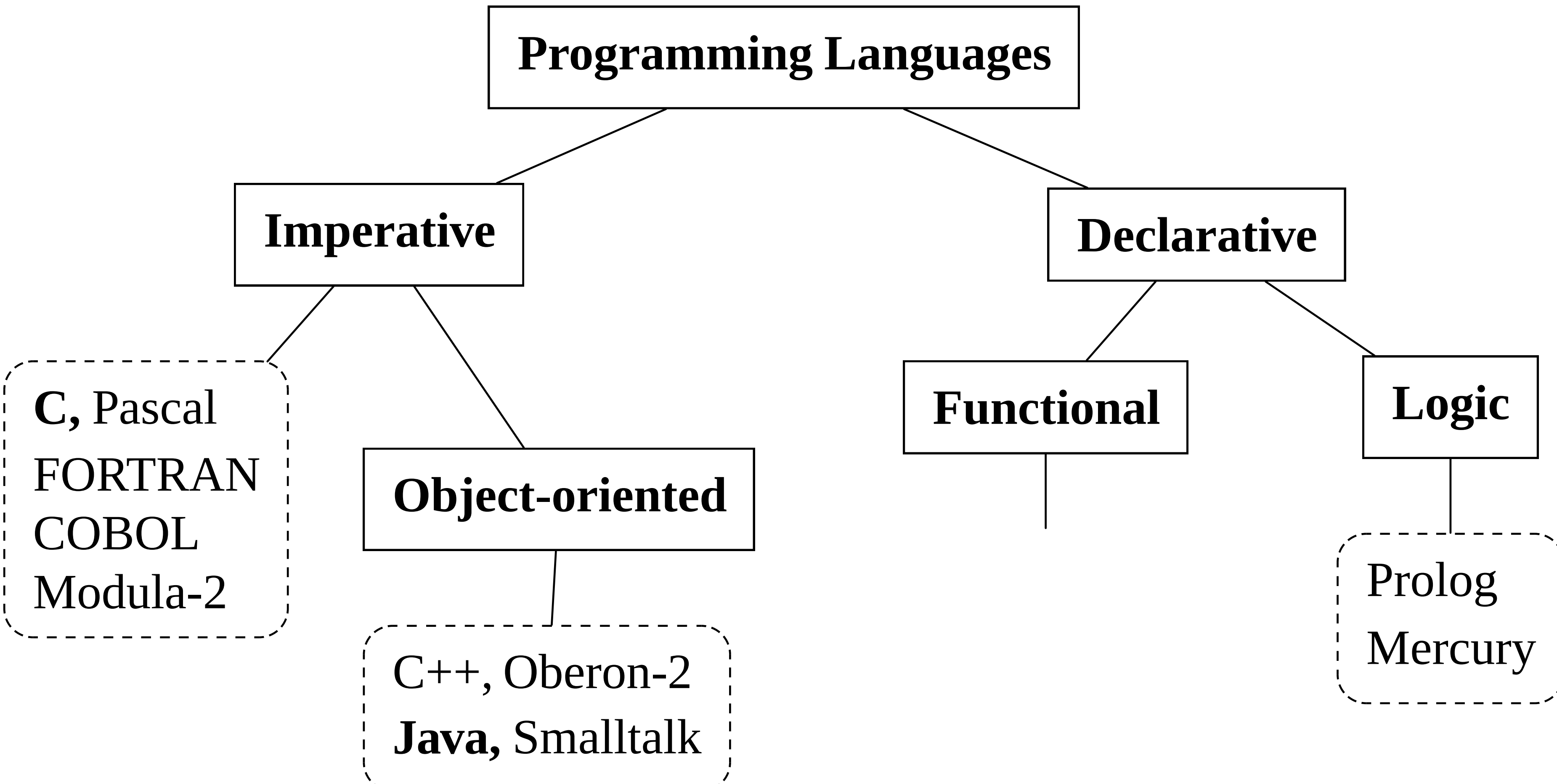

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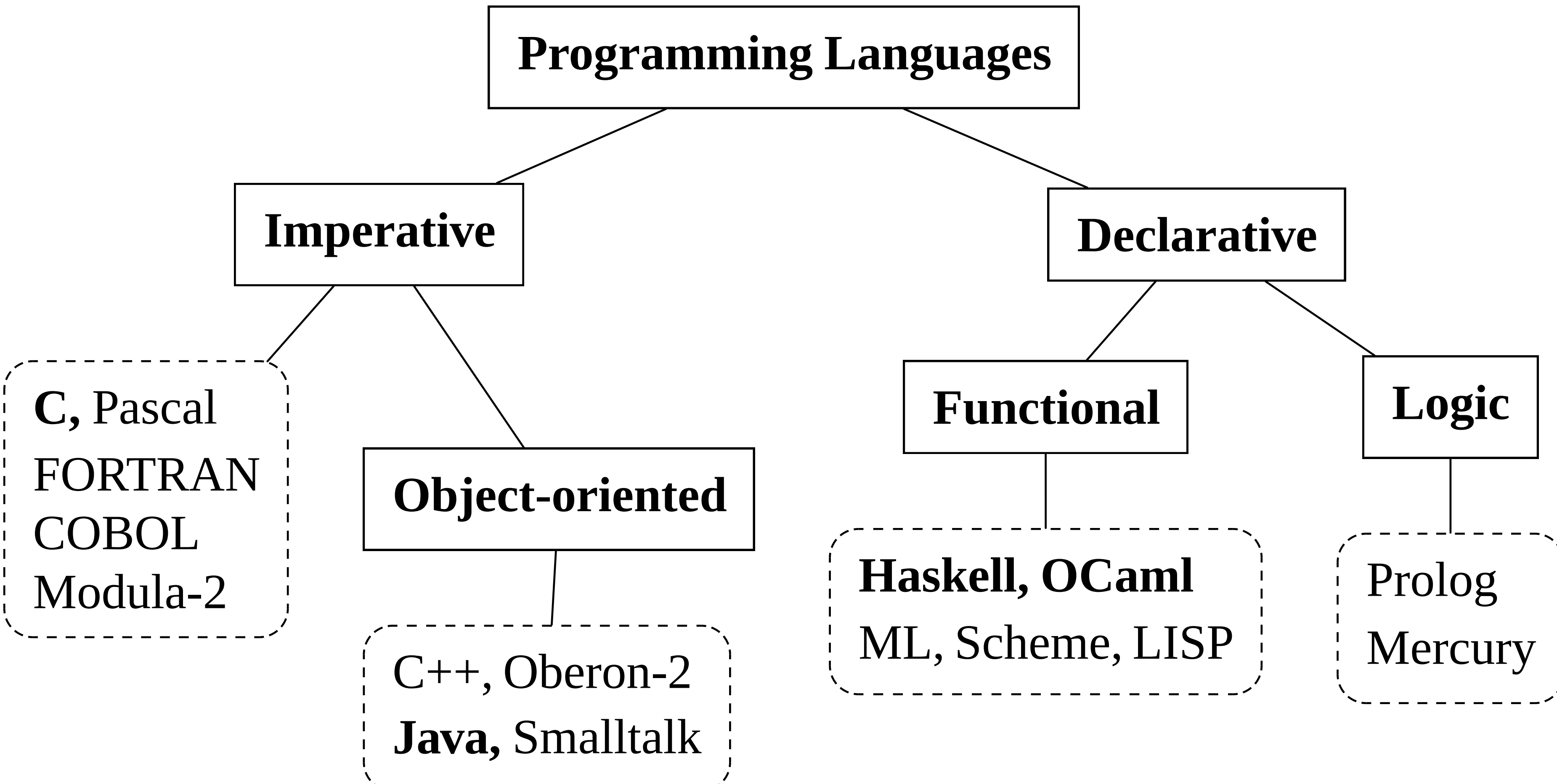

Programming Language Paradigms

Imperative Programming Languages

Statement oriented languages

Every statement changes the machine state

Object-oriented languages

Organising the state into *objects* with individual state and behaviour

Message passing paradigm (instead of subprogram call)

Rule-Based (Logical) Programming Languages

Specify rule that specifies problem solution (Prolog, BNF Parsing)

Other examples: Decision procedures, Grammar rules (BNF)

Programming consists of specifying the attributes of the answer

Functional (Applicative) Programming Languages

Goal is to understand the function that produces the answer

Function composition is major operation

Programming consists of building the function that computes the answer

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Goal of language development:

– Developers concentrate on design (or even just specification)

– Programming is trivial or handled by computer

(*executable specification languages, rapid prototyping*)

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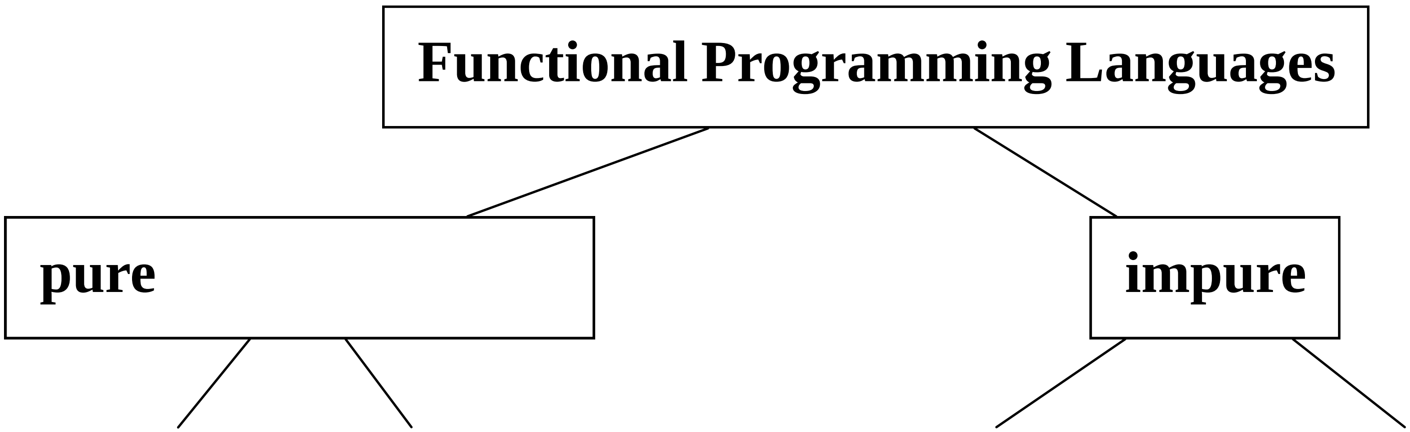

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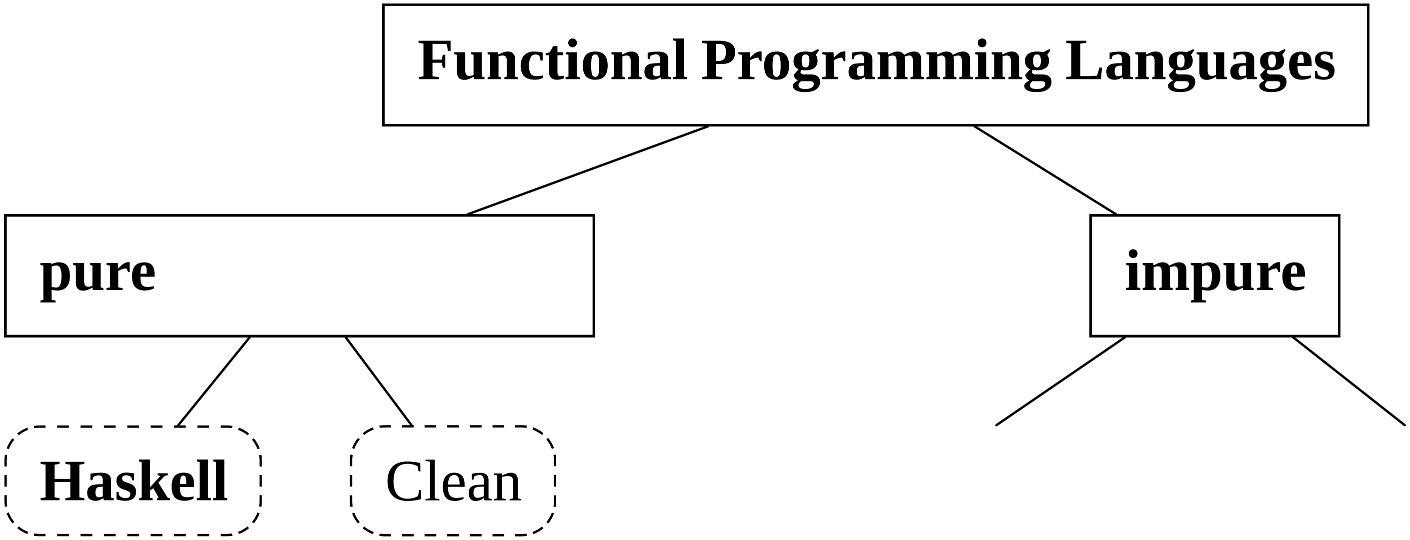

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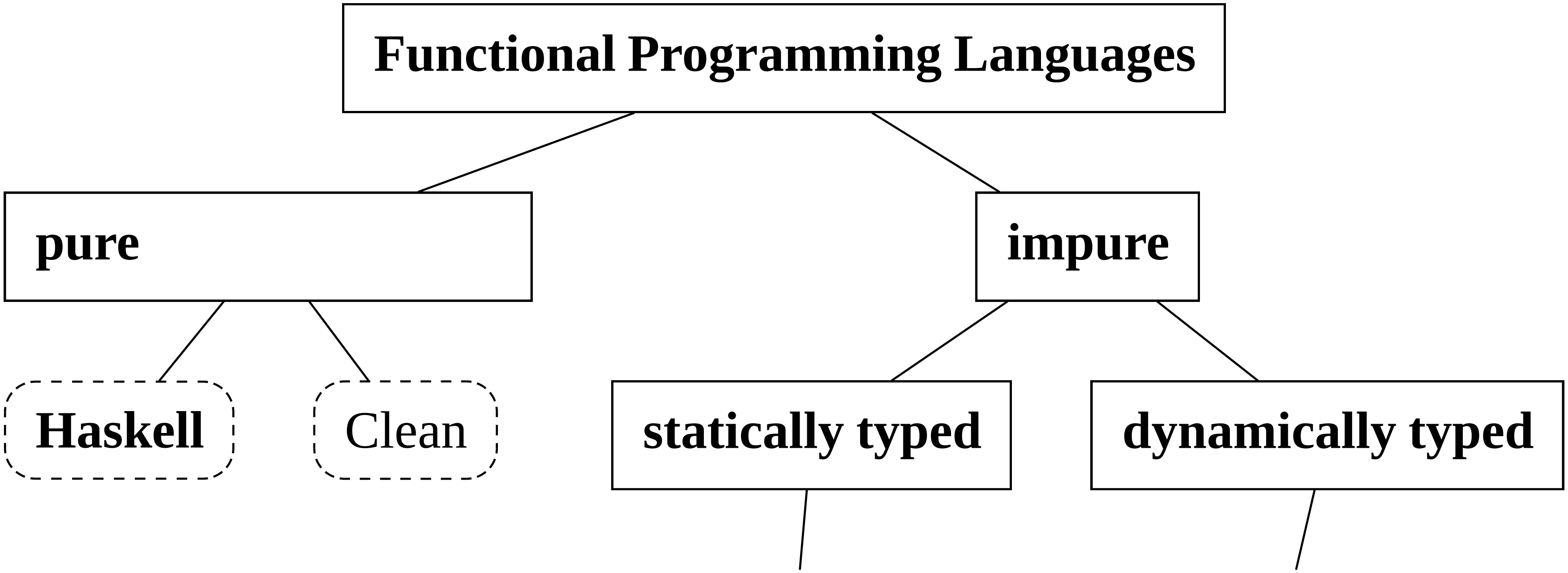

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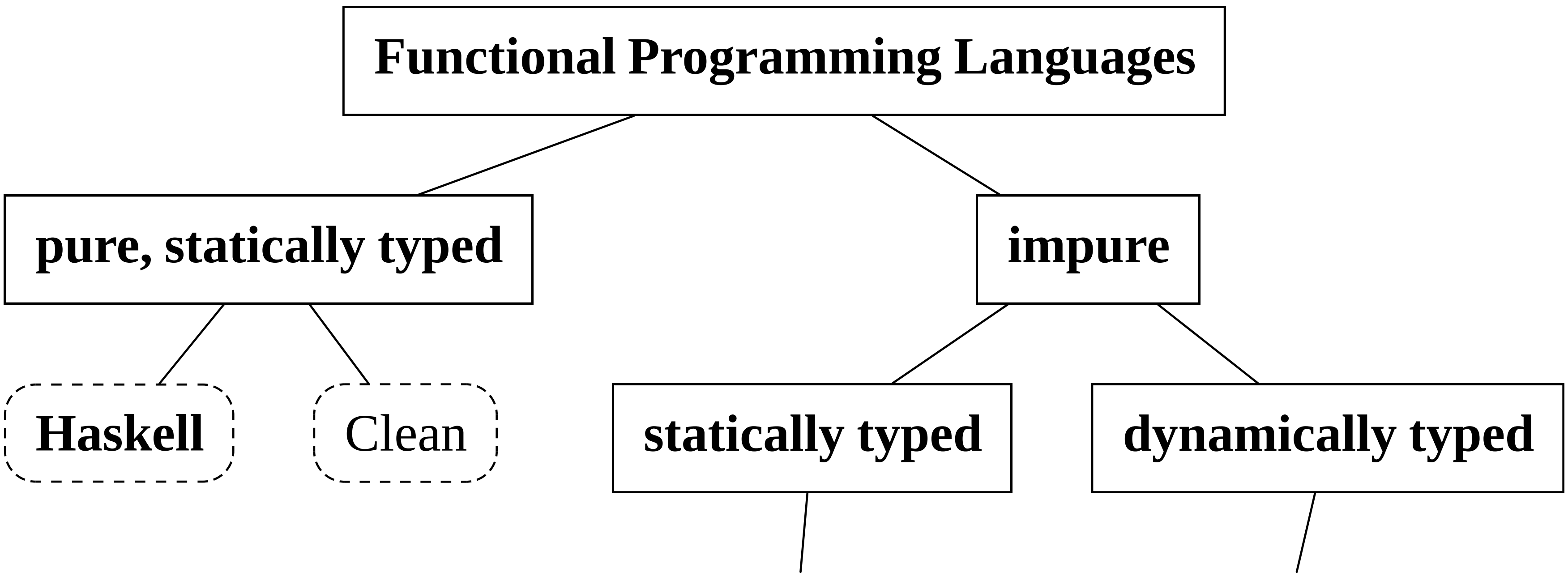

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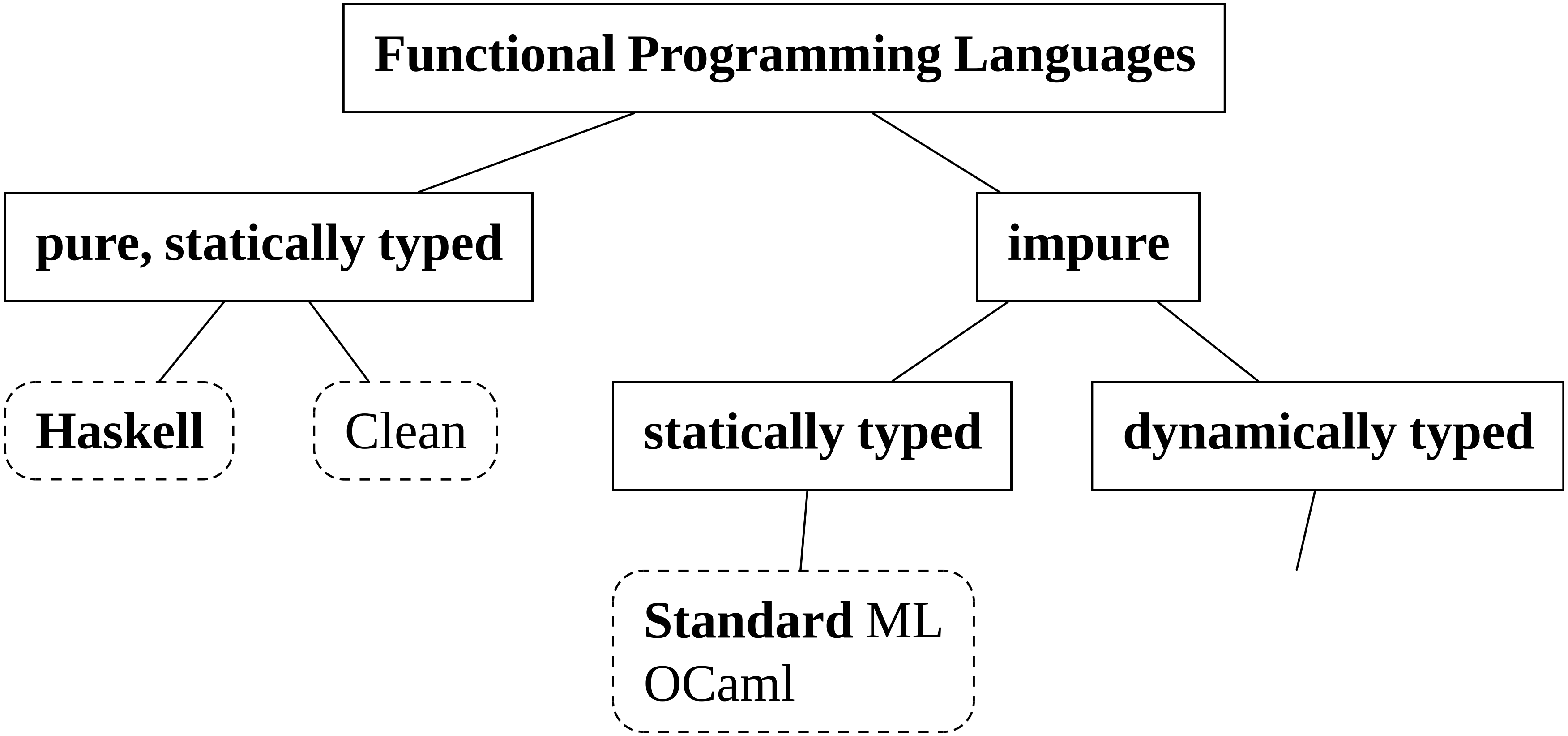

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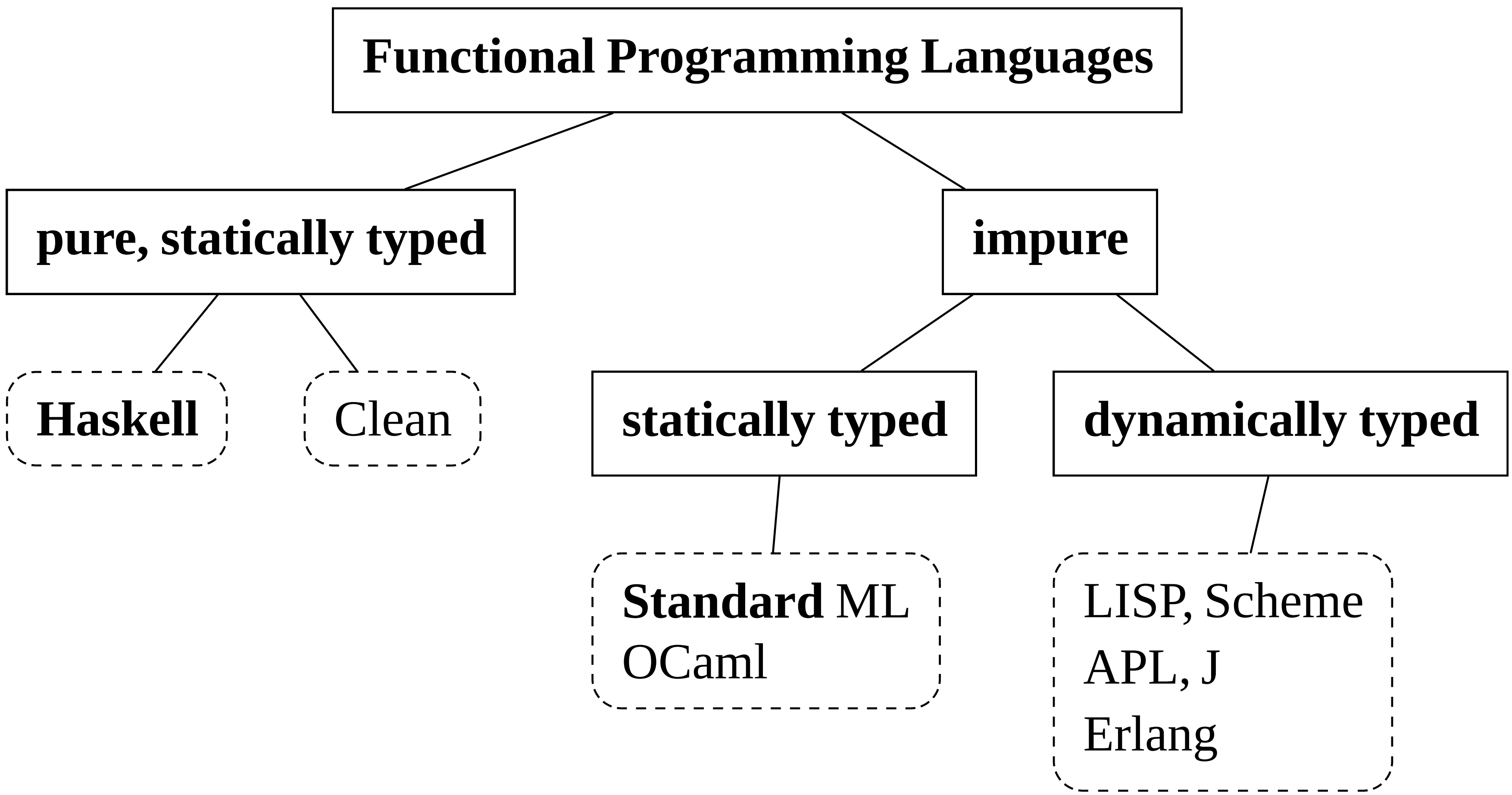

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  In many cases, advanced Haskell programmers will use other datastructures, for example *Set*s, or *FiniteMap*s instead of association lists.

Simple Expression Evaluation

The Haskell interpreters `hugs`, `ghci`, and `hi` accept any expression at their prompt and print (after the first ENTER) the value resulting from *evaluation* of that expression.

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Prelude> 4*(5+6)-2
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Expression evaluation proceeds by applying rules to subexpressions:

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  - **Standard arithmetic operators** are **strict in both arguments**: $0 * \mathit{undefined} = \mathit{undefined}$

Unfolding Definitions

Assume the following definitions to be in scope:

```
answer = 42
magic = 7
```

Expression evaluation will **unfold** (or **expand**) definitions:

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 = 41 * (7 * 42 - 23)                        (magic)
 = 41 * (294 - 23)                           (multiplication)
 = 41 * 271                                  (subtraction)
 = 11111                                     (multiplication)
```

How did I find those numbers?

Easy!

```
Prelude> [ n | n <- [1 .. 400] , 11111 ‘mod‘ n == 0 ]
[1,41,271]
```

This is a **list comprehension**:

- return all `n`
- where `n` is taken from then list `[1 .. 400]`
- and a result is returned only if `n` divides `11111`.

Conditional Expressions

Prelude> **if** 11111 ‘*mod*‘ 41 == 0 **then** 11111 ‘*div*‘ 41 **else** 5
271

The pattern is:

if *condition* **then** *expression1* **else** *expression2*

– If the condition evaluates to **True**, the conditional expression evaluates to the value of *expression1*.

– If the condition evaluates to **False**, the conditional expression evaluates to the value of *expression2*.

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Therefore: **“if _ then _ else”** is **strict in the condition**.

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Therefore: **“if _ then _ else”** is **strict in the condition**.

In C: (*condition* ? *expression1* : *expression2*)

Expanding Function Definitions

```
fact :: Integer -> Integer
fact n = if n == 0 then 1 else n * fact (n-1)
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```

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 = 3 * 2 * 1 * if True then 1 else 0 * fact (0-1)
 = 3 * 2 * 1 * 1
```

Expanding Function Definitions

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 = 3 * 2 * if (2-1) == 0 then 1 else (2-1) * fact ((2-1)-1)
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 = 3 * 2 * 1 * fact (1-1)
 = 3 * 2 * 1 * if (1-1) == 0 then 1 else (1-1) * fact ((1-1)-1)
 = 3 * 2 * 1 * if 0 == 0 then 1 else 0 * fact (0-1)
 = 3 * 2 * 1 * if True then 1 else 0 * fact (0-1)
 = 3 * 2 * 1 * 1
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```

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 = 3 * 2 * 1 * 1
 = 3 * 2 * 1
 = 3 * 2
```

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 = 3 * 2 * 1 * if 0 == 0 then 1 else 0 * fact (0-1)
 = 3 * 2 * 1 * if True then 1 else 0 * fact (0-1)
 = 3 * 2 * 1 * 1
 = 3 * 2 * 1
 = 3 * 2
 = 6
```

Matching Function Definitions

```
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fact 0 = 1
fact n = n * fact (n-1)
```

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```

```
fact 3
```

Matching Function Definitions

```
fact :: Integer -> Integer
fact 0 = 1
fact n = n * fact (n-1)
```

```
   fact 3
 = 3 * fact (3-1)                (fact n)
```

Matching Function Definitions

```
fact :: Integer -> Integer
fact 0 = 1
fact n = n * fact (n-1)
```

```
   fact 3
 = 3 * fact (3-1)
 = 3 * fact 2
```

`fact 3`	
`= 3 * fact (3-1)`	(`fact n`)
`= 3 * fact 2`	(determining which `fact` rule matches)

Matching Function Definitions

```
fact :: Integer -> Integer
fact 0 = 1
fact n = n * fact (n-1)
```

```
   fact 3
 = 3 * fact (3-1)              (fact n)
 = 3 * fact 2                  (determining which fact rule matches)
 = 3 * (2 * fact (2-1))        (fact n)
```

Matching Function Definitions

```
fact :: Integer -> Integer
fact 0 = 1
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```

`  fact 3`	
`= 3 * fact (3-1)`	(`fact n`)
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`= 3 * (2 * fact (2-1))`	(`fact n`)
`= 3 * (2 * fact 1)`	(determining which `fact` rule matches)

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`= 3 * fact (3-1)`	(`fact n`)
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`= 3 * (2 * (1 * fact (1-1)))`	(`fact n`)

Matching Function Definitions

```
fact :: Integer -> Integer
fact 0 = 1
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`= 3 * (2 * fact 1)`	(determining which `fact` rule matches)
`= 3 * (2 * (1 * fact (1-1)))`	(`fact n`)
`= 3 * (2 * (1 * fact 0))`	(determining which `fact` rule matches)
`= 3 * (2 * (1 * 1))`	(`fact 0`)

Matching Function Definitions

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fact 0 = 1
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`= 3 * fact (3-1)`	(`fact n`)
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`= 3 * (2 * fact (2-1))`	(`fact n`)
`= 3 * (2 * fact 1)`	(determining which `fact` rule matches)
`= 3 * (2 * (1 * fact (1-1)))`	(`fact n`)
`= 3 * (2 * (1 * fact 0))`	(determining which `fact` rule matches)
`= 3 * (2 * (1 * 1))`	(`fact 0`)
`= 3 * (2 * 1)`	(multiplication)

Matching Function Definitions

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fact :: Integer -> Integer
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 = 3 * fact (3-1)                    (fact n)
 = 3 * fact 2                        (determining which fact rule matches)
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 = 3 * (2 * 1)                       (multiplication)
 = 3 * 2                             (multiplication)
```

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fact :: Integer -> Integer
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= 3 * (2 * fact 1)                 (determining which fact rule matches)
= 3 * (2 * (1 * fact (1-1)))       (fact n)
= 3 * (2 * (1 * fact 0))           (determining which fact rule matches)
= 3 * (2 * (1 * 1))                (fact 0)
= 3 * (2 * 1)                      (multiplication)
= 3 * 2                            (multiplication)
= 6                                (multiplication)
```

Lists

- **List display:** between square brackets explicitly listing all elements, separated by commas:

```
[1,4,9,16,25]
```

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```

- **Enumeration lists:** denoted by ellipsis "`..`" inside square brackets; defined by beginning (and end, if applicable):

```
[1 .. 10]    = [1,2,3,4,5,6,7,8,9,10]
[1,3 .. 10] = [1,3,5,7,9]
[1,3 .. 11] = [1,3,5,7,9,11]
[11,9 .. 1] = [11,9,7,5,3,1]
[11 .. 1]    = []
[1 .. ]      = [1,2,3,4,5,6,7,8,9,10, …]  -- infinite list
[1,3 .. ]    = [1,3,5,7,9,11, …]           -- infinite list
```

List Construction

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`3 : []` = `[3]`

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```
3 : []        =        [3]
2 : [3]
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3 : []     =     [3]
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3 : []          =          [3]
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```

As an infix operator, ":" *associates to the right*

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  Reason: `1` and `[2]` cannot be members of the same list (*type error*).

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– Haskell code fragments will frequently be presented like above in a form that is more readable than plain typewriter text — in that case, the "comes from" arrow "`<-`" in generators turns into "$\leftarrow$"

The Type Language

Haskell has a full-fledged **type language**, with

- Simple predefined datatypes: `Bool`, `Char`, `Integer`, …
- Predefined **type constructors**: lists, tuples, functions, …
- Type synonyms
- User-defined datatypes and type constructors
- Type variables — to express **parametric polymorphism**
- …

Simple Predefined Datatypes

`Bool`	truth values	False, True
`Char`	"Unicode" characters	(in GHC: ISO-10646)
`Integer`	integers	**arbitrary precision**
`Int`	"machine integers"	$\geq$ 32 bits
`Float`	real floating point	single precision
`Double`	real floating point	double precision
`Complex Float`	complex floating point	single precision
`Complex Double`	complex floating point	double precision

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- `(limit, ("???", 'X')) :: ???`

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If t and u are types, then the **product type** (t, u) is the type of **pairs** with first component of type t and second component of type u (mathematically: $t \times u$).

Examples:

- `(answer, limit) :: (Integer, Int)`
- `(limit, answer) :: (Int, Integer)`
- `("???", answer) :: ([Char], Integer)`
- `("???", (limit, answer)) :: ([Char], (Int, Integer))`
- `("???", 'X') :: ([Char], Char)`
- `(limit, ("???", 'X')) :: (Int, ([Char], Char))`

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- `("???", answer) :: ([Char], Integer)`
- `("???", (limit, answer)) :: ([Char], (Int, Integer))`
- `("???", 'X') :: ([Char], Char)`
- `(limit, ("???", 'X')) :: (Int, ([Char], Char))`
- `(True, [("X",limit),("Y",5)]) :: ???`

Product Types (Pairs)

If t and u are types, then the **product type** (t, u) is the type of **pairs** with first component of type t and second component of type u (mathematically: $t \times u$).

Examples:

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- `("???", answer) :: ([Char], Integer)`
- `("???", (limit, answer)) :: ([Char], (Int, Integer))`
- `("???", 'X') :: ([Char], Char)`
- `(limit, ("???", 'X')) :: (Int, ([Char], Char))`
- `(True, [("X",limit),("Y",5)]) :: (Bool, [([Char], Int)])`

Tuple Types

If $n \neq 1$ is a natural number and $t_1, \ldots, t_n$ are types, then the **tuple type** `(` t_1 `,` $\ldots$ `,` t_n `)` is the type of n-**tuples** with the ith component of type t_i.

Examples:

- `(answer, ’c’, limit) :: (Integer, Char, Int)`
- `(answer, ’c’, limit, "all") :: (Integer, Char, Int, [Char])`
- `() :: ()`

  — there is exactly one **zero-tuple**.

  The type `()` of zero-tuples is also called the **unit type**.

Simple Type Synonyms

If t is a type not containing any type variables, and *Name* is an identifier with a capital first letter, then

`type` *Name* `=` t

defines *Name* as a **type synonym** for t, i.e., *Name* can now be used interchangeably with t.

Examples:

```
type String = [Char]                        -- predefined

type Point = (Double, Double)               -- (1.5, 2.7)

type Triangle = (Point, Point, Point)

type CharEntity = (Char, String)            -- ('Ã¼', "&uuml;")

type Dictionary = [(String,String)] -- [("day","jour")]
```

Type Variables and Polymorphic Types

- Identifiers with lower-case first letter can be used as type variables.
- Type variables can be used like other types in the construction of types, e.g.:

```
[(a,b)]
(Bool, (a, Int))
[ ( String, [(key, val)] ) ]
```

- A type containing at least one type variable is called **polymorphic**
- Polymorphic types can be instantiated by instantiating type variables with types, e.g.:

```
[(a,b)]    ⇒    [(Char,b)]
[(a,b)]    ⇒    [(Char,Int)]
[(a,b)]    ⇒    [(a,[(String,Int)])]
[(a,b)]    ⇒    [(a,[(String,c)])]
```

Typing of List Construction

- The empty list can be used at any list type: `[] :: [a]`
- If an element `x :: a` and a list `xs :: [a]` are given, then

```
(x : xs) :: [a]
```

Examples:

```
2                                   :: Int
[]                                  :: [Int]
[2] = 2 : []                        :: [Int]
[[3,4,5], [6,7]]                    :: [[Int]]
[2] : [[3,4,5], [6,7]]              :: [[Int]]
1 : ([2] : [[3,4,5], [6,7]])
```
– – ***cannot be typed!***

Function Types and Function Application

If t and u are types, then the **function type** t`->`u is the type of all **functions** accepting arguments of type t and producing results of type u (mathematically: $t \rightarrow u$).

Then:

- If a function `f :: a -> b` and an argument `x :: a` are given, then we have `(f x) :: b`.
- If a function `f :: a -> b` is given and we know that `(f x) :: b`, then the argument `x` is used at type `a`.
- If an argument `x :: a` is given and we know that `(f x) :: b`, then the function `f` is used at type `a -> b`.

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x

fst ('c', False)
```

Type Inference Examples

```
fst :: (a,b) -> a
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fst :: (a,b) -> a
fst (x,y) = x

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["hello", fst (x, 17)]
```

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x
```

```
fst ('c', False)                :: Char
```

```
["hello", fst (x, 17)]     ⇒     x :: String
```

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x

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["hello", fst (x, 17)]      ⇒     x :: String

f p = limit + fst p
```

Type Inference Examples

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fst (x,y) = x

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["hello", fst (x, 17)]       ⇒    x :: String

f p = limit + fst p          ⇒    p :: (Int,a)
```

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x

fst ('c', False)                          :: Char

["hello", fst (x, 17)]          ⇒          x :: String

f p = limit + fst p             ⇒          p :: (Int,a)
                                           f :: (Int,a) -> Int
```

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x


fst ('c', False)                    :: Char


["hello", fst (x, 17)]      ⇒     x :: String


f p = limit + fst p         ⇒     p :: (Int,a)
                                  f :: (Int,a) -> Int


g h = fst (h "") : [limit]
```

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x

fst ('c', False)                    :: Char

["hello", fst (x, 17)]       ⇒     x :: String

f p = limit + fst p          ⇒     p :: (Int,a)
                                   f :: (Int,a) -> Int

g h = fst (h "") : [limit]
    ⇒      h ::  String -> (Int,a)
```

Let’s Play the Evaluation Game Again — 1

```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Let’s Play the Evaluation Game Again — 1

```
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```

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```
g h1
```

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```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Then:

```
g h1
= fst (h1 "") : [limit]
```

Let’s Play the Evaluation Game Again — 1

```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Then:

```
g h1
= fst (h1 "") : [limit]
= fst (length "", ’ ’ : "") : [limit]
```

Let’s Play the Evaluation Game Again — 1

```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Then:

```
g h1
= fst (h1 "") : [limit]
= fst (length "", ’ ’ : "") : [limit]
= length "" : [limit]
```

Let’s Play the Evaluation Game Again — 1

```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Then:

```
g h1
= fst (h1 "") : [limit]
= fst (length "", ’ ’ : "") : [limit]
= length "" : [limit]
= 0 : [limit]
```

Let’s Play the Evaluation Game Again — 1

```
h1 :: String -> (Int, String)
h1 str = (length str, ’ ’ : str)

g h = fst (h "") : [limit]
```

Then:

```
g h1
= fst (h1 "") : [limit]
= fst (length "", ’ ’ : "") : [limit]
= length "" : [limit]
= 0 : [limit]
= [0, 100]
```

Let’s Play the Evaluation Game Again — 2

```
h2 :: String -> (Int, Char)
h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Let’s Play the Evaluation Game Again — 2

```
h2 :: String -> (Int, Char)
h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
```

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notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
= fst (h2 "") : [limit]
```

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```
h2 :: String -> (Int, Char)
h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
= fst (h2 "") : [limit]
= fst (sum (map ord (notOccCaps "")), head "") : [limit]
```

Let’s Play the Evaluation Game Again — 2

```
h2 :: String -> (Int, Char)
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```

Then:

```
g h2
= fst (h2 "") : [limit]
= fst (sum (map ord (notOccCaps "")), head "") : [limit]
= sum (map ord (notOccCaps "")) : [limit]
```

Let’s Play the Evaluation Game Again — 2

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h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
= fst (h2 "") : [limit]
= fst (sum (map ord (notOccCaps "")), head "") : [limit]
= sum (map ord (notOccCaps "")) : [limit]
= …
```

Let’s Play the Evaluation Game Again — 2

```
h2 :: String -> (Int, Char)
h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
= fst (h2 "") : [limit]
= fst (sum (map ord (notOccCaps "")), head "") : [limit]
= sum (map ord (notOccCaps "")) : [limit]
= …
= 2015 : [limit]
```

Let’s Play the Evaluation Game Again — 2

```
h2 :: String -> (Int, Char)
h2 str = (sum (map ord (notOccCaps str)), head str)

notOccCaps :: String -> String
notOccCaps str = filter (‘notElem‘ str) [’A’ .. ’Z’]

g h = fst (h "") : [limit]
```

Then:

```
g h2
= fst (h2 "") : [limit]
= fst (sum (map ord (notOccCaps "")), head "") : [limit]
= sum (map ord (notOccCaps "")) : [limit]
= …
= 2015 : [limit]
= [2015, 100]
```

Higher-Order Functions

```
g h = fst (h "") : [limit]
```

Higher-Order Functions

```
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Functional Programming: **Functions are first-class citizens**

Higher-Order Functions

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Functional Programming: Functions are first-class citizens

- Functions can be **arguments of other functions**: `g h2`
- Functions can be **components of data structures**: `(7,h1)`, `[h1, h2]`

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Functional Programming: Functions are first-class citizens

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- Functions can be **results of function application**: `succ . succ`

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A **first-order function** accepts only non-functional values as arguments.

A **higher-order function** expects functions as arguments.

Higher-Order Functions

```
g h = fst (h "") : [limit]
```

Functional Programming: **Functions are first-class citizens**

- Functions can be **arguments of other functions**: `g h2`
- Functions can be **components of data structures**: `(7,h1)`, `[h1, h2]`
- Functions can be **results of function application**: `succ . succ`

A **first-order function** accepts only non-functional values as arguments.

A **higher-order function** expects functions as arguments.

`g` is a second-order function: it expects first-order functions like `h1`, `h2` as arguments.

Type Inference Examples

```
fst :: (a,b) -> a
fst (x,y) = x

fst ('c', False)                    :: Char

["hello", fst (x, 17)]      ⇒    x :: String

f p = limit + fst p         ⇒    p :: (Int,a)
                                 f :: (Int,a) -> Int

g h = fst (h "") : [limit]
     ⇒      h ::  String -> (Int,a)
```

Type Inference Examples

```
fst :: (a,b) -> a
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["hello", fst (x, 17)]      ⇒     x :: String

f p = limit + fst p         ⇒     p :: (Int,a)
                                  f :: (Int,a) -> Int

g h = fst (h "") : [limit]
    ⇒      h ::  String -> (Int,a)
           g :: (String -> (Int,a)) -> [Int]
```

Curried Functions

- **Function application associates to the left**, i.e.,

```
f x y  =  (f x) y
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- Multi-argument functions in Haskell are typically defined as **curried** function, i.e., "they accept their arguments one at a time":

```
cylVol r h = (pi :: Double) * r * r * h
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  Since the right-hand side, `r`, and `h` obviously all have type `Double`, we have;

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(cylVol r) :: ???
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  cylVol r h = (pi :: Double) * r * r * h
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  (cylVol r) :: Double -> Double
  ```

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cylVol     :: ???
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Since the right-hand side, `r`, and `h` obviously all have type `Double`, we have;

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(cylVol r) :: Double -> Double
cylVol     :: Double -> (Double -> Double)
```

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- Multi-argument functions in Haskell are typically defined as **curried** function, i.e., “they accept their arguments one at a time”:

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  Since the right-hand side, `r`, and `h` obviously all have type `Double`, we have;

```
(cylVol r) :: Double -> Double
cylVol     :: Double -> (Double -> Double)
```

- **Function type construction associates to the right**, i.e.,

```
a -> b -> c  =  a -> (b -> c)
```

“Partial Application”

Let values with the following types be given:

$f \ :: \ a \rightarrow b \rightarrow c$

$x \ :: \ a$

$y \ :: \ b$

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Let values with the following types be given:

$f :: a \to b \to c$

$x :: a$

$y :: b$

The type of f is the function type $a \to (b \to c)$

“Partial Application”

Let values with the following types be given:

$f :: a \to b \to c$
$x :: a$
$y :: b$

The type of f is the function type $a \to (b \to c)$, with

- argument type a,
- result type $b \to c$.

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Let values with the following types be given:

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- argument type a,
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Therefore, we can apply f to x and obtain:

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The application of a “two-argument function” to a single argument is a “one-argument function”

"Partial Application"

Let values with the following types be given:

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$x :: a$

$y :: b$

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- argument type a,
- result type $b \to c$.

Therefore, we can apply f to x and obtain:

$(f\ x) :: b \to c$

The application of a "two-argument function" to a single argument is a "one-argument function", which can then be applied to a second argument:

$(f\ x)\ y :: c \qquad = \quad f\ x\ y$

Partial Application — Example

$$g :: (String \rightarrow (Int, a)) \rightarrow [Int]$$

$$g\ h = fst\ (h\ "")\ :\ [limit]$$

Partial Application — Example

$$g \;::\; (\mathit{String} \rightarrow (\mathit{Int}\,,\; a)) \rightarrow [\mathit{Int}\,]$$
$$g\;\; h \;=\; \mathit{fst}\;\; (h\;\text{""}) \;:\; [\mathit{limit}\,]$$

$$k\;\; n\;\; \mathit{str} \;=\; (n \;*\; (\mathit{length}\;\; \mathit{str} \;+\; 1)\,,\; \mathit{unwords}\;\; (\mathit{replicate}\;\; n\;\; \mathit{str}\,))$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "")\ :\ [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

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$$g :: (String \rightarrow (Int, a)) \rightarrow [Int]$$
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$$g\ (k\ 3)$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \to String \to (Int, String)$$
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$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "") : [limit]$$

Partial Application — Example

$$g :: (String \rightarrow (Int, a)) \rightarrow [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \rightarrow String \rightarrow (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "") : [limit]$$
$$= fst\ (3 * (length\ "" + 1), unwords\ (replicate\ 3\ "")) : [limit]$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "") : [limit]$$
$$= fst\ (3 * (length\ "" + 1), unwords\ (replicate\ 3\ "")) : [limit]$$
$$= (3 * (length\ "" + 1)) : [limit]$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "")\ :\ [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "")\ :\ [limit]$$
$$= fst\ (3 * (length\ "" + 1), unwords\ (replicate\ 3\ ""))\ :\ [limit]$$
$$= (3 * (length\ "" + 1))\ :\ [limit]$$
$$= (3 * (0 + 1))\ :\ [limit]$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ 3\ str))$$

$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "") : [limit]$$
$$= fst\ (3 * (length\ "" + 1), unwords\ (replicate\ n\ "")) : [limit]$$
$$= (3 * (length\ "" + 1)) : [limit]$$
$$= (3 * (0 + 1)) : [limit]$$
$$= (3 * 1) : [limit]$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

$$g\ (k\ 3)$$
$$= fst\ (k\ 3\ "") : [limit]$$
$$= fst\ (3 * (length\ "" + 1), unwords\ (replicate\ 3\ "")) : [limit]$$
$$= (3 * (length\ "" + 1)) : [limit]$$
$$= (3 * (0 + 1)) : [limit]$$
$$= (3 * 1) : [limit]$$
$$= 3 : [limit]$$

Partial Application — Example

$$g :: (String \to (Int, a)) \to [Int]$$
$$g\ h = fst\ (h\ "") : [limit]$$

$$k :: Int \to String \to (Int, String)$$
$$k\ n\ str = (n * (length\ str + 1), unwords\ (replicate\ n\ str))$$

$$
\begin{aligned}
& g\ (k\ 3) \\
= {} & fst\ (k\ 3\ "") : [limit] \\
= {} & fst\ (3 * (length\ "" + 1), unwords\ (replicate\ 3\ "")) : [limit] \\
= {} & (3 * (length\ "" + 1)) : [limit] \\
= {} & (3 * (0 + 1)) : [limit] \\
= {} & (3 * 1) : [limit] \\
= {} & 3 : [limit] \\
= {} & [3, 100]
\end{aligned}
$$

Operations on Functions

```
id :: a -> a                                      -- identity function
id x = x

(.) :: (b -> c) -> (a -> b) -> (a -> c)           -- function composition
(f . g) x  =  f (g x)

flip :: (a -> b -> c) -> (b -> a -> c)            -- argument swapping
flip f x y  =  f y x

curry :: ((a,b) -> c) -> (a -> b -> c)            -- currying
curry g x y = g (x,y)

uncurry :: (a -> b -> c) -> ((a,b) -> c)
uncurry f (x,y) = f x y
```

Exercise (*necessary!*): Copy only the definitions to a sheet of paper, and then infer the types yourself!

Operator Sections

- Infix operators are turned into functions by surrounding them with parentheses:

```
(+) 2 3  =  2 + 3
```

- This is necessary in type declarations:

```
(+)  :: Int -> Int -> Int      -- not the "natural" type of (+)
(:)  ::  a  -> [a] -> [a]
(++) :: [a] -> [a] -> [a]
```

- It is also possible to supply only one argument (which has to be an atomic expression):

```
    (2   +) 3    =    2   + 3       =  (+ 3  ) 2
    (8,3 /) 2.5  =    8.3 / 2.5     =  (/ 2.5) 8.3
    (7   :) []   =    7   : []      =  (: [] ) 7
((2^17) :) (16:[]) = (2^17) : 16 : [] = (: (16:[])) (2^17)
```

Turning Functions into Infix Operators

Surrounding a function name by **backquotes** turns it into an infix operator.

Frequently used examples (not the “natural” types throughout):

```
div, mod, max, min :: Int -> Int -> Int
elem :: Int -> [Int] -> Bool

12 ‘div‘  7             =    1
12 ‘mod‘  7             =    5
12 ‘max‘  7             =   12
12 ‘min‘  7             =    7
12 ‘elem‘ [1 .. 10]     =  False
```

Defining Functions Over Lists by Pattern Matching

Some functions taking lists as arguments can be defined directly via **pattern matching**:

$$\mathit{null} \quad :: \; [\,a\,] \rightarrow \mathit{Bool}$$

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$$\mathit{head} \quad :: [a] \to a$$

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$$\mathit{tail} \quad :: [a] \rightarrow [a]$$

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&\mathit{tail}\ (x : xs) &&= xs
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$$

(*head* and *tail* are **partial functions** — both are undefined on the empty list.)

Defining Functions Over Lists by Structural Induction

Many functions taking lists as arguments can be defined via **structural induction**:

$length :: [a] \to Int$

$concat :: [[a]] \to [a]$

$product :: [Integer] \to Integer$

$(++) :: [a] \to [a] \to [a]$

$sum :: [Integer] \to Integer$

$(\text{`}elem\text{`}) :: Int \to [Int] \to Bool$

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(All these functions are in the standard prelude.)

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\mathit{product}\ [\,] & = & 0 \\
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$$
\begin{array}{lcl}
x\ \text{`}\mathit{elem}\text{`}\ [\,] & = & \mathbf{False} \\
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$$\begin{aligned} &x\ \text{`}\mathit{elem}\text{`}\ [\,] &&= \mathbf{False} \\ &x\ \text{`}\mathit{elem}\text{`}\ (y : ys) &&= x \equiv y \ \|\ x\ \text{`}\mathit{elem}\text{`}\ ys \end{aligned}$$

(All these functions are in the standard prelude.)

Guarded Definitions

$$
\begin{array}{llll}
sign\ x & \mid x > 0 & = & 1 \\
 & \mid x == 0 & = & 0 \\
 & \mid x < 0 & = & -1
\end{array}
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$$

$$\mathit{choose} :: \mathit{Ord}\ a \Rightarrow (a, b) \rightarrow (a, b) \rightarrow b$$

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\mathit{choose}\ (x, v)\ (y, w)} \\
\quad \mid x > y & = & v \\
\quad \mid x < y & = & w \\
\quad \mid \mathit{otherwise} & = & \mathit{error}\ \text{"I cannot decide!"}
\end{array}
$$

Guarded Definitions

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&sign\ x \quad | \ x > 0 \ = \ 1 \\
&\qquad\quad\ \ | \ x == 0 \ = \ 0 \\
&\qquad\quad\ \ | \ x < 0 \ = \ \text{-}1
\end{aligned}
$$

$$
\begin{aligned}
&choose :: Ord\ a \Rightarrow (a,b) \rightarrow (a,b) \rightarrow b \\
&choose\ (x,v)\ (y,w) \\
&\qquad |\ x > y \quad = \ v \\
&\qquad |\ x < y \quad = \ w \\
&\qquad |\ otherwise \ = \ error\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&take\_while\ p\ (x : xs)\ |\ p\ x \ = \ x : take\_while\ p\ xs \\
&take\_while\ p\ xs \qquad\qquad\quad\ = \ [\,]
\end{aligned}
$$

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\begin{aligned}
sign\ x \quad & \mid x > 0 \;=\; 1 \\
& \mid x == 0 \;=\; 0 \\
& \mid x < 0 \;=\; -1
\end{aligned}
$$

$$
\begin{aligned}
& choose :: Ord\ a \Rightarrow (a, b) \rightarrow (a, b) \rightarrow b \\
& choose\ (x, v)\ (y, w) \\
& \quad \mid x > y \;=\; v \\
& \quad \mid x < y \;=\; w \\
& \quad \mid otherwise \;=\; error\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
& take\_while\ p\ (x : xs) \mid p\ x \;=\; x : take\_while\ p\ xs \\
& take\_while\ p\ xs \;=\; [\,]
\end{aligned}
$$

$$take\_while\ (< 5)\ [1, 2, 3]$$

Guarded Definitions

$$\begin{aligned} sign\ x\ \ & \mid x > 0 = 1 \\ & \mid x == 0 = 0 \\ & \mid x < 0 = -1 \end{aligned}$$

$$\begin{aligned} &choose :: Ord\ a \Rightarrow (a,b) \rightarrow (a,b) \rightarrow b \\ &choose\ (x,v)\ (y,w) \\ &\quad \mid x > y \quad = v \\ &\quad \mid x < y \quad = w \\ &\quad \mid otherwise = error \text{ "I cannot decide!"} \end{aligned}$$

If no guard succeeds, the next pattern is tried:

$$\begin{aligned} &take\_while\ p\ (x : xs) \mid p\ x = x : take\_while\ p\ xs \\ &take\_while\ p\ xs \qquad\qquad\quad = [\,] \end{aligned}$$

$$\begin{aligned} &take\_while\ (<5)\ [1, 2, 3] \\ &= take\_while\ (<5)\ (1 : 2 : 3 : [\,]) \end{aligned}$$

Guarded Definitions

$$
\begin{array}{llll}
sign\ x & \mid x > 0 & = & 1 \\
 & \mid x == 0 & = & 0 \\
 & \mid x < 0 & = & -1
\end{array}
$$

$$
\begin{array}{l}
choose :: Ord\ a \Rightarrow (a, b) \rightarrow (a, b) \rightarrow b \\
choose\ (x, v)\ (y, w) \\
\quad \mid x > y \quad = v \\
\quad \mid x < y \quad = w \\
\quad \mid otherwise = error\ \text{"I cannot decide!"}
\end{array}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{array}{ll}
take\_while\ p\ (x : xs) \mid p\ x & = x : take\_while\ p\ xs \\
take\_while\ p\ xs & = [\,]
\end{array}
$$

$$
\begin{array}{l}
take\_while\ (< 5)\ [1, 2, 3] \\
= take\_while\ (< 5)\ (1 : 2 : 3 : [\,]) \\
= 1 : take\_while\ (< 5)\ (2 : 3 : [\,])
\end{array}
$$

Guarded Definitions

$$
\begin{aligned}
\mathit{sign}\ x\ &|\ x > 0 = 1\\
&|\ x == 0 = 0\\
&|\ x < 0 = -1
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{choose} :: \mathit{Ord}\ a \Rightarrow (a, b) \rightarrow (a, b) \rightarrow b\\
&\mathit{choose}\ (x, v)\ (y, w)\\
&\quad |\ x > y = v\\
&\quad |\ x < y = w\\
&\quad |\ \mathit{otherwise} = \mathit{error}\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&\mathit{take\_while}\ p\ (x : xs)\ |\ p\ x = x : \mathit{take\_while}\ p\ xs\\
&\mathit{take\_while}\ p\ xs = [\,]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{take\_while}\ (< 5)\ [1, 2, 3]\\
&= \mathit{take\_while}\ (< 5)\ (1 : 2 : 3 : [\,])\\
&= 1 : \mathit{take\_while}\ (< 5)\ (2 : 3 : [\,])\\
&= 1 : 2 : \mathit{take\_while}\ (< 5)\ (3 : [\,])
\end{aligned}
$$

Guarded Definitions

$$
\begin{aligned}
sign\ x\ \ & \mid x > 0 = 1 \\
& \mid x == 0 = 0 \\
& \mid x < 0 = -1
\end{aligned}
$$

$$
\begin{aligned}
& choose :: Ord\ a \Rightarrow (a, b) \rightarrow (a, b) \rightarrow b \\
& choose\ (x, v)\ (y, w) \\
& \quad \mid x > y \quad = v \\
& \quad \mid x < y \quad = w \\
& \quad \mid otherwise = error\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
& take\_while\ p\ (x : xs) \mid p\ x = x : take\_while\ p\ xs \\
& take\_while\ p\ xs \qquad\qquad\quad = [\,]
\end{aligned}
$$

$$
\begin{aligned}
& take\_while\ (<5)\ [1, 2, 3] \\
& = take\_while\ (<5)\ (1 : 2 : 3 : [\,]) \\
& = 1 : take\_while\ (<5)\ (2 : 3 : [\,]) \\
& = 1 : 2 : take\_while\ (<5)\ (3 : [\,]) \\
& = 1 : 2 : 3 : take\_while\ (<5)\ [\,]
\end{aligned}
$$

Guarded Definitions

$$
\begin{aligned}
\mathit{sign}\ x\ \ &|\ x > 0 \ = \ 1\\
&|\ x == 0 \ = \ 0\\
&|\ x < 0 \ = \ \text{-}1
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{choose} :: \mathit{Ord}\ a \Rightarrow (a, b) \to (a, b) \to b\\
&\mathit{choose}\ (x, v)\ (y, w)\\
&\quad |\ x > y \quad = \ v\\
&\quad |\ x < y \quad = \ w\\
&\quad |\ \mathit{otherwise} \ = \ \mathit{error}\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&\mathit{take\_while}\ p\ (x : xs)\ |\ p\ x \ = \ x : \mathit{take\_while}\ p\ xs\\
&\mathit{take\_while}\ p\ xs \qquad\qquad\quad = \ [\,]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{take\_while}\ (<5)\ [1, 2, 3]\\
&= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : [\,])\\
&= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : [\,])\\
&= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : [\,])\\
&= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ [\,]\\
&= 1 : 2 : 3 : [\,]
\end{aligned}
$$

Guarded Definitions

$$
\begin{aligned}
sign\ x\ &|\ x > 0 = 1 \\
&|\ x == 0 = 0 \\
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$$

$$
\begin{aligned}
&choose :: Ord\ a \Rightarrow (a,b) \rightarrow (a,b) \rightarrow b \\
&choose\ (x,v)\ (y,w) \\
&\quad |\ x > y \quad = v \\
&\quad |\ x < y \quad = w \\
&\quad |\ otherwise = error\ \text{"I cannot decide!"}
\end{aligned}
$$

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&take\_while\ p\ (x : xs)\ |\ p\ x = x : take\_while\ p\ xs \\
&take\_while\ p\ xs \qquad\qquad\quad = [\,]
\end{aligned}
$$

$$
\begin{aligned}
&take\_while\ (< 5)\ [1, 2, 3] \\
&= take\_while\ (< 5)\ (1 : 2 : 3 : [\,]) \\
&= 1 : take\_while\ (< 5)\ (2 : 3 : [\,]) \\
&= 1 : 2 : take\_while\ (< 5)\ (3 : [\,]) \\
&= 1 : 2 : 3 : take\_while\ (< 5)\ [\,] \\
&= 1 : 2 : 3 : [\,] \\
&= [1, 2, 3]
\end{aligned}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{array}{lcl}
\mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\
\mathit{take\_while}\ p\ xs & = & [\,]
\end{array}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{array}{lcl} \mathit{take\_while}\ p\ (x : xs)\ \mid\ p\ x & = & x : \mathit{take\_while}\ p\ xs \\ \mathit{take\_while}\ p\ xs & = & [\,] \end{array}$$

$$\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6]$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{array}{lcl} \mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\ \mathit{take\_while}\ p\ xs & = & [\,] \end{array}$$

$$\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6]$$
$$= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,])$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{array}{lcl} \mathit{take\_while}\ p\ (x : xs)\ \mid\ p\ x & = & x : \mathit{take\_while}\ p\ xs \\ \mathit{take\_while}\ p\ xs & = & [\,] \end{array}$$

$$\begin{array}{l} \mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\ = \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ = 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \end{array}$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{aligned} &\mathit{take\_while}\ p\ (x : xs) \mid p\ x && = x : \mathit{take\_while}\ p\ xs \\ &\mathit{take\_while}\ p\ xs && = [\,] \end{aligned}$$

$$\begin{aligned} &\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\ &= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ &= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ &= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \end{aligned}$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{array}{lcl}
\mathit{take\_while}\ p\ (x : xs)\ |\ p\ x & = & x : \mathit{take\_while}\ p\ xs \\
\mathit{take\_while}\ p\ xs & = & [\,]
\end{array}
$$

$$
\begin{array}{l}
\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\
= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,])
\end{array}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{array}{lcl} \mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\ \mathit{take\_while}\ p\ xs & = & [\,] \end{array}$$

$$\begin{array}{l} \mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\ = \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ = 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ = 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ = 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\ = 1 : 2 : 3 : 2 : \mathit{take\_while}\ (<5)\ (3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \end{array}$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{aligned} &\mathit{take\_while}\ p\ (x : xs) \mid p\ x &&= x : \mathit{take\_while}\ p\ xs \\ &\mathit{take\_while}\ p\ xs &&= [\,] \end{aligned}$$

$$\begin{aligned} &\mathit{take\_while}\ (<5)\ [1,2,3,2,3,4,3,4,5,4,3,4,5,6] \\ &= \mathit{take\_while}\ (<5)\ (1:2:3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\ &= 1 : \mathit{take\_while}\ (<5)\ (2:3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\ &= 1:2 : \mathit{take\_while}\ (<5)\ (3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\ &= 1:2:3 : \mathit{take\_while}\ (<5)\ (2:3:4:3:4:5:4:3:4:5:6:[\,]) \\ &= 1:2:3:2 : \mathit{take\_while}\ (<5)\ (3:4:3:4:5:4:3:4:5:6:[\,]) \\ &= 1:2:3:2:3 : \mathit{take\_while}\ (<5)\ (4:3:4:5:4:3:4:5:6:[\,]) \end{aligned}$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&\mathit{take\_while}\ p\ (x : xs)\ |\ p\ x &&= x : \mathit{take\_while}\ p\ xs \\
&\mathit{take\_while}\ p\ xs &&= [\,]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{take\_while}\ (<5)\ [1,2,3,2,3,4,3,4,5,4,3,4,5,6] \\
&= \mathit{take\_while}\ (<5)\ (1:2:3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : \mathit{take\_while}\ (<5)\ (2:3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : 2 : \mathit{take\_while}\ (<5)\ (3:2:3:4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2:3:4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : 2 : 3 : 2 : \mathit{take\_while}\ (<5)\ (3:4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : 2 : 3 : 2 : 3 : \mathit{take\_while}\ (<5)\ (4:3:4:5:4:3:4:5:6:[\,]) \\
&= 1 : 2 : 3 : 2 : 3 : 4 : \mathit{take\_while}\ (<5)\ (3:4:5:4:3:4:5:6:[\,])
\end{aligned}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{aligned}
&\mathit{take\_while}\ p\ (x : xs) \mid p\ x &&= x : \mathit{take\_while}\ p\ xs \\
&\mathit{take\_while}\ p\ xs &&= [\,]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\
&= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : 3 : 2 : \mathit{take\_while}\ (<5)\ (3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : 3 : 2 : 3 : \mathit{take\_while}\ (<5)\ (4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : 3 : 2 : 3 : 4 : \mathit{take\_while}\ (<5)\ (3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
&= 1 : 2 : 3 : 2 : 3 : 4 : 3 : \mathit{take\_while}\ (<5)\ (4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,])
\end{aligned}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$\begin{array}{lcl} \mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\ \mathit{take\_while}\ p\ xs & = & [\,] \end{array}$$

$$\begin{array}{l}
\mathit{take_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\
= \mathit{take_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : \mathit{take_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : \mathit{take_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
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= 1 : 2 : 3 : 2 : \mathit{take_while}\ (<5)\ (3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : \mathit{take_while}\ (<5)\ (4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : \mathit{take_while}\ (<5)\ (3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : \mathit{take_while}\ (<5)\ (4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : \mathit{take_while}\ (<5)\ (5 : 4 : 3 : 4 : 5 : 6 : [\,])
\end{array}$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{array}{lcl}
\mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\
\mathit{take\_while}\ p\ xs & = & [\,]
\end{array}
$$

$$
\begin{array}{l}
\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\
= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : \mathit{take\_while}\ (<5)\ (3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : \mathit{take\_while}\ (<5)\ (4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : \mathit{take\_while}\ (<5)\ (3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : \mathit{take\_while}\ (<5)\ (4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : \mathit{take\_while}\ (<5)\ (5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : [\,]
\end{array}
$$

Guarded Definitions — Fall-Through

If no guard succeeds, the next pattern is tried:

$$
\begin{array}{lll}
\mathit{take\_while}\ p\ (x : xs) \mid p\ x & = & x : \mathit{take\_while}\ p\ xs \\
\mathit{take\_while}\ p\ xs & = & [\,]
\end{array}
$$

$$
\begin{array}{l}
\mathit{take\_while}\ (<5)\ [1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 3, 4, 5, 6] \\
= \mathit{take\_while}\ (<5)\ (1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : \mathit{take\_while}\ (<5)\ (2 : 3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : \mathit{take\_while}\ (<5)\ (3 : 2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : \mathit{take\_while}\ (<5)\ (2 : 3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : \mathit{take\_while}\ (<5)\ (3 : 4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : \mathit{take\_while}\ (<5)\ (4 : 3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : \mathit{take\_while}\ (<5)\ (3 : 4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : \mathit{take\_while}\ (<5)\ (4 : 5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : \mathit{take\_while}\ (<5)\ (5 : 4 : 3 : 4 : 5 : 6 : [\,]) \\
= 1 : 2 : 3 : 2 : 3 : 4 : 3 : 4 : [\,] \\
= [1, 2, 3, 2, 3, 4, 3, 4]
\end{array}
$$

`case` Expressions

```
sign x = case compare x 0 of
           GT ->  1
           EQ ->  0
           LT -> -1
```

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```
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The prelude datatype *Ordering* has three elements

data *Ordering* = *LT* | *EQ* | *GT*

`case` Expressions

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The prelude datatype *Ordering* has three elements and is used mostly as result type of the prelude function *compare*:

data *Ordering* = *LT* | *EQ* | *GT*

$compare :: Ord\ a \Rightarrow a \rightarrow a \rightarrow Ordering$

`case` Expressions

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sign x = case compare x 0 of
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```

The prelude datatype *Ordering* has three elements and is used mostly as result type of the prelude function *compare*:

data *Ordering* = *LT* | *EQ* | *GT*

$compare :: Ord\ a \Rightarrow a \rightarrow a \rightarrow Ordering$

Another example:

choose (*x*,*v*) (*y*,*w*) = **case** *compare x y* **of**
    GT $\rightarrow$ *v*
    LT $\rightarrow$ *w*
    EQ $\rightarrow$ *error* "I cannot decide!"

`if` … `then` … `else` … and `case` Expressions

The type *Bool* can be considered as a two-element enumeration type:

data *Bool* = **False** | **True**

`if` ... `then` ... `else` ... and `case` Expressions

The type *Bool* can be considered as a two-element enumeration type:

data *Bool* = **False** | **True**

Conditional expressions are "syntactic sugar" for **case** expressions over *Bool*:

if *condition* **then** *expr1* **else** *expr2*	$\equiv$	**case** *condition* **of** **True** $\rightarrow$ *expr1* **False** $\rightarrow$ *expr2*

`if` … `then` … `else` … **and** `case` **Expressions**

The type *Bool* can be considered as a two-element enumeration type:

data *Bool* = **False** | **True**

Conditional expressions are "syntactic sugar" for **case** expressions over *Bool*:

if *condition* **then** *expr1* **else** *expr2*	$\equiv$	**case** *condition* **of** **True** $\rightarrow$ *expr1* **False** $\rightarrow$ *expr2*

Two ways of defining functions:

Pattern Matching	`case`
not **True** = **False** *not* **False** = **True**	*not* *b* = **case** *b* **of** **True** $\rightarrow$ **False** **False** $\rightarrow$ **True**

`case` Expressions are "Anonymous" Pattern Matching

$$
\begin{array}{l}
\mathit{commaWords} :: [\mathit{String}] \to \mathit{String} \\
\mathit{commaWords}\ [\,] = [\,] \\
\mathit{commaWords}\ (x : \mathit{xs}) = x +\!\!+\ \textbf{case}\ \mathit{xs}\ \textbf{of} \\
\qquad\qquad\qquad\qquad\qquad\qquad [\,] \to [\,] \\
\qquad\qquad\qquad\qquad\qquad\qquad \_ \ \to \texttt{","} : \mathit{commaWords}\ \mathit{xs}
\end{array}
$$

`case` Expressions are "Anonymous" Pattern Matching

$$
\begin{aligned}
&\mathit{commaWords} :: [\mathit{String}] \to \mathit{String} \\
&\mathit{commaWords}\ [\,] = [\,] \\
&\mathit{commaWords}\ (x : xs) = x +\!\!+\ \textbf{case}\ xs\ \textbf{of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad [\,] \to [\,] \\
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`where` clauses are visible **only** within their enclosing clause, here "$\mathit{commaWords}\ (x : xs) = \ldots$"

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\end{array}
$$

`where` clauses are visible **only** within their enclosing clause, here "$\mathit{commaWords}\ (x : \mathit{xs}) = \ldots$"

`where` clauses are visible within all guards:

```
f x y | y >  z  =  …
      | y == z  =  …
      | y <  z  =  …
  where z = x * x
```

`let` Expressions

Local definitions can also be part of expressions:

```
f k n = let m = k ‘mod‘ n
        in if m == 0
           then n
           else f n m
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Definitions can use **pattern bindings**:

```
g k n = let (d,m) = divMod k n
        in if d == 0
           then [m]
           else g d n ++ [m]
```

`let` Expressions

Local definitions can also be part of expressions:

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Definitions can use **pattern bindings**:

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        in if d == 0
           then [m]
           else g d n ++ [m]
```

Guards, `let` and `where` bindings, and `case` cases all are **layout sensitive**!

`let` or `where`?

`let` **or** `where`**?**

- `let` *bindings* `in` *expression*

  is an **expression**

`let` **or** `where`**?**

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- *fname patterns guardedRHSs* `where` *bindings*

  is a clause that is part of a **definition**

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- (`where` clauses can also modify `case` cases)

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  is an **expression**

- *fname patterns guardedRHSs* `where` *bindings*

  is a clause that is part of a **definition**

- (`where` clauses can also modify `case` cases)

Frequently, the choice between `let` and `where` is a matter of *style*:

- `where` clauses result in a top-down presentation
- `let` expressions lend themselves also to bottom-up presentations

Some Prelude Functions — Elementary List Access

```
head                 :: [a] -> a
head (x:_)            = x

last                 :: [a] -> a
last [x]              = x
last (_:xs)           = last xs

tail                 :: [a] -> [a]
tail (_:xs)           = xs

init                 :: [a] -> [a]
init [x]              = []
init (x:xs)           = x : init xs

null                 :: [a] -> Bool
null []               = True
null (_:_)            = False
```

Some Prelude Functions — List Indexing

```
length                :: [a] -> Int
length                 = foldl' (\n _ -> n + 1) 0

(!!)                  :: [b] -> Int -> b
(x:_)  !! 0            = x
(_:xs) !! n | n>0      = xs !! (n-1)
(_:_)  !! _            = error "PreludeList.!!: negative index"
[]     !! _            = error "PreludeList.!!: index too large"
```

Some Prelude Functions — Positional List Splitting

```
take                    :: Int -> [a] -> [a]
take 0 _                 = []
take _ []                = []
take n (x:xs) | n>0      = x : take (n-1) xs
take _ _                 = error "take: negative argument"

drop                    :: Int -> [a] -> [a]
drop 0 xs                = xs
drop _ []                = []
drop n (_:xs) | n>0      = drop (n-1) xs
drop _ _                 = error "drop: negative argument"

splitAt                     :: Int -> [a] -> ([a], [a])
splitAt 0 xs                 = ([],xs)
splitAt _ []                 = ([],[])
splitAt n (x:xs) | n>0 = (x:xs’,xs”)
                 where (xs’,xs”) = splitAt (n-1) xs
splitAt _ _            = error "splitAt: negative argument"
```

Some Prelude Functions — Concatenation, Iteration

```
(++) :: [a] -> [a] -> [a]
[]     ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)

concat :: [[a]] -> [a]
concat = foldr (++) []

iterate                :: (a -> a) -> a -> [a]
iterate f x             = x : iterate f (f x)

repeat                 :: a -> [a]
repeat x                = xs  where  xs = x:xs
{– repeat x             = x : repeat x –}       – – for understanding

replicate              :: Int -> a -> [a]
replicate n x           = take n (repeat x)

cycle                  :: [a] -> [a]
cycle xs                = xs’  where  xs’ = xs ++ xs’
```

Separation of Concerns: Generation and Consumption

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```
replicate 3 '!'
```

Separation of Concerns: Generation and Consumption

```
replicate 3 '!'
= take 3 (repeat '!')                    -- replicate
```

Separation of Concerns: Generation and Consumption

```
replicate 3 '!'
= take 3 (repeat '!')                        -- replicate
= take 3 ('!' : repeat '!')                  -- repeat
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Separation of Concerns: Generation and Consumption

```
replicate 3 '!'
= take 3 (repeat '!')                    -- replicate
= take 3 ('!' : repeat '!')              -- repeat
= '!' : take (3 - 1) (repeat '!')        -- take (iii)
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= '!' : take 2 ('!' : repeat '!')                  -- repeat
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= '!' : '!' : '!' : []                                 -- take (i)
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= '!' : '!' : take 1 (repeat '!')                   -- subtraction
= '!' : '!' : take 1 ('!' : repeat '!')             -- repeat
= '!' : '!' : '!' : take (1 - 1) (repeat '!')       -- take (iii)
= '!' : '!' : '!' : take 0 (repeat '!')             -- subtraction
= '!' : '!' : '!' : []                              -- take (i)
= "!!!"
```

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- **Argument passing:** not by value or reference, but by name
- **Powerful datatypes** with simple interface: *Integer*, lists, lists of lists of …
- **Non-local control** (evaluation on demand): modularity (e.g., generate / prune)

Some Prelude Functions — List Splitting with Predicates

```
takeWhile          :: (a -> Bool) -> [a] -> [a]
takeWhile p []      = []
takeWhile p (x:xs)
      | p x         = x : takeWhile p xs
      | otherwise = []

dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p xs@(x:xs')
      | p x         = dropWhile p xs'
      | otherwise = xs

span, break        :: (a -> Bool) -> [a] -> ([a],[a])
span p []           = ([],[])
span p xs@(x:xs')
      | p x         = let (ys,zs) = span p xs' in (x:ys,zs)
      | otherwise = ([],xs)

break p             = span (not . p)
```

as-Patterns

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []       = []
dropWhile p xs@(x:xs’)
      | p x          = dropWhile p xs’
      | otherwise = xs
```

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```

Consider matching of the third clause against *dropWhile* (< 5) $[1,2,3]$:

as-Patterns

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dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []       = []
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```

Consider matching of the third clause against *dropWhile* (<5) $[1,2,3]$:

- $p =$
- $xs =$
- $x =$
- $xs' =$

as-Patterns

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
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Consider matching of the third clause against *dropWhile* (< 5) $[1,2,3]$:

- $p = (< 5)$
- $xs = [1,2,3]$
- $x = 1$
- $xs' =$

as-Patterns

```
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []     = []
dropWhile p xs@(x:xs')
      | p x        = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (< 5) $[1,2,3]$:

- $p = (< 5)$
- $xs = [1,2,3]$
- $x = 1$
- $xs' = [2,3]$

as-Patterns

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
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```

Consider matching of the third clause against $\mathit{dropWhile}\ (<5)\ [1,2,3]$:

- $p = (<5)$
- $xs = [1,2,3]$
- $x = 1$
- $xs' = [2,3]$
- $p\ x = (<5)\ 1 = 1 < 5 = \mathbf{True}$

as-Patterns

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dropWhile          :: (a -> Bool) -> [a] -> [a]
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```

Consider matching of the third clause against *dropWhile* (< 5) $[1,2,3]$:

- $p = (< 5)$
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- $x = 1$
- $xs' = [2,3]$
- $p\ x = (< 5)\ 1 = 1 < 5 =$ **True**

Therefore: *dropWhile* (< 5) $[1,2,3] =$

as-Patterns

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- $p = (<5)$
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- $x = 1$
- $xs' = [2,3]$
- $p\ x = (<5)\ 1 = 1 < 5 = \mathbf{True}$

Therefore: $\mathit{dropWhile}\ (<5)\ [1,2,3] = \mathit{dropWhile}\ (<5)\ [2,3]$

as-Patterns — 2

```
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p xs@(x:xs')
      | p x         = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* `(< 5) [5,4,3]`:

as-Patterns — 2

```
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p xs@(x:xs’)
      | p x         = dropWhile p xs’
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* `(< 5) [5,4,3]`:

- *p* =
- *xs* =
- *x* =
- *xs’* =

as-Patterns — 2

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []       = []
dropWhile p xs@(x:xs')
      | p x          = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (< 5) [5,4,3]:

- $p = (< 5)$
- $xs =$
- $x =$
- $xs' =$

as-Patterns — 2

```
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []     = []
dropWhile p xs@(x:xs')
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      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* `(< 5) [5,4,3]`:

- p = `(< 5)`
- xs = `[5,4,3]`
- x =
- xs' =

as-Patterns — 2

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p xs@(x:xs')
      | p x         = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (< 5) [5,4,3]:

- *p* = (< 5)
- *xs* = [5,4,3]
- *x* = 5
- *xs'* =

as-Patterns — 2

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []       = []
dropWhile p xs@(x:xs')
      | p x          = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* `(< 5) [5,4,3]`:

- *p* = `(< 5)`
- *xs* = `[5,4,3]`
- *x* = `5`
- *xs'* = `[4,3]`

as-Patterns — 2

```
dropWhile           :: (a -> Bool) -> [a] -> [a]
dropWhile p []       = []
dropWhile p xs@(x:xs')
      | p x          = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (<5) $[5,4,3]$:

- $p = (<5)$
- $xs = [5,4,3]$
- $x = 5$
- $xs' = [4,3]$
- $p\ x = (<5)\ 5 = 5 < 5 =$ **False**

as-Patterns — 2

```
dropWhile            :: (a -> Bool) -> [a] -> [a]
dropWhile p []        = []
dropWhile p xs@(x:xs')
      | p x           = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (< 5) $[5,4,3]$:

- $p = (< 5)$
- $xs = [5,4,3]$
- $x = 5$
- $xs' = [4,3]$
- $p\ x = (< 5)\ 5 = 5 < 5 =$ **False**

Therefore: *dropWhile* (< 5) $[5,4,3] =$

as-Patterns — 2

```
dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []     = []
dropWhile p xs@(x:xs')
      | p x        = dropWhile p xs'
      | otherwise = xs
```

Consider matching of the third clause against *dropWhile* (< 5) [5,4,3]:

- *p* = (< 5)
- *xs* = [5,4,3]
- *x* = 5
- *xs*' = [4,3]
- *p x* = (< 5) 5 = 5 < 5 = **False**

Therefore: *dropWhile* (< 5) [5,4,3] = [5,4,3]

Some Prelude Functions — List Splitting with Predicates

```
takeWhile          :: (a -> Bool) -> [a] -> [a]
takeWhile p []      = []
takeWhile p (x:xs)
      | p x         = x : takeWhile p xs
      | otherwise = []

dropWhile          :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p xs@(x:xs')
      | p x         = dropWhile p xs'
      | otherwise = xs

span, break        :: (a -> Bool) -> [a] -> ([a],[a])
span p []           = ([],[])
span p xs@(x:xs')
      | p x         = let (ys,zs) = span p xs' in (x:ys,zs)
      | otherwise = ([],xs)

break p             = span (not . p)
```

Some Prelude Functions — Text Processing

```
lines        :: String -> [String]
lines ""      = []
lines s       = let (l,s') = break ('\n'==) s
                in l : case s' of []      -> []
                                  (_:s'') -> lines s''

words        :: String -> [String]
words s       = case dropWhile isSpace s of
                     "" -> []
                     s' -> w : words s''
                           where (w,s'') = break isSpace s'

unlines      :: [String] -> String
unlines []        = []
unlines (l:ls)    = l ++ '\n' : unlines ls

unwords      :: [String] -> String
unwords []        =  ""
unwords [w]       = w
unwords (w:ws)    = w ++ ' ' : unwords ws
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a -> Bool) -> ([a] -> [a])
filter p [] = []
filter p (x : xs) = if p x then x : rest else rest
  where rest = filter p xs
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a -> Bool) -> ([a] -> [a])
filter p [] = []
filter p (x : xs) = if p x then x : rest else rest
  where rest = filter p xs
```

These functions could also be defined via list comprehension:

```
map    f xs = [ f x | x <- xs ]
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a -> Bool) -> ([a] -> [a])
filter p [] = []
filter p (x : xs) = if p x then x : rest else rest
  where rest = filter p xs
```

These functions could also be defined via list comprehension:

```
map    f xs = [ f x | x <- xs ]
filter p xs = [   x | x <- xs, p x ]
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a -> Bool) -> ([a] -> [a])
filter p [] = []
filter p (x : xs) = if p x then x : rest else rest
  where rest = filter p xs
```

These functions could also be defined via list comprehension:

```
map    f xs = [ f x | x <- xs ]
filter p xs = [   x | x <- xs, p x ]
```

Examples:

```
map   (7 *) [1 .. 6] = [7, 14, 21, 28, 35, 42]
```

`map` and `filter`

```
map :: (a -> b) -> ([a] -> [b])
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a -> Bool) -> ([a] -> [a])
filter p [] = []
filter p (x : xs) = if p x then x : rest else rest
  where rest = filter p xs
```

These functions could also be defined via list comprehension:

```
map    f xs = [ f x | x <- xs ]
filter p xs = [   x | x <- xs, p x ]
```

Examples:

```
map   (7 *) [1 .. 6] = [7, 14, 21, 28, 35, 42]
filter even [1 .. 6] = [2, 4, 6]
```

foldr1

```
foldr1              :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]      = x
foldr1 (⊗) (x:xs)   = x ⊗ (foldr1 (⊗) xs)
```

foldr1

```
foldr1              :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]        = x
foldr1 (⊗) (x:xs)     = x ⊗ (foldr1 (⊗) xs)
```

foldr1 (⊗) [x_1, x_2, x_3, x_4, x_5]

foldr1

```
foldr1                :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]        = x
foldr1 (⊗) (x:xs)     = x ⊗ (foldr1 (⊗) xs)
```

foldr1 (⊗) $[x_1, x_2, x_3, x_4, x_5]$

= $x_1 \otimes$ (foldr1 (⊗) $[x_2, x_3, x_4, x_5]$)

foldr1

```
foldr1               :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]       = x
foldr1 (⊗) (x:xs)    = x ⊗ (foldr1 (⊗) xs)
```

foldr1 (⊗) [x_1, x_2, x_3, x_4, x_5]

= x_1 ⊗ (foldr1 (⊗) [x_2, x_3, x_4, x_5])

= x_1 ⊗ (x_2 ⊗ (foldr1 (⊗) [x_3, x_4, x_5]))

foldr1

```
foldr1                  :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]          = x
foldr1 (⊗) (x:xs)       = x ⊗ (foldr1 (⊗) xs)
```

foldr1 $(\otimes)$ $[x_1, x_2, x_3, x_4, x_5]$

$= x_1 \otimes (\text{foldr1 } (\otimes)\ [x_2, x_3, x_4, x_5])$

$= x_1 \otimes (x_2 \otimes (\text{foldr1 } (\otimes)\ [x_3, x_4, x_5]))$

$= x_1 \otimes (x_2 \otimes (x_3 \otimes (\text{foldr1 } (\otimes)\ [x_4, x_5])))$

foldr1

```
foldr1                  :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]          = x
foldr1 (⊗) (x:xs)       = x ⊗ (foldr1 (⊗) xs)
```

foldr1 ($\otimes$) [x_1, x_2, x_3, x_4, x_5]

= x_1 $\otimes$ (foldr1 ($\otimes$) [x_2, x_3, x_4, x_5])

= x_1 $\otimes$ (x_2 $\otimes$ (foldr1 ($\otimes$) [x_3, x_4, x_5]))

= x_1 $\otimes$ (x_2 $\otimes$ (x_3 $\otimes$ (foldr1 ($\otimes$) [x_4, x_5])))

= x_1 $\otimes$ (x_2 $\otimes$ (x_3 $\otimes$ (x_4 $\otimes$ (foldr1 ($\otimes$) [x_5]))))

foldr1

```
foldr1                 :: (a -> a -> a) -> [a] -> a
foldr1 (⊗) [x]         = x
foldr1 (⊗) (x:xs)      = x ⊗ (foldr1 (⊗) xs)
```

foldr1 ($\otimes$) [x_1, x_2, x_3, x_4, x_5]

= $x_1 \otimes$ (foldr1 ($\otimes$) [x_2, x_3, x_4, x_5])

= $x_1 \otimes$ ($x_2 \otimes$ (foldr1 ($\otimes$) [x_3, x_4, x_5]))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ (foldr1 ($\otimes$) [x_4, x_5])))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ (foldr1 ($\otimes$) [x_5]))))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes x_5$)))

foldr

```
foldr                 :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []          = z
foldr (⊗) z (x:xs)      = x ⊗ (foldr (⊗) z xs)
```

foldrX

$$
\begin{aligned}
&\mathit{foldrX} && :: (a \to b \to b) \to b \to [a] \to b \\
&\mathit{foldrX}\ (***)\ z\ [\,] && = z \\
&\mathit{foldrX}\ (***)\ z\ (x : xs) && = x\ ***\ (\mathit{foldrX}\ (***)\ z\ xs)
\end{aligned}
$$

foldr

```
foldr                   :: (a -> b -> b) -> b -> [a] -> b
foldr ( ⊗ ) z []           = z
foldr ( ⊗ ) z (x:xs)  = x  ⊗  (foldr ( ⊗ ) z xs)
```

foldr ($\otimes$) z [x_1, x_2, x_3, x_4, x_5]

foldr

```
foldr                 :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []        = z
foldr (⊗) z (x:xs)    = x ⊗ (foldr (⊗) z xs)
```

foldr ($\otimes$) z [x_1, x_2, x_3, x_4, x_5]

= x_1 $\otimes$ (foldr ($\otimes$) z [x_2, x_3, x_4, x_5])

foldr

```
foldr                 :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []        = z
foldr (⊗) z (x:xs)    = x ⊗ (foldr (⊗) z xs)
```

foldr (⊗) z [x_1, x_2, x_3, x_4, x_5]

= x_1 ⊗ (foldr (⊗) z [x_2, x_3, x_4, x_5])

= x_1 ⊗ (x_2 ⊗ (foldr (⊗) z [x_3, x_4, x_5]))

foldr

```
foldr                  :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []         = z
foldr (⊗) z (x:xs)     = x ⊗ (foldr (⊗) z xs)
```

foldr ($\otimes$) z [x_1, x_2, x_3, x_4, x_5]

= $x_1 \otimes$ (foldr ($\otimes$) z [x_2, x_3, x_4, x_5])

= $x_1 \otimes$ ($x_2 \otimes$ (foldr ($\otimes$) z [x_3, x_4, x_5]))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ (foldr ($\otimes$) z [x_4, x_5])))

foldr

```
foldr                   :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []          = z
foldr (⊗) z (x:xs)      = x ⊗ (foldr (⊗) z xs)
```

foldr (⊗) z $[x_1, x_2, x_3, x_4, x_5]$

$= x_1 \otimes$ (foldr (⊗) z $[x_2, x_3, x_4, x_5]$)

$= x_1 \otimes (x_2 \otimes$ (foldr (⊗) z $[x_3, x_4, x_5]$))

$= x_1 \otimes (x_2 \otimes (x_3 \otimes$ (foldr (⊗) z $[x_4, x_5]$)))

$= x_1 \otimes (x_2 \otimes (x_3 \otimes (x_4 \otimes$ (foldr (⊗) z $[x_5]$))))

foldr

```
foldr                 :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []        = z
foldr (⊗) z (x:xs)    = x ⊗ (foldr (⊗) z xs)
```

foldr (⊗) z [x_1, x_2, x_3, x_4, x_5]

= $x_1 \otimes$ (foldr (⊗) z [x_2, x_3, x_4, x_5])

= $x_1 \otimes$ ($x_2 \otimes$ (foldr (⊗) z [x_3, x_4, x_5]))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ (foldr (⊗) z [x_4, x_5])))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ (foldr (⊗) z [x_5]))))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ ($x_5 \otimes$ (foldr (⊗) z []))))))

foldr

```
foldr                  :: (a -> b -> b) -> b -> [a] -> b
foldr (⊗) z []          = z
foldr (⊗) z (x:xs)      = x ⊗ (foldr (⊗) z xs)
```

foldr (⊗) z [x_1, x_2, x_3, x_4, x_5]

= $x_1 \otimes$ (foldr (⊗) z [x_2, x_3, x_4, x_5])

= $x_1 \otimes$ ($x_2 \otimes$ (foldr (⊗) z [x_3, x_4, x_5]))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ (foldr (⊗) z [x_4, x_5])))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ (foldr (⊗) z [x_5]))))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ ($x_5 \otimes$ (foldr (⊗) z [])))))

= $x_1 \otimes$ ($x_2 \otimes$ ($x_3 \otimes$ ($x_4 \otimes$ ($x_5 \otimes$ z))))

List Folding

```
foldr               :: (a -> b -> b) -> b -> [a] -> b
foldr f z []         = z
foldr f z (x:xs)     = f x (foldr f z xs)

foldr1              :: (a -> a -> a) -> [a] -> a
foldr1 f [x]         = x
foldr1 f (x:xs)      = f x (foldr1 f xs)

foldl               :: (a -> b -> a) -> a -> [b] -> a
foldl f z []         = z
foldl f z (x:xs)     = foldl f (f z x) xs

foldl1              :: (a -> a -> a) -> [a] -> a
foldl1 f (x:xs)      = foldl f x xs
```

Unfolding Definitions

A simple definition:

limit = 10 ^ 2

Unfolding Definitions

A simple definition:

$limit = 10 \wedge 2$

Expanding this definition:

$4 * (limit + 1)$

Unfolding Definitions

A simple definition:

$\mathit{limit} = 10 \wedge 2$

Expanding this definition:

$4 * (\mathit{limit} + 1)$
$= 4 * ((10 \wedge 2) + 1)$

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$limit = 10 \wedge 2$

Expanding this definition:

$4 * (limit + 1)$
$= 4 * ((10 \wedge 2) + 1)$
$= \ldots$

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A simple definition:

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Another definition:

$concat = foldr\ (+\!\!+)\ [\,]$

Unfolding Definitions

A simple definition:

$limit = 10 \wedge 2$

Expanding this definition:

$4 * (limit + 1)$
$= 4 * ((10 \wedge 2) + 1)$
$= \ldots$

Another definition:

$concat = foldr\ (++)\ [\,]$

Expanding this definition:

$concat\ [[1,2,3],[4,5]]$

Unfolding Definitions

A simple definition:

$limit = 10 \wedge 2$

Expanding this definition:

$4 * (limit + 1)$
$= 4 * ((10 \wedge 2) + 1)$
$= \ldots$

Another definition:

$concat = foldr\ (+\!\!+)\ [\,]$

Expanding this definition:

$concat\ [[1,2,3],[4,5]]$
$= (foldr\ (+\!\!+)\ [\,])\ [[1,2,3],[4,5]]$
$= \ldots$

Enumeration Type Definitions

$\textbf{data}\ \mathit{Bool} = \textbf{False} \mid \textbf{True} \qquad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}, \mathit{Read}, \mathit{Show})$

$\textbf{data}\ \mathit{Ordering} = \mathit{LT} \mid \mathit{EQ} \mid \mathit{GT} \qquad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}, \mathit{Read}, \mathit{Show})$

$\textbf{data}\ \mathit{Suit} = \mathit{Diamonds} \mid \mathit{Hearts} \mid \mathit{Spades} \mid \mathit{Clubs} \quad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord})$

Pattern matching:

$\mathit{not}\ \textbf{False} = \textbf{True}$
$\mathit{not}\ \textbf{True} = \textbf{False}$

$\mathit{lexicalCombineOrdering} :: \mathit{Ordering} \to \mathit{Ordering} \to \mathit{Ordering}$
$\mathit{lexicalCombineOrdering}\ \mathit{LT}\ \_ = \mathit{LT}$
$\mathit{lexicalCombineOrdering}\ \mathit{EQ}\ x = x$
$\mathit{lexicalCombineOrdering}\ \mathit{GT}\ \_ = \mathit{GT}$

Simple data Type Definitions

$\textbf{data}\ \mathit{Point} = \mathit{Pt}\ \mathit{Int}\ \mathit{Int}\ \ \textbf{deriving}\ (\mathit{Eq})$ —— screen coordinates

$\textbf{data}\ \mathit{Transport} = \mathit{Feet}$
$\quad\quad |\ \mathit{Bike}$
$\quad\quad |\ \mathit{Train}\ \mathit{Int}$ —— price in cent

This defines at the same time **data constructors**:

$\mathit{Pt} :: \mathit{Int} \rightarrow \mathit{Int} \rightarrow \mathit{Point}$

$\mathit{Feet} :: \mathit{Transport}$
$\mathit{Bike} :: \mathit{Transport}$
$\mathit{Train} :: \mathit{Int} \rightarrow \mathit{Transport}$

Pattern matching:

$\mathit{addPt}\ (\mathit{Pt}\ x1\ y1)\ (\mathit{Pt}\ x2\ y2) = \mathit{Pt}\ (x1 + x2)\ (y1 + y2)$

$\mathit{cost}\ \mathit{Feet} = 0$
$\mathit{cost}\ \mathit{Bike} = 0$
$\mathit{cost}\ (\mathit{Train}\ \mathit{Int}) = \mathit{Int}$

Simple Polymorphic data Type Definitions

The prelude **type constructors** *Maybe*, *Either*, *Complex* are defined as follows:

$\textbf{data}\ \mathit{Maybe}\ a = \mathit{Nothing} \mid \mathit{Just}\ a \qquad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}, \mathit{Read}, \mathit{Show})$
$\textbf{data}\ \mathit{Either}\ a\ b = \mathit{Left}\ a \mid \mathit{Right}\ b$

$\textbf{data}\ \mathit{Complex}\ r = r\ {:}{+}\ r\ \ \textbf{deriving}\ (\mathit{Eq}, \mathit{Read}, \mathit{Show})$

This defines at the same time **data constructors**:

$\mathit{Nothing} :: \mathit{Maybe}\ a$
$\mathit{Just} :: a \to \mathit{Maybe}\ a$

$\mathit{Left} :: a \to \mathit{Either}\ a\ b$
$\mathit{Right} :: b \to \mathit{Either}\ a\ b$

$({:}{+}) :: r \to r \to \mathit{Complex}\ r$