

# Computer Science 1FC3

## Lab 9 – Recurrence Relations (theory)

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Earlier we discussed how functions could be defined recursively. Recall that a recursive definition specifies one or more base cases and a recursive step. The rule for finding the terms that are generated from functions is called a recurrence relation.

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### DEFINITION OF RECURRENCE RELATION

A *sequence* is a non-empty set of ordered homogenous elements where  $a_n$  refers to the  $n^{\text{th}}$  term of a sequence. A *recurrence relation* for the sequence  $\{a_1, \dots, a_n\}$  is an equation that expresses  $a_n$  in terms of one or more of the previous terms (i.e.  $a_n = a_{n-1} + a_{n-2}$ ). A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

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### WORKING WITH RECURRENCE RELATIONS

Let  $a_n = a_{n-1} - a_{n-2}$  define a recurrence relation, where  $a_0 = 3$  and  $a_1 = 5$ . And suppose that we would like to find the values for  $a_2$  and  $a_3$ .

*Solution:*

$$a_2 = a_{2-1} - a_{2-2}$$

$$a_2 = a_1 - a_0$$

$$a_2 = 5 - 3$$

$$a_2 = 2$$

$$a_3 = a_{3-1} - a_{3-2}$$

$$a_3 = a_2 - a_1$$

$$a_3 = 2 - 5$$

$$a_3 = -3$$

Observe how when we substitute a value for  $n$  in the recurrence relation the relation becomes quite clear. The set of all elements generated in this fashion would constitute a sequence that is a solution to this recurrence relation.

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### VERIFYING SOLUTIONS

Define the recurrence relation  $a_n = 2a_{n-1} - a_{n-2}$  and set  $A = \{a_n \mid a_n = 3n\} = \{0, 1, 6, 9 \dots 3n\}$ . Is set  $A$  a solution to the recurrence relation?

*Solution:*

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(3(n-1)) - 3(n-2) \\ &= 6n - 6 - 3n + 6 \\ &= 3n \\ &= a_n \end{aligned}$$

So set A is a solution since it fulfills the criteria of the recurrence relation, however, this is not the only relation. Consider the set  $B = \{5, 5, 5, 5, 5, 5, \dots, 5\} = \{a_n \mid a_n = 5\}$  does this constitute a solution?

*Solution:*

$$\begin{aligned}
 a_n &= 2a_{n-1} - a_{n-2} \\
 &= 2(5) - 3(5) \\
 &= 10 - 5 \\
 &= 5 \\
 &= a_n
 \end{aligned}$$

As you see set B is also a solution.

## SETTING UP RECURRENCE RELATIONS

*and why everyone always talks about Fibonacci and his rabbits*

Leonardo di Pisa, also known as Fibonacci, was a thirteenth century mathematician who produced an interesting solution to a simple problem. Consider a young pair of rabbits (one of each sex) placed on an island. Rabbits can breed once they are two months old at which time they pair off to give birth to two other rabbits each month. How many rabbits will we have after  $n$  months, assuming no rabbits ever die.

*Solution:*

Let  $f_n$  be the number of pairs of rabbits after  $n$  months. Since the rabbits do not breed for the first two months we have  $f_1 = 1$  and  $f_2 = 1$ . To find the number of pairs after  $n$  months, add the number of rabbits on the island the previous month,  $f_{n-1}$ , and the number of newborn pairs, which equals  $f_{n-2}$  since each newborn pair comes from a pair at least two months old. We then are left with the relation:

Fibonacci Sequence

$$f_{n+2} = f_{n-1} + f_{n-2}$$

Which has some very interesting properties such as  $\Phi$  or the Golan ratio equaling the absolute difference of two consecutive terms in this sequence as  $n \rightarrow \infty$ .

## BACTERIA

Suppose that a population of bacteria,  $x$ , triples every hour. How much bacteria do we have after  $n$  hours?

*Solution:*

$$a_1 = x \qquad a_n = 3a_{n-1}$$

Which is intuitively obvious since we will have three times more bacteria then the hour before.

## SUMMATION

Consider  $\sum_{k=1}^n (2k - 1)$ , where we would like to instead have a recursive definition. As we know we could

re-write this as  $2(n) - 1 + \sum_{k=1}^{n-1} (2k - 1)$  so the recursive definition would follow from this.

$$\sum_{k=1}^1 (2k - 1) = 2(1) - 1 = 1 \rightarrow a_1 = 1$$

$$\sum_{k=1}^n (2k - 1) = a_n$$

$$2(n) - 1 + \sum_{k=1}^{n-1} (2k - 1) = 2(n) - 1 + a_{n-1}$$

$$a_n = 2(n) - 1 + a_{n-1}$$

In a more general sense:  $\sum_{k=1}^n f(k) = \begin{cases} a_1 = f(1) \\ a_n = f(n) + a_{n-1} \end{cases}$  which can be proven using induction. This is left as an exercise for the student.

As a side note there is a much better way to represent  $\sum_{k=1}^n (2k - 1)$ . Notice that this is the sum of all even numbers,  $1, 3, 5, 7, 9, \dots, 2n - 1$ . Considering some of the earlier sums we see a clear pattern:

$$\begin{array}{l} 1 = 1 \\ 4 = 1 + 3 \\ 9 = 1 + 3 + 5 \\ 16 = 1 + 3 + 5 + 7 \\ 25 = 1 + 3 + 5 + 7 + 9 \\ \cdot \\ \cdot \\ n^2 = 1, 3, 5, \dots, 2n - 1 \end{array}$$

So a better definition would have been  $a_n = n^2$ .

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## SOLVING RECURRENCE RELATIONS

Given a recurrence relation it is possible to convert the recursive definition to a function which would be non-recursive. Having a function instead of a relation allows us to easily take limits and determine its behavior. Doing something like this is considered solving the relation.

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## LINEAR HOMOGENOUS RECURRENCE RELATIONS<sup>1</sup>

Linear homogenous recurrence relations are those relations in the form

linear homogenous form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where  $c_1, c_2, \dots, c_k \in \mathbb{R}$

The basic approach for solving linear homogenous recurrence relations is to look for solutions in the form of  $a_n = r^n$ , where  $r$  is a constant. Note that  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$  if and only if  $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$  (by direct substitution).

If we divide both sides of this equation by  $r^{n-k}$  we get

$$\frac{r^n}{r^{n-k}} = \frac{c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}}{r^{n-k}}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_{k-1} r + c_k \quad (*)$$

$$0 = r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k$$

As a consequence to this equation the sequence  $\{a_n\}$  with  $a_n = r^n$  is a solution if and only if  $r$  is a solution of (\*). We call (\*) and its roots the *characteristic equation* and *characteristic roots* of the recurrence relation.

For now we will limit ourselves to recurrence relations of the form  $r^n = c_1 r^{n-1} + c_2 r^{n-2}$ . First, consider the case when there are two distinct characteristic roots.

Theorem 1

Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1 r - c_2 = 0$  has two distinct roots  $r_1$  and  $r_2$ . Then the sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$  if and only if  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$  for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

Lets apply this theory to an example.

What is the solution of the recurrence relation  $a_n = a_{n-1} + 2a_{n-2}$  where  $a_0 = 2$  and  $a_1 = 7$ ?

*Solution:*

the characteristic polynomial is,  $\frac{r^n}{r^{n-2}} = \frac{r^{n-1}}{r^{n-2}} + \frac{2r^{n-2}}{r^{n-2}} \rightarrow r^2 = r + 2 \rightarrow r^2 - r - 2 = 0$  which has roots  $r = 2, -1$ .

Hence the sequence  $\{a_n\}$  is a solution to the recurrence relation if and only if  $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$  (\*), for some constants  $\alpha_1$  and  $\alpha_2$ . From our initial conditions, it follows that

$$a_0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0 \rightarrow 2 = \alpha_1 + \alpha_2$$

$$a_1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1 \rightarrow 7 = 2\alpha_1 - \alpha_2$$

Solving for these two equations gives  $\alpha_1 = 3$  and  $\alpha_2 = -1$ .

Plugging these values into (\*) yields the solution,  $a_n = \alpha_1 2^n + \alpha_2 (-1)^n \rightarrow a_n = 3 \cdot 2^n - (-1)^n$ .

<sup>1</sup> this section was modeled after section 6.2 of Kenneth H. Rosen "Discrete Mathematics and Its Applications – fifth edition"

Theorem 1 however does not handle the case when there is only one characteristic root or the recurrence relation. Theorem 2 shows how to handle this case.

Theorem 2

Let  $c_1$  and  $c_2$  be real numbers with  $c_2 \neq 0$ . Suppose that  $r^2 - c_1r - c_2 = 0$  has only one root  $r_0$ . A sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  if and only if  $a_n = \alpha_1 r_0^n + \alpha_2 r_0^n$ , for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

As an example, what is the solution of the recurrence relation  $a_n = 6a_{n-1} - 9a_{n-2}$  with initial conditions  $a_0 = 1$  and  $a_1 = 6$ ?

*Solution:*

the characteristic polynomial is,  $\frac{r^n}{r^{n-2}} = \frac{6r^{n-1}}{r^{n-2}} - \frac{9r^{n-2}}{r^{n-2}} \rightarrow r^2 = 6r - 9 \rightarrow r^2 - 6r + 9 = 0$  which has root  $r = 3$ .

Hence, the solution to this recurrence relation is  $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$  (\*) for some constants  $\alpha_1$  and  $\alpha_2$ . Using the initial conditions as we did before

$$a_0 = 1 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^0 \rightarrow 1 = \alpha_1$$

$$a_1 = 6 = \alpha_1 \cdot 3^1 + \alpha_2 \cdot 1 \cdot 3^1 \rightarrow 6 = 3\alpha_1 + 3\alpha_2$$

Solving for these two equations gives  $\alpha_1 = 1$  and  $\alpha_2 = 1$ .

Plugging these values into (\*) yields the solution,  $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n \rightarrow a_n = 3^n + n3^n$ .

## GENERAL METHOD FOR SOLVING RECURENCE RELATIONS

*don't get to hung up on this section if you don't understand*

We will leave you with the general theory for solving linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots.

Theory 3

Let  $c_1, c_2, \dots, c_k$  be real numbers. Suppose that the characteristic equation

$$r^k - c_1r^{k-1} - \dots - c_k = 0$$

has  $k$  distinct roots  $r_1, r_2, \dots, r_k$ . Then a sequence  $\{a_n\}$  is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

for  $n = 0, 1, 2, \dots$ , where  $\alpha_1, \alpha_2, \dots, \alpha_k$  are constants.

## SELF TEST PROBLEMS

- 1) What is the recurrence relation for  $\sum_{k=0}^n \left(\frac{\sin(k)}{k}\right)^k$ .
- 2) Define a recurrence relation for the number of ways to permute  $n$  objects. (\*note\* you can permute  $n$  objects  $n!$  ways)
- 3) Give a recurrence relation for  $\binom{n+1}{k}$  given that  $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$  and 
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
- 4) Show that the sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = -3a_{n-1} + 4a_{n-2}$  if
  - a.  $a_n=0$
  - b.  $a_n=1$
  - c.  $a_n=(-4)^n$
  - d.  $a_n=2(-4)^n+3$
- 5) Is the sequence  $\{a_n\}$  a solution of the recurrence relation  $a_n = 8a_{n-1} - 16a_{n-2}$  if
  - a.  $a_n=0$
  - b.  $a_n=2^n$
  - c.  $a_n=n4^n$
  - d.  $a_n=n^24^n$
- 6) Solve these recurrence relations
  - a.  $a_n=a_{n-1}+6a_{n-2}$  for  $n \geq 2$ ,  $a_0=3$ ,  $a_1=6$
  - b.  $a_n=6a_{n-1}-8a_{n-2}$  for  $n \geq 2$ ,  $a_0=4$ ,  $a_1=10$
  - c.  $a_n=a_{n-2}$  for  $n \geq 2$ ,  $a_0=5$ ,  $a_1=-1$
  - d.  $a_{n+2}=-4a_{n+1}+5a_n$  for  $n \geq 0$ ,  $a_0=2$ ,  $a_1=8$