

A canonical form for some piecewise defined functions

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ABSTRACT

We define a canonical form for piecewise defined functions. We show that this has a wider range of application as well as better complexity properties than previous work.

1. INTRODUCTION

Piecewise defined functions are ubiquitous in mathematics, starting from the Kronecker Delta function, through characteristic functions for sets, on to functions such as `signum` and `floor`. Although all of these are certainly interesting, this paper will concentrate on those functions defined over a linearly ordered domain (like \mathbb{R} or $\overline{\mathbb{R}}$) and with a finite number of pieces (unlike `floor` say).

There has been previous work in this area, most notably that of von Mohreschildt [7]. There, a normal form was defined for a large class of piecewise-defined expressions through the use of a very simple set of primitive elements; as well, clear steps were given to modify this normal form to give a canonical form. In our approach, the primitive elements are much more complex; however this allows all the algorithms to be greatly simplified. Furthermore, we obtain substantial arithmetic complexity improvements. We can also handle a wider domain of definition. This form has been independently rediscovered by several authors, see for example [1] and [2]. However both of those papers are about applications of these extended piecewise functions to optimisation, to Fenchel coordinates in particular. To our knowledge, the underlying theory of piecewise functions over linearly ordered spaces has never been published.

It is important to note that, outside of [7] (and the references therein), there seems to be no reference to a formalization of the concept of a piecewise function. This is probably because the usual notation is so suggestive that no one ever thought to question if the concept was ever properly defined.

The results we obtain in this work are deceptively simple, but this is largely because a considerable amount of effort has been put into ensuring that all the definitions are “just

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right”.

This paper benefited from some discussions of the contents with Alexander Potapchik of Maplesoft Inc. He also implemented, in Maple 7, many of the ideas contained in this paper, and this is what Maple now uses for simplification and normalization of piecewise functions.

2. PIECEWISE

2.1 Observations

Although the most common piecewise defined functions are of the type

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & \text{otherwise.} \end{cases} \quad (1)$$

where x is (implicitly) understood to be real, we also encounter functions of the kind

$$f(x) = \begin{cases} x^2 & y < 0 \\ x^3 & \text{otherwise} \end{cases} \quad (2)$$

where x and y are also (implicitly) understood to be real. The notation in the second case above is poor, as the dependence on y is not well indicated, but in frequent use nevertheless. This leads us to observe that, in both cases, there really are two different kinds of variables at play: those that need to satisfy a boolean condition, and those that occur in an arithmetic context (y and x respectively). This “separation of concerns” leads to an important conceptual simplification of the requirements for a piecewise defined function.

Another observation is that, at least in computer algebra systems, it is common to take the derivative of objects like

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Accordingly, the resulting object

$$f'(x) = \begin{cases} 0 & x \neq 0 \\ \perp & \text{otherwise} \end{cases}$$

should really be within the realm of objects that can be talked about. Furthermore it should be possible to correctly compute with such partial functions, as well as with functions on extended domains. The normal form of [7] explicitly requires a ring for the range.

A third and final observation is that, for linearly ordered domains like \mathbb{R} , adding and even multiplying two functions

that are each defined by formulas valid on some finite union of intervals is very easy and can be done in linear arithmetic cost (with respect to the total number of intervals). From these three observations, the author believes that a keen reader should be able to derive the rest of the paper!

2.2 Definition of piecewise

We will start with a relatively simple case of piecewise-defined function, one which is defined on a unique linearly ordered domain.

Definition 1. A set S is said to be *linearly ordered* if there exists a relation $<$ on S such that for all $a, b \in S$, $a \neq b$ either $a < b$ or $b < a$ holds.

From now on, let Λ be a linearly ordered set. We will also need the concept of **range partition** of such a set. This is one of the crucial ingredients.

Definition 2. A *range partition* \mathcal{R} of a linearly ordered set Λ is a finite set B of points $\lambda_1 < \lambda_2 < \dots < \lambda_n$, along with the natural decomposition of Λ into disjoint subsets subsets $\Lambda_1, \dots, \Lambda_{n+1}$ where

$$\Lambda_1 := \{x \in \Lambda \mid x < \lambda_1\}$$

$$\Lambda_i := \{x \in \Lambda \mid \lambda_{i-1} < x < \lambda_i\}, i = 2, \dots, n$$

$$\Lambda_{n+1} := \{x \in \Lambda \mid \lambda_n < x\}.$$

Note that the λ_i themselves are outside these subsets, that

$$\Lambda = \left(\bigcup_{i=1}^{n+1} \Lambda_i \right) \cup \{\lambda_1, \dots, \lambda_n\} = \bigcup \mathcal{R},$$

and that it is the *ordered* version of this complete decomposition of Λ which is the range partition. For a given Λ , we will often just give the set of points λ_i that generate a range partition. It is sometimes useful to consider Λ itself to be a degenerate range partition with the empty set \emptyset as the generating set. We will sometimes refer to the generating set B of a range partition as a set of **breakpoints**.

Perhaps surprisingly, it is expressions like (2) that are simplest to deal with.

Definition 3. A *piecewise expression* is a function from a range partition to a set S .

Example 4. Taking $\Lambda = \mathbb{R}$, $B = \{0\}$, and $S = \{x^2, x^3\}$ then $f : \mathcal{R} \rightarrow S$ defined by

$$f(z) = \begin{cases} x^2 & z = \Lambda_1 \\ x^3 & z = 0 \\ x^3 & z = \Lambda_2, \end{cases}$$

is a piecewise expression.

Of course this is a rather pedantic definition as this clearly does not represent an object of common mathematical interest. Nevertheless it is a very useful definition as it encodes the core computational concept necessary for the sequel succinctly and unambiguously. With just a little more work, we will soon be able to define an object which will be much closer to the usual piecewise functions encountered in textbooks.

Proposition 5. Let Λ be a linearly ordered set and \mathcal{R} a range partition. Then there exists a function $\mathcal{X} : \Lambda \rightarrow \mathcal{R}$ which associates to each $\lambda \in \Lambda$ the unique element r of \mathcal{R} such that $\lambda \in r$.

Corollary 6. Assuming that $=$ and $<$ are decidable and take $O(1)$ time, then for $\lambda \in \Lambda$, $\mathcal{X}(\lambda)$ can be computed using at most $O(\log_2(n))$ operations, where $n = |B|$.

PROOF. Since Λ is linearly ordered, we can store \mathcal{R} in a contiguous sorted array and use an adapted binary search on its $2n + 1$ elements to find $\mathcal{X}(\lambda)$.

The assumption that $=$ and $<$ are decidable over all of Λ can be weakened to merely assuming that they are decidable for the evaluation point λ relative to be set of breakpoints B . This is why in practice these functions can be effectively evaluated even though the zero equivalence problem is undecidable.

From now on we will assume that all range partitions are stored in a contiguous sorted 1-dimensional array; we will sometimes simply say use the term 'list' to refer to this data-structure.

Using \mathcal{X} , and a little bit of abusive notation, we get a much more familiar expression for $f_B = f \circ \mathcal{X} : \Lambda \rightarrow S$ where we explicitly indicate the range partition generator B . For the previous example, this unravels to:

$$f_{\{0\}}(z) = \begin{cases} x^2 & z < 0 \\ x^3 & z = 0 \\ x^3 & z > 0. \end{cases}$$

There is clearly a bijection between the set of f_B and the set of piecewise expression defined previously. Next, we really want to be able to treat expressions like

$$f(x) = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0. \end{cases} \quad (3)$$

where want the evaluation bindings to be such that $f(-5) = 5$ and not $-x$. Our definition of piecewise expressions, using terms from a set S as above, would indeed give $-x$ because there is no relationship between the elements of Λ and those of S . This is definitely not what is wanted. If we used expressions with strict evaluation rules, this particular problem would be solved. However, that is not quite enough because we would still have problems with singular expressions in "other" branches. To fix both of these problems at once, what we really need to do is to treat S as a set of functions instead of a set of (first order) values. To avoid spurious evaluations, we are going to steal a standard trick¹ from functional programming² and use currying to solve our problems. This leads us to define a somewhat more general concept than a piecewise function, but the extra generality is exactly what allows us to solve the problem mentioned

¹also known to logicians as lambda-lifting

²we could have also used some fancy version of lazy evaluation, but that would have introduced new problems whose solution would have distracted greatly from the main points of this paper. A very specialized version of lazy evaluation is what was later implemented in Maple 8 for this purpose but, in this author's opinion, the downsides of integrating this in an eager language outweigh the apparent benefits of being able to use a simpler representation.

above. Furthermore, it specializes easily and correctly to the intuitive notions of piecewise functions, as we will prove in the next section.

Definition 7. Let S be a set of functions, then a *piecewise operator* is a piecewise expression $f : \mathcal{R} \rightarrow S$.

We can thus rewrite example 3, using $\tilde{S} = \{y \mapsto -y, y \mapsto 0, y \mapsto y\}$, the curried, relabelled version of S to get

$$\tilde{f}(x) = \begin{cases} y \mapsto -y & x < 0 \\ y \mapsto 0 & x = 0 \\ y \mapsto y & x > 0. \end{cases} \quad (4)$$

Then we have that $\tilde{f}(-5)(\sqrt{2}) = -\sqrt{2}$. This is actually progress! What we really want is $\tilde{f}(-5)(-5) = 5$. This last ingredient is exactly what we need to define piecewise functions that behave as expected mathematically as well as when implemented.

Definition 8. Given a piecewise operator $f : \mathcal{R} \rightarrow S$ where S is a set of functions $s : \Lambda \rightarrow V$ call $\bar{f} : \Lambda \rightarrow V$ defined by

$$\bar{f}(\lambda) := f(\mathcal{X}(\lambda))(\lambda) = f_B(\lambda)(\lambda)$$

a *piecewise function*.

Note that the notation \bar{f} is sufficient since all of \mathcal{R} , \mathcal{X} and B can be recovered from a *representation* of f . Also note that there are no restrictions on V at all. When multiple piecewise functions defined on different range partitions (but the same Λ) are being discussed, we will denote them p_B , making the generating set of the range partition explicit. A strict notation for piecewise functions would be given by

$$f(x) := \begin{cases} g_1(x) & x \in \Lambda_1 \\ g_2(x) & x = \lambda_1 \\ g_3(x) & x \in \Lambda_2 \\ g_4(x) & x = \lambda_2 \\ \vdots & \vdots \\ g_{2n}(x) & x = \lambda_n \\ g_{2n+1}(x) & x \in \Lambda_{n+1} \end{cases}$$

with $g_i \in S$. It is worthwhile noting that giving Λ, B and g_1, \dots, g_{2n+1} (as ordered sets) are sufficient to fully determine f .

As Λ is linearly ordered, and the g_i 's for i even are actually only evaluated at one point, this is customarily written as

$$f(x) := \begin{cases} h_1(x) & x < \lambda_1 \\ \beta_1 & x = \lambda_1 \\ \vdots & \vdots \\ \beta_n & x = \lambda_n \\ h_{n+1}(x) & \lambda_n < x \end{cases}$$

where that last condition is often written as the word **otherwise**, and where $h_i = g_{2i-1}$ and $\beta_i = g_{2i}(\lambda_i)$. This notation can at times be problematic as it mixes ground values (the β_i 's) and functions (the h_i) at the same "level", even though they have different types. This is why we prefer to lift the constants up to functions explicitly.

It is important to notice that a piecewise function is a function that uses its argument twice, for very different purposes. It is the separation of these two concerns that make

many of the subsequent algorithms simple yet general. In the next sections it will be important to keep track of which properties hold in the general case of piecewise operators and which need to be specialized for piecewise functions.

2.3 Definition of domains

In order to be able to define a canonical form, we will require somewhat more structure on the range S of functions of a piecewise operator.

Definition 9. An *effective domain* D is a pair (F, \sim) , where

1. $F : O^n \rightarrow V$ is a set of functions (of varied arity n) from a set O to a set V
2. \sim is a binary function on F that decides extensional equivalence.

Definition 10. Two (n -ary) functions $f, g \in F$ are said to be *extensionally equivalent* if for all $x \in O^n$, either f and g are both defined and $f(x) = g(x)$, or neither f nor g are defined. Denote this by $f \simeq g$.

It is very important to note that

1. the functions in F can be partial,
2. \simeq denotes equivalence, not equality,
3. \sim is defined for F , not O nor V ,
4. \sim decides equivalence, where \simeq denotes equivalence.

In most practical cases, \sim will necessarily be compatible with a (possibly partial) equivalence of elements of V since there is a canonical identification between the functions g_{2i} of the breakpoints of a piecewise operator and the constants they represent. Of course, since \sim is a decision procedure, this implies that the constant functions present in F must in fact come from a subset of constants of V over which a similar decision procedure exists. But the point is that we should be able to tell that $(x+1)^2$ and $x^2 + 2x + 1$ (over \mathbb{R} say) are equivalent. What is crucial here is that we can tell this completely independently from any representation issue of the underlying domain. In other words, this works just as well over the usual uncountable \mathbb{R} as it does with constructive \mathbb{R} .

Given an effective domain D and a computable total function $C : F \rightarrow F$ such that $C(s) \sim s$, $C(s) = 0 \iff s \simeq 0$, and $\forall s, t \in F, C(s) = C(t) \iff s \sim t$, we will call the triple $\{F, \sim, C\}$ a **strong effective domain**. It is worthwhile noting that given $\{F, C\}$ one can always obtain a strong effective domain by defining \sim to be $(a, b) \mapsto C(a) = C(b)$ whenever equality is decidable on F .

Proposition 11. Let $\{F, \sim, C\}$ be a strong effective domain. Then $C \circ C = C$. In other words, C is a canonical form for F .

PROOF. Let $s \in F$, and $t = C(s)$. Since $t = C(s) \sim s$, then $C(t) = C(C(s)) \sim C(s) \sim s$. But $t = C(s) \sim s$, so $C(C(s)) = C(t) = C(s)$.

One cannot under-estimate the power of such a C : it gives a *canonical form* for functions in F . It is important to notice that it is defined *globally*, in other words, it treats (partial) functions of the whole domain. It is outside of the scope of the current work, but roughly speaking such canonical forms only (seem to?) exist for very rigid objects, like meromorphic functions or, more generally for functions for an ω -minimal structure.

Example 12. *The polynomial functions over \mathbb{Z} , coded as $D = \{\mathbb{Z}[\bar{x}], =, \text{expand}\}$, is a strong effective domain.*

In fact, we can replace \mathbb{Z} with \mathcal{RA} , the real algebraic numbers, and still get a strong effective domain, see [6] for the details. This example also shows why it is important to deal with equivalence rather than equality, as well as the fact that a canonical form induces a (computable) equivalence test. Perhaps more important still, at least to symbolic computation, is the next example.

Example 13. *Let P be a term algebra (of rational functions) containing the rationals \mathbb{Q} , the symbol x , the binary operations $+$, \times , and composition. Let T be the term algebra defined by the grammar $P|\sin(P)|\cos(P)|T + T|T * T$. Let $T' = \{f : \mathbb{R} \rightarrow \mathbb{R} \text{ where } f : x \mapsto t, t \in T\}$ be the corresponding set of functions. Then $\{T, \text{normal}\}$ with **normal** the expanded normal form defined in [8], is a strong effective domain.*

While it is possible to further generalize the above example, the term algebra T is already very close to the one used in the undecidability results of [3, 9], and thus we cannot expect to be able to continue with pure decision procedures much further, although it would be interesting to see in which ways holonomic functions can be mixed with piecewise functions and retain decidability. Semi-decision procedures and even heuristics can however be quite useful in practice.

A **weakly effective domain** is a pair $\{F, \sim\}$ where \sim only decides equivalence to a distinguished element of F (typically $x \mapsto 0$). This is often the case when associated to F we have a normal form operator N for elements of F instead of a canonical form.

2.4 Spaces of piecewise operators

Definition 14. *Let S be a set, then denote by $\mathbf{Fin}(S)$ the set $\{p \in \mathcal{P}(S) \mid \#p < \infty\}$ of finite subsets of S , where we denote the power set of S by $\mathcal{P}(S)$.*

Definition 15. *Let $\mathbb{P}(\mathbf{Fin}(\Lambda), F)$ denote the set of all piecewise operators defined on the range partitions of Λ generated by all its (finite) subsets with values in F .*

Of particular interest will be the case where $\{F, \sim\}$ is (at least) a weakly effective domain. As we will often discuss multiple piecewise functions at once, it is convenient to define $B : \mathbb{P}(\mathbf{Fin}(\Lambda), F) \rightarrow \mathbf{Fin}(\Lambda)$ as the function which given a (representation of a) piecewise operator will return its set of breakpoints.

2.5 Redundancies and refinement

It is important to notice that F is canonically embedded in $\mathbb{P}(\mathbf{Fin}(\Lambda), F)$ since $\emptyset \in \mathbf{Fin}(\Lambda)$ generates piecewise operators extensionally equivalent to those in F . However, this space also contains a lot of redundancies. If we let $\Lambda = \mathbb{R}$ and F the space of all continuous functions $C(\mathbb{R}, \mathbb{R})$, then

$$p(\lambda) := \begin{cases} 1 & \lambda < 0 \\ 1 & \lambda = 0 \\ 1 & \lambda > 0 \end{cases}$$

is clearly an element of $\mathbb{P}(\mathbf{Fin}(\mathbb{R}), C(\mathbb{R}, \mathbb{R}))$ which is extensionally equivalent to $x \mapsto 1 \in F$. We will deal with this redundancy later. This redundancy is in fact very useful,

and is the key to efficient arithmetic in \mathbb{P} ! As increasing the redundancy of the representation of a piecewise operator can be quite useful, we will encode this in a definition.

Definition 16. *A **refinement** of a piecewise operator p is another operator q such that $p(\lambda) = q(\lambda)$ for all $\lambda \in \Lambda$, and the set of breakpoints of p is a subset of that of q . We will call a refinement **strict** if the set of breakpoints of p is a strict subset of that of q .*

Note that we used $p = q$ and not $p \sim q$ in this definition. It is in fact possible to do this either way, but since we will always be using explicit refinements, this would be an unnecessary complication. Most often, we will actually want to specify the (new) set of breakpoints of a refinement:

Definition 17. *For any ordered finite set $A \subset \Lambda$, a **A-refinement** of a piecewise operator p is another operator q such that q is a refinement of p , and $A \subset B(q)$. We will say a **A-refinement** q is **exact** if $B(q) = A \cup B(p)$.*

Given a finite ordered set A and a piecewise operator p as above, one can use the usual **linear merge** algorithm to generate q in time $O(|A \cup B(p)|)$.

We have glossed over one very important point: we can perform a linear merge of two ordered finite lists of breakpoints if and only if we can effectively decide $<$ and $=$ for each of the breakpoints. In other words, for all of our algorithms we need to make the assumption that whenever we need to compute a common refinement q of two piecewise functions p_1, p_2 , then the union B of their respective sets of breakpoints B_1, B_2 must be such that $B \subset \mathcal{O} \subset \Lambda$ where $<$ and $=$ are decidable on \mathcal{O} . For this purpose, we introduce following variation on **Fin**.

Definition 18. *Let D be a linearly ordered domain, \mathcal{O} a subset of D over which $<$ and $=$ are decidable, then denote by $\mathbf{Fin}(S)$ the set $\{p \in \mathcal{P}(\mathcal{O}) \mid \#p < \infty\}$ of finite subsets of \mathcal{O} .*

All definitions of piecewise functions, piecewise operators and operations on them should be understood to use **Fin** in place of **Fin** whenever computability and decidability are needed. We will not systematically do so since the *mathematical* definitions of many of the concepts work equally well without this restriction.

Another aspect to notice is that since we are dealing with piecewise operators (and not functions) even at breakpoints, so that the underlying functions in the representation of p are not evaluated to give q . For example, the $\{1\}$ -refinement of the trivial piecewise operator $p(\mathbb{R}) := (x \mapsto 0)$ is

$$q(\lambda) = \begin{cases} x \mapsto 0 & \lambda < 1 \\ x \mapsto 0 & \lambda = 1 \\ x \mapsto 0 & \lambda > 1. \end{cases}$$

2.6 Denesting

There are two different ways in which “nesting” of piecewise expressions can arise: functional composition and definitional nesting. This is easiest to understand via examples: consider the piecewise operators

$$t(\lambda) = \begin{cases} x \mapsto x^2 - 3 & \lambda < 1 \\ x \mapsto -5 & \lambda = 1 \\ x \mapsto x^3 - 7x^2 + 16x - 12 & \lambda > 1. \end{cases} \quad (5)$$

and the absolute value function as the piecewise operator f of example 4, along with the corresponding piecewise functions \bar{f}, \bar{t} . Then $\bar{f}(\bar{t}(\lambda)) = |\bar{t}(\lambda)|$ is an example of functional composition. Expanding the definitions gives

$$\bar{f}(\bar{t}(\lambda)) = \begin{cases} -\bar{t}(\lambda) & \bar{t}(\lambda) < 0 \\ 0 & \bar{t}(\lambda) = 0 \\ \bar{t}(\lambda) & \bar{t}(\lambda) > 0 \end{cases}$$

$$= \begin{cases} -\bar{t}(\lambda) & \left(\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda > 1 \end{cases} < 0 \right) \\ 0 & \left(\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda > 1 \end{cases} = 0 \right) \\ \bar{t}(\lambda) & \left(\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda > 1 \end{cases} > 0 \right) \end{cases}$$

which, after quite a number of non-trivial computations (see [7] for the details) gives

$$\bar{f}(\bar{t}(\lambda)) = \begin{cases} \lambda^2 - 3 & \lambda < -\sqrt{3} \\ 0 & \lambda = -\sqrt{3} \\ -\lambda^2 + 3 & \lambda < 1 \\ 5 & \lambda = 1 \\ -\lambda^3 + 7\lambda^2 - 16\lambda + 12 & \lambda < 3 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & 3 \leq \lambda \end{cases}$$

were we would have to expand the first and last cases further if we wanted to write this more formally. The most difficult parts of this computation involve extracting a range partition from conditions like

$$\left(\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda > 1 \end{cases} < 0 \right)$$

The case of definitional nesting is considerably simpler.

$$\left\{ \begin{array}{l} \left(\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda > 1 \end{cases} \right) \\ 3 \end{array} \right. \begin{array}{l} \lambda < 3 \\ \lambda = 3 \end{array}$$

$$\left\{ \begin{array}{l} \left(\begin{cases} -\lambda & \lambda < 0 \\ 0 & \lambda = 0 \\ \lambda & \lambda > 0 \end{cases} \right) \\ \lambda \end{array} \right. \begin{array}{l} \lambda > 1 \end{array}$$

only involves simple set-theoretic intersections to obtain the equivalent

$$\begin{cases} \lambda^2 - 3 & \lambda < 1 \\ -5 & \lambda = 1 \\ \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda < 3 \\ 3 & \lambda = 3 \\ \lambda & \lambda > 3 \end{cases}$$

3. ARITHMETIC

We first show how to do arithmetic with piecewise functions. Very few assumptions are needed to just perform

arithmetic. For this section, let Λ be a fixed linearly ordered set, and F a set of functions from Λ to some set M . Let $\mathbb{P} = \mathbb{P}(\mathbf{Fin}(\Lambda), F)$ be the corresponding space of piecewise operators. Furthermore suppose we have a function $\psi : F \rightarrow F$, we want to lift this to a function on \mathbb{P} .

Definition 19. Let $\psi : F \rightarrow F$ be a unary function on F . For $p \in \mathbb{P}$ determined by a breakpoint set B and functions g_i , $1 \leq i \leq 2|B| + 1$, define $\bar{\psi}(p)$ by the same breakpoint set B and $\psi(g_i)$, $1 \leq i \leq 2|B| + 1$.

We should prove that this properly lifts the unary functions from those of F onto \mathbb{P} :

Theorem 20. $\bar{\psi}(p)$ and $\lambda \mapsto \psi(p(\lambda))$ are extensionally equivalent.

PROOF. Let $\lambda \in \Lambda_i$. Then

$$\begin{aligned} \bar{\psi}(p)(\lambda) &= \psi(g_{2i+1}) && \text{(by definition)} \\ &= \psi(p(\lambda)) && \text{(by definition)} \end{aligned}$$

as required. Similarly, let $\lambda = \lambda_j$, then $\bar{\psi}(p)(\lambda) = \psi(g_{2i}) = \psi(p(\lambda))$.

We can lift any n-ary function on O to one on \mathbb{P}^n . For the case of $\psi : F \times F \rightarrow F$ and p^1, p^2 with common breakpoint set B and associated functions g_i^l for $l = 1, 2$, $1 \leq i \leq 2|B| + 1$, $\bar{\psi}(p^1, p^2)$ is defined by the same breakpoint set and $\psi(g_i^1, g_i^2)$. The details for this and the n-ary case are left to the reader - they are not difficult, but notationally hideous, and no new insight is gained from the exercise.

For the case $\psi : F \times F \rightarrow F$ and p^1, p^2 with different breakpoint sets B_1, B_2 , we must first transform p_1, p_2 to B -refinements q_1, q_2 (with $B = B_1 \cup B_2$), and then we can apply the previous construction. However we can no longer work over $\mathbf{Fin}(\Lambda)$ but must work only over $\mathbf{Fin}(\Lambda)$ for the refinement algorithm to be effective.

Corollary 21. Addition from a linearly ordered ring R can be lifted to addition of piecewise-defined polynomials. More generally, any ring R gives rise to well defined operations on $\mathbb{P}(\mathbf{Fin}(|R|), R[x])$ with $|R|$ the underlying set of elements of the ring R .

PROOF. The lifting of the addition from any ring R to addition on $R[x]$ is classical. Treating $R[x]$ as a set of functions, one can use the previous construction to lift addition (from $+$ to $\bar{+}$) up to $\mathbb{P}(\mathbf{Fin}(|R|), R[x])$.

Clearly the same can be done for negation and multiplication, and so on.

Corollary 22. Let R be a linearly ordered ring, and denote $\mathbf{Hom}(R, R)$ the space of homomorphisms from R to R . Then we can make $\mathbb{P}(\mathbf{Fin}(R), \mathbf{Hom}(R, R))$ into a ring.

PROOF. Lifting the ring operations from R to $\mathbf{Hom}(R, R)$ is classical: $(f + g)(x)$ is defined to be $f(x) + g(x)$, etc. The functions $\mathbf{0} = x \mapsto 0$ and $\mathbf{1}_x = x \mapsto x$ are the additive and multiplicative unit respectively. Letting $F = \mathbf{Hom}(R, R)$, simple verification shows that $(\mathbb{P}(\mathbf{Fin}(|R|), F), \mathbf{0}, \mathbf{1}_x, \bar{+}, \bar{*})$ is a ring.

The previous corollary hints at an even more general result: that our construction is actually functorial. We will not go into the details since this is not needed. Before we move on, it is useful to explicitly turn these theoretical results in to algorithms. For example, using binary operators, we get

Proposition 23. *Let $R = \mathbb{P}(\mathbf{Fin}(|O|, F))$ be a ring of piecewise functions. Let $f^1, f^2 \in R$ be given explicitly. Then $f^+ = f^1 + f^2$ can be computed explicitly by*

1. *Forming the B -refinements of f^1 and f^2 , with $B = B(f^1) \cup B(f^2)$.*
2. *Letting the component functions g_i^+ of f^+ be $g_i^1 + g_i^2$, where the g_i^j come from the B -refinements above.*

Clearly we can replace $+$ by any other binary operation. The reader will recognize the above as being the *linear merge* algorithm.

4. CANONICAL FORM

Simply defining arithmetic is not the end of the story. For example, consider $|x|^2 - x^2$ over \mathbb{R} . Translating the absolute value function to its piecewise equivalent (see 3), the results of carrying out the arithmetic as above gives

$$\begin{cases} x \mapsto 0 & \lambda < 0 \\ x \mapsto 0 & \lambda = 0 \\ x \mapsto 0 & \lambda > 0. \end{cases}$$

which is extensionally equivalent to 0, but not intensionally equal to 0. Thus we need a further normalization step which would combine the above redundancies.

Definition 24. *Let $D = (F, \sim, C)$ be a strong effective domain of functions, where $F : O \rightarrow V$ and O is a linearly ordered domain. We call $\mathbb{P}_D(\mathbf{Fin}(|O|, F))$ an effective piecewise domain.*

Note that we are not assuming that all the functions in F are total - only that we have an effective method (\sim) for deciding equivalence. The first simplification algorithm is then very simple to describe: apply C to each part of a piecewise function p giving a new function q with the same breakpoint set, and then merge (in increasing order) adjoining triples $(g_{2i-1}, g_{2i}, g_{2i+1})$ if they are all equal. More precisely, Figure 25 gives Ocaml code to implement this. *normal* is the normalizing function C of D . Note that we have used a record structure to *statically* enforce the fact that any piecewise function with n breakpoints must consist of $2n + 1$ functions, with adjoining regions alternating between connected open sets and a point upper bound, and ending with a single (upward) unbounded piece. It is possible to use a simpler data-structure for this and simplify the code, but we would lose the ability to statically enforce some important invariants.

Proposition 26. *Let $f \in \mathbb{P} = \mathbb{P}_D(\mathbf{Fin}(|O|, F))$, where \mathbb{P} is an effective piecewise domain, then $\mathbf{pseudonormalform}(N, f)$ and f are extensionally equal, where N is any function $N : F \mapsto F$ which preserves \simeq . Additionally, if N is idempotent, then so is $\mathbf{pseudonormalform}$.*

A complete proof can be found in Appendix A. Unfortunately, this simple algorithm does not actually give a normal form, never mind a canonical form, even if we restrict our input functions to polynomials over \mathbb{Z} . Consider for example

$$\begin{cases} x \mapsto 0 & \lambda < 0 \\ x \mapsto x^2 & \lambda = 0 \\ x \mapsto 0 & \lambda > 0. \end{cases}$$

Algorithm 25. *Pseudo normal form*

```

type ('a,'b) condpair =
  {left_fn : ('a -> 'b);
   pt_fn : ('a -> 'b); right_pt : 'a}
and ('a,'b) endpiece = {fn : ('a -> 'b)}
and ('a,'b) piecewise =
  (('a,'b) condpair) array * ('a,'b) endpiece ;;

let pseudonormalform (normal:('a->'b) -> ('a->'b))
  ((a,e):('a,'b) piecewise) : ('a,'b) piecewise =
  let pnormal y = {y with left_fn = normal y.left_fn;
                  pt_fn = normal y.pt_fn}
  and canmerge a b =
    a.left_fn == a.pt_fn && a.pt_fn == b.left_fn
  and merge a b = {left_fn = a.left_fn;
                  pt_fn = b.pt_fn; right_pt = b.right_pt}
  in
  let (b,newe) = (Array.map pnormal a, {fn = normal e.fn})
  and j = ref 0
  and n = Array.length a in
  if n=0 then (b,newe)
  else begin
    for i=1 to n-1 do
      if canmerge b.(!j) b.(i) then
        b.(!j) <- merge b.(!j) b.(i)
      else
        j := !j + 1;
    done;
    if b.(!j).left_fn==b.(!j).pt_fn &&
       b.(!j).pt_fn==newe.fn then
      (Array.sub b 0 !j, newe)
    else
      (Array.sub b 0 (!j+1), newe)
  end;;

```

which “simplifies” to itself. Of course, the above function is extensionally equal to 0, so we do not in fact have a complete normal form. However, for some restricted classes of functions, this does give a normal form.

Proposition 27. *Let $f \in \mathbb{P}_D(\mathbf{Fin}(|O|, F))$ be such that for all g_{2i} components of f defined on the points of the range partition associated with f , then either $g_{2i} \simeq g_{2i-1}$ or $g_{2i} \simeq g_{2i+1}$. For such f , the *pseudonormalform* algorithm gives a normal form.*

The proof is straightforward. The proposition can be understood to say that if the function we are dealing with has a representation into pieces that are somehow compatible with each other (i.e. applying C is enough to recognize this), then we have a normal form. To get a complete normal form, we have to figure out if, at the breakpoint, the function is “compatible” with its neighbours. To understand why this is not so simple, consider

$$\begin{cases} x \mapsto 0 & \lambda < 0 \\ x \mapsto \delta_0(x) & \lambda = 0 \\ x \mapsto 0 & \lambda > 0, \end{cases}$$

where $\delta_a(x)$ is the usual characteristic function of the point a . To be able to properly handle such cases, de-nesting of piecewise-defined functions is necessary.

Consider our first algorithm 25, but with the *canmerge* function defined as

```

let canmerge' a b = ((a.left_fn = b.left_fn) &&
                    (a.pt_fn a.right_pt == b.left_fn a.right_pt))

```

In other words, we merge 2 consecutive pieces if and only if the normal forms for the functions on the two open intervals

are the same and if the point function and left hand function evaluate to the same value. More precisely,

Algorithm 28. *Let canonform $p = \text{pseudonormalform}'$ (denest p) where $\text{pseudonormalform}'$ is obtained from pseudonormalform by replacing `canmerge` with `canmerge'`.*

The `denest` function is a simple linear traversal (specified by example in subsection 2.6) which brings (definitionally) nested piecewise functions to the surface. This does not increase the *total* number of breakpoints, although it usually increases the number of outer breakpoints.

Theorem 29. *Let $f \in \mathbb{P} = \mathbb{P}_D(\mathbf{Fin}(|O|), F)$, where \mathbb{P} is an effective piecewise domain, and f is such that for all breakpoints $b \in \mathbf{Fin}(|O|)$, there exists a decision oracle $=_V$ for equality of values. In other words, for all $g_1, g_2 \in F$ and all $b \in \mathbf{Fin}(|O|)$, it is possible to decide if $g_1(b) = g_2(b)$ with $=_V$. Then algorithm 28 is a canonical form algorithm.*

The full proof is in Appendix A. While the above may appear to give a qualified normal form, it nevertheless turns out to be extremely useful in practice, as very wide classes of examples are covered. Instead of using a function $=_V$ on values, one instead uses a semi-decision procedure for \neq_V , and only structural equivalence for $=_V$. While this no longer gives a normal (or canonical) form, for many practical examples this appears to be sufficient.

Corollary 30. *Let $O = \mathbb{R}$, restrict $\mathbf{Fin}(|O|)$ to the real algebraic numbers, and F to be rational functions, then algorithm 28 gives a canonical form algorithm.*

5. COMPLEXITY

We are primarily interested in comparing complexity results between our approach and that of [7], and thus we will restrict ourselves to a setting where this comparison can (fairly) be made. Although we would have preferred to make this paper self-contained, repeating all the necessary definitions from [7] would take us too far afield, and we will be forced to assume that the reader has a certain familiarity with its contents.

Without loss of generality, we can assume that arithmetic operations on function *representations* are $O(1)$, and that the normal form operation C on function representations is $O(M(n))$ where n is the size of the representation. It is then easy to obtain that

Proposition 31. *Let f be a piecewise function (as per Theorem 29) with d breakpoints, with each g_i bounded in size by n . Then Algorithm 28 runs in $O(dM(n))$.*

Naturally for complex expressions, $M(n)$ can still be the driving factor in the overall cost. The cost above is trivial to establish as Algorithm 25 only does at most 4 linear traversals of the expression (once for denesting, the `Array.map`, the middle loop, and the final `Array.sub`). Only the middle loop needs to perform non-trivial computations.

Proposition 32. *Under the same assumptions, von Mohrenschildt’s algorithm [7] runs in $O(2^d M(n))$.*

The reason for this is that the algorithm of section 6.1 of [7] expands piecewise expressions into terms which the normal form algorithm steps (3.4) and (3.6) (section 3) further expand.

6. REMARKS

For lack of space, we did not include here the full algorithm for definitional denesting. However this is quite straightforward. Denesting of composed piecewise functions is considerably more difficult; however, the key ideas are in von Mohrenschildt’s work [7], and these can be combined with our the ones in the present work. The principal difficulty here remains that of “inverting” functions to create a *finite* set of breakpoints. This is why [7] restricts the inner functions to be polynomials.

7. CONCLUSIONS AND FURTHER WORK

In the current work, we make the following contributions: a simple yet general exposition of piecewise functions that cleanly separates the decision aspects from the value aspects of these functions; this allows us to leverage the underlying linear structure to give faster algorithms (linear instead of exponential in the number of breakpoints); a clean separation of concerns between the requirements on the domain and the range of piecewise functions; and a clearer picture of the kinds of normal and canonical forms needed from the base domains to be able to build piecewise functions.

While all our examples are over the \mathbb{R} , it is clear that our work also applies to finite domains (which can be linearly ordered). Finite unions of linearly ordered domains can also be handled - one can just pick an arbitrary order between the domains, where none of the domains “touch”; we can combine the decision procedure $=_V$ for each of the sub-domains to a decision procedure for the full domain.

For example, by using a logic which can deal with partial functions and undefinedness [4, 5], the functions we deal with can be partial. This was our original motivation for looking into this problem! The issue with von Mohrenschildt’s work is that it needs a *ring* in both the value and range domains. Here, we only require sets with operations and a normalization procedure in the range, and ordering properties in the domain.

In the future, we hope to move from linearly ordered domains to domains with finite presentations and algorithmic combination properties. The main examples, of course, being algebraic and semi-algebraic sets in \mathbb{R}^n , where respectively Gröbner Bases and CAD are the algorithmic tools. Another source of generalization might be to work with implicit characteristic functions, so as to be able to handle functions like `floor` and `trunc`.

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APPENDIX

A. PROOFS

Proposition (26). *Let $f \in \mathbb{P} = \mathbb{P}_D(\mathbf{Fin}(|O|), F)$, where \mathbb{P} is an effective piecewise domain, then $\mathbf{pseudonormalform}(N, f)$ and f are extensionally equal, where N is any function $N : F \mapsto F$ which preserves \simeq . Additionally, if N is idempotent, then so if $\mathbf{pseudonormalform}$.*

PROOF. Let f have the following form:

$$\begin{cases} g_1(x) & x \in \Lambda_1 \\ g_2(x) & x = \lambda_1 \\ \vdots & \vdots \\ g_{2n}(x) & x = \lambda_n \\ g_{2n+1}(x) & x \in \Lambda_{n+1} \end{cases}$$

We will work on the abstract mathematical form as above, since the mapping between it and the data-structure (`'a`, `'b`) **piecewise** is obvious, but the mathematical notation is clearer. Remember that the decomposition above satisfies the following pre-condition,

$$\forall x \in \Lambda_i, x \leq \lambda_i \wedge \forall 1 \leq j < n, \lambda_j < \lambda_{j+1}$$

and that the λ_i 's form a range partition for Λ .

The first step of the algorithm transforms this to a function f_1 (stored in `(b, newe)` in the code) of the same form but with g_i replaced with $\tilde{g}_i := N(g_i)$; however, by definition, N preserves \simeq , and thus $f_1 \simeq f$.

We then have 3 different cases: $n = 0$, $n = 1$, and $n > 1$. For $n = 0$, then our piecewise function is already in one piece, no reduction can be done, and so the normalized form can be returned.

For $n = 1$, the **for** loop is empty. The **if** condition is a specialized version of **canmerge** for comparing a general piece to the end piece. The condition amounts to checking that all 3 functions involved are the same, in which case we can simply make the range partition null, which is what is done via `Array.sub b 0 j` (the sub-array of `b` starting at index 0 of length `j`) as `j = 0`.

For $n > 1$, the **for** loop actually encodes a fold function in an imperative manner. It is easy to see that, for any i , $j \leq i$ and that the function defined by the first i pieces of f and the one contained in the j pieces of b are extensionally

equivalent (by construction and the properties of **canmerge** and **merge**). This is the invariant maintained by the loop. On exit of the loop, we perform a last step on the last general piece and the end piece, as already analyzed above in the $n = 1$ case.

To prove idempotency, assuming that N is idempotent, the same proof skeleton as above works. The only difference is in the case of $n > 1$, where we must show that a second (linear) pass through the pieces of f_1 would not merge any more segments. But if a second pass were to merge two pieces, we can show that these pieces would have been merged already in the first pass.

Theorem (29). *Let $f \in \mathbb{P} = \mathbb{P}_D(\mathbf{Fin}(|O|), F)$, where \mathbb{P} is an effective piecewise domain, and f is such that for all breakpoints $b \in \mathbf{Fin}(|O|)$, there exists a decision oracle $=_V$ for equality of values. In other words, for all $g_1, g_2 \in F$ and all $b \in \mathbf{Fin}(|O|)$, it is possible to decide if $g_1(b) = g_2(b)$ with $=_V$. Then algorithm 28 is a canonical form algorithm.*

PROOF. First, we can re-use the proof of 26, but with the modified **canmerge'** to show that extensional equivalence is maintained. Since `a.pt_fn = b.left_fn` implies `a.pt_fn a.right_pt == b.left_fn a.right_pt`, the new algorithm will definitely merge all previously merged pieces, and may merge more. But, similarly to the previous proof, extensional equivalence is always maintained. In fact, the modified **canmerge'** checks this quite explicitly for the breakpoints. Actually, using the same proof, since C preserves decidable equivalence, we have the stronger result that \sim is preserved.

We have to prove that if f_1 and f_2 are two functions in $\mathbb{P}_D(\mathbf{Fin}(|O|), F)$ then $\mathbf{canonform}(f_1) = \mathbf{canonform}(f_2)$ if and only if $f_1 \sim f_2$.

Without loss of generality, we can assume that neither f_1 nor f_2 are nested.

\Rightarrow . Suppose $\mathbf{canonform}(f_1) = \mathbf{canonform}(f_2) = \tilde{f}$. But we have already shown that $\mathbf{canonform}(f_1) \sim f_1$ and $\mathbf{canonform}(f_2) \sim f_2$, and as \sim is symmetric, this implies that $f_1 \sim f_2$.

\Leftarrow . Suppose $f_1 \sim f_2$. But $\mathbf{canonform}(f_1) \sim f_1$ and $\mathbf{canonform}(f_2) \sim f_2$, and by symmetry, $\mathbf{canonform}(f_1) \sim \mathbf{canonform}(f_2)$. However, we need to prove that these are in fact equal. Suppose that they are not. They either they have a different set of breakpoints or (assuming that they have the same breakpoints), that the underlying functions differ. Let $k_1 = \mathbf{canonform}(f_1)$ and $k_2 = \mathbf{canonform}(f_2)$.

First, suppose that the set of breakpoints is different. Without loss of generality, assume that it is k_1 and is in fact the first breakpoint λ_1 . Let k_1 have the same form as f in the preceding proof (with sub-functions labelled g_i), and use h_j for the labels of the sub-functions of k_2 . Now the g_i 's and the h_i 's are breakpoint-free functions for which we have a canonical form C (by assumption). Since $g_1 \sim h_1$, $C(g_1) = C(h_1)$; $g_2(\lambda_1) =_V h_1(\lambda_1)$ (since $f_1 \sim f_2$); and $g_3 \sim h_1$ implies $C(g_3) = C(h_1) = C(g_1)$. But since $C(g_1) = C(g_3)$ and h_1 is defined at λ_1 , so is g_1 . By extensionality and the fact that h_1 is breakpoint free, **canmerge'** applied to the first breakpoint of k_1 would have merged this part – contradiction.

Second, suppose that the breakpoints are the same, but the sub-functions are different. This is not possible either because C is, by definition, a canonical form for the underlying sub-functions.