Definite Folds

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∂ everywhere

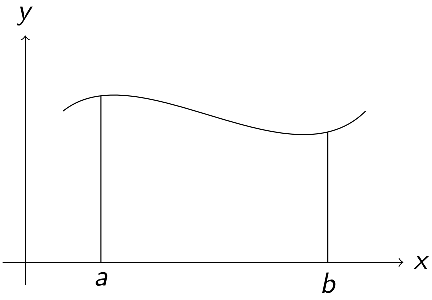
so...

what about

$\int$

?

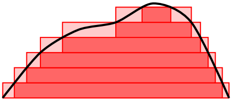
1D calculus



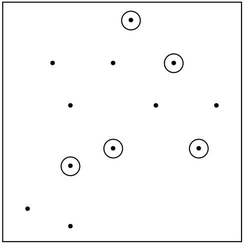
Bad for intuition!

- Continuity
- Limited geometry
- Same domain and codomain

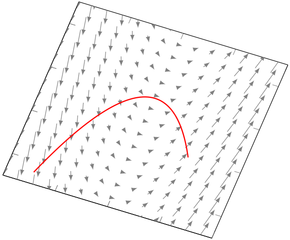
Intuition / analogy generators



Lebesgue Integration



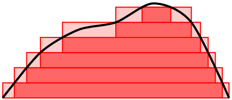
Discrete Geometry



Manifolds

✗ Under-studied ✓

Intuition / analogy generators



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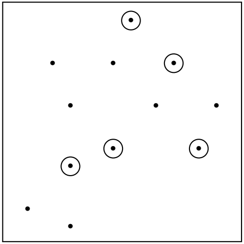

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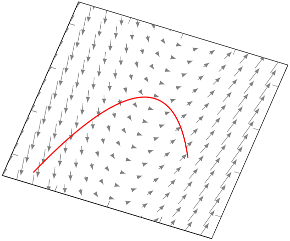


Manifolds

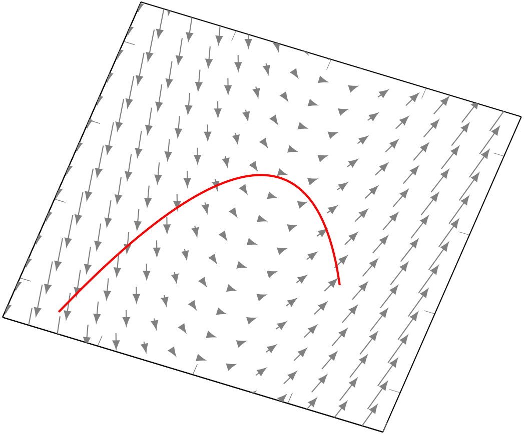
✗

Under-studied

✓

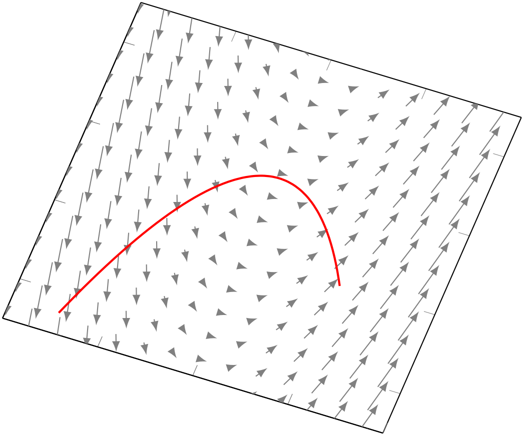
✗ locally Euclidean

✗ continuity



$$f : M \to V$$
$$\gamma : [0, 1] \to M$$
$$\int_\gamma f(\gamma)$$

- path (route)
- coordinates
- values



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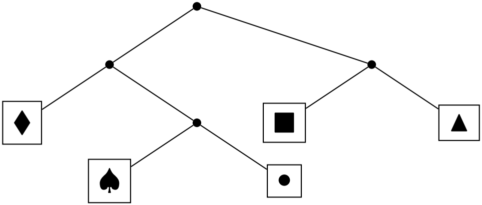
Abstract view of integration: information accumulation along a route

Mapping to data-structures

Coordinates

- Labels
- Locations
- Holes
- Positions ⇐



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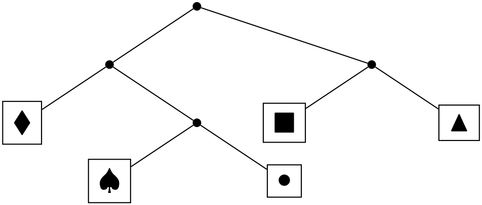

⟹ Data-structures as instances of *containers*

Definition (Route)

A **route** is a list of positions.

$$[\![S \triangleright P]\!]\, A \;=\; \sum_{s:S} \underbrace{(\, \underbrace{P\ s}_{\text{positions}} \;\to\; \underbrace{A}_{\text{values}} \,)}_{\text{Function!}}$$

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In general: $\int(s, f)\, ps =$ `foldr op b ps` where op $p\ b = f\ \ p \otimes b$.

Theorems on Folds

Notation: $\int_r f$ for r a *route* and f the function "under" a structure.

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If $A = B$ **and** $(B, \oplus, b)$ **is a left unital semigroup then:**

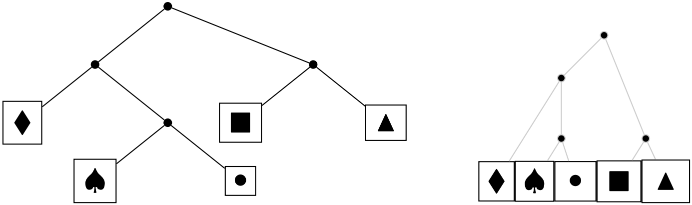
$$\int_{r_1} f \;\oplus\; \int_{r_2} f = \int_{r_1 \mathbin{++} r_2} f$$

$$\int_{[]} f = b$$

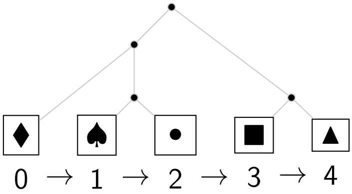
If B **is a monoid:**

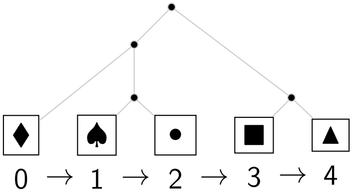
$$\int_r (\lambda y \to f\ y \;\oplus\; g\ y) = \int_r f \oplus \int_r g$$

Whole-space folds?



Enumerability






```
record IsEnumerable {x : Level} (X : Set x) : Set x where
  field
    size : ℕ
    enum : Fin size ↔ X
```

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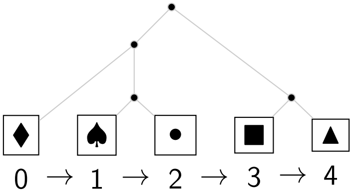


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- Container $\uplus$, $\times$, and indexed sums & products preserve enumerability.
- Usual finitary data-structures have enumerable types of positions.

Definite Folds

If $\ell = b - a$ and $b \geqslant a$ then:

$$\int_a^b f(x)dx = \int_a^{a+\ell} f(x)dx$$

So:

$$\int_{[a \cdot\cdot\, \ell]} f$$

Position → a; #s steps → ℓ

```
⟦_ ·· _⟧_ : (a, ℓ, size : ℕ) → List ( Fin size)
  ⟦a ·· ℓ⟧ size  =  take ℓ ( drop a (allFin size))
```

Definite Folds (2)

If `let r = map (fwd enum)` $[\![a \cdot\cdot \ell]\!]$ *size* then:

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If underlying B is a *Group* then we can extend by setting

$$\int_{p_1 \rightsquigarrow p_2} = - \int_{p_2 \rightsquigarrow p_1}$$

where p_1 comes after p_2.

Theorems of definite folds

For $\left(B, \oplus, 0^b\right)$ a monoid,

$$\int_{[a \,..\, 0]} f \;=\; 0^b$$

$$\int_{[a \,..\, \ell_1]} f \;\oplus\; \int_{[a+\ell_1 \,..\, \ell_2]} f \;=\; \int_{[a \,..\, \ell_1+\ell_2]} f$$

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Usual `foldr` corresponds to

$$\int_{[0 \,..\, size]} f$$

Derivatives?

$$T = [\![S \triangleright P]\!]$$

$$\text{Zipper}(TX) = X \times \delta T\, X$$

$$\text{Pointed}(TX) = \Sigma(T\, X)((s, \_) \to P\, s)$$

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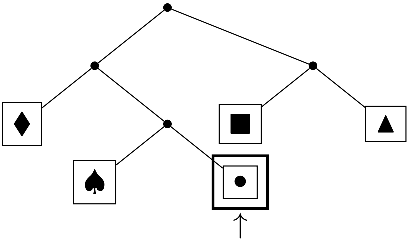
Theorem

$P\ :\ Dec \Rightarrow Zipper \simeq Pointed$



More goodies fall out

Incremental

```
incrfold : ((s , _) : T X) → ℕ → (B × P  s) → Either (B × P s)  B
```

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```

Parallel

```
record Partition : (n : ℕ) : Set where
 field
  size : ℕ
  pieces : Vec ℕ size
  is-partition : sum pieces ≡ n
```

fold over fold

Fundamental Theorem of Calculus?

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$F(b) = F(a) + \int_a^b f(x)dx$$

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Definition

Indefinite

$$\mathtt{indef}(f) = \ell \mapsto \int_{[0\,..\,\ell]} f$$

$$\underbrace{\int_{[0\,..\,\ell_1+\ell_2]} f}_{\text{F(b)}} = \underbrace{\int_{[0\,..\,\ell_1]} f}_{\text{F(a)}} \oplus \int_{[\ell_1\,..\,\ell_2]} f$$

Where a is at ℓ_1 and b is at $\ell_1 + \ell_2$.

Dictionary

Calculus	**Programming**
Coordinates	Positions
Path	Route
Functions	Data-structures
Integral over total space	Fold
Definite Integral	Definite Fold
Path Append	Route Concatenation
Iso $[0,1] \to M$	Enumerable
Euler differential	Zipper / Pointed
Indefinite	$\ell \mapsto \int_{[0\,..\,\ell]} f$