

4. RESOLUTION PRINCIPLE

4.1. REFUTATION PRINCIPLE

B set of closed formulas

G a closed formula

? How to prove $B \vdash G$

Th. (Refutation principle). Let B be a set of closed formulas, G a closed formula.

$B \vdash G$ iff $B \cup \{\neg G\}$ is inconsistent

4.2. CLAUSES

Closed formula F



Prenex conjunction normal form $(Q_1 x_1)(Q_2 x_2) \dots (Q_n x_n) M$

- (1) Eliminate $\Rightarrow \Leftrightarrow$
- (2) Bring \neg immediately before atomic formulas

$\neg \neg F$	$\Leftrightarrow F$
$\neg (F \& G)$	$\Leftrightarrow \neg F \vee \neg G$
$\neg (F \vee G)$	$\Leftrightarrow \neg F \& \neg G$
$\neg (\forall x) F(x)$	$\Leftrightarrow (\exists x) \neg F(x)$
$\neg (\exists x) F(x)$	$\Leftrightarrow (\forall x) \neg F(x)$

(3) Move the quantifiers to the left

$$(Qx) F(x) \& G \Leftrightarrow (Qx) (F(x) \& G)$$

$$(\forall x) F(x) \& (\forall x) H(x) \Leftrightarrow (\forall x) (F(x) \& H(x))$$

$$(\exists x) F(x) \vee (\exists x) H(x) \Leftrightarrow (\exists x) (F(x) \vee H(x))$$

$$(Q_1x) F(x) \& (Q_2z) H(z) \Leftrightarrow (Q_1x)(Q_2z) (F(x) \& H(z))$$

(4) Use the distributive law to get conjunction normal form:

$$F \vee (G \& H) \Leftrightarrow (F \vee G) \& (F \vee H)$$



Skolem normal form $(\forall x_1) (\forall x_2) \dots (\forall x_k) N$

(1) If $(\exists x)$ is the left-most quantifier, choose a new constant c and replace all x appearing in the formula by c , and delete $(\exists x)$

(2) If $(\forall y_1), \dots, (\forall y_m)$ are all the universal quantifiers appearing before $(\exists x)$, choose a new m -ary functor f , replace all x in the formula by $f(y_1, \dots, y_m)$, and delete $(\exists x)$



Set of clauses

(1) Remove all universal quantifiers and rewrite the formula

$$F \equiv C_1 \& C_2 \& C_3$$

into the set of clauses

$$B = \{ C_1, C_2, C_3 \}$$

(2) Rename variables in B so that each clause has different variables.

Clause is a finite disjunction of zero or more literals.

Empty clause \square is a clause which contains no literal.

- $F = C_1 \& C_2 \& C_3$ is inconsistent iff $B = \{ C_1, C_2, C_3 \}$ is inconsistent.

Example.

$$\begin{aligned}
 F & \equiv (\exists x) [P(x) \& (\forall y) (D(y) \Rightarrow L(x,y))] \\
 & (\exists x) [P(x) \& (\forall y) (\neg D(y) \vee L(x,y))] \\
 & (\exists x) (\forall y) [P(x) \& (\neg D(y) \vee L(x,y))] \\
 & (\forall y) [P(c) \& (\neg D(y) \vee L(c,y))] \\
 & \quad \underbrace{P(c)}_{C_1} \& \underbrace{(\neg D(y) \vee L(c,y))}_{C_2}
 \end{aligned}$$

$$B = \{ P(c), \neg D(y) \vee L(c,y) \}$$

Example.

$$F \equiv (\forall x) (\forall y) [(\exists z)(P(x,z) \& P(y,z)) \Rightarrow (\exists z) Q(x,y,z)]$$

$$(\forall x) (\forall y) [\neg (\exists z)(P(x,z) \& P(y,z)) \vee (\exists z) Q(x,y,z)]$$

$$(\forall x) (\forall y) [(\forall z) (\neg P(x,z) \vee \neg P(y,z)) \vee (\exists z) Q(x,y,z)]$$

$$(\forall x) (\forall y) (\forall z) (\exists u) [\neg P(x,z) \vee \neg P(y,z) \vee Q(x,y,u)]$$

$$(\forall x) (\forall y) (\forall z) [\neg P(x,z) \vee \neg P(y,z) \vee Q(x,y, f(x,y,z))]$$

$$\underbrace{\neg P(x,z) \vee \neg P(y,z) \vee Q(x,y, f(x,y,z))}_{C_1}$$

C_1

$$B = \{C_1\}$$

4.3. RESOLUTION PRINCIPLE

Modus ponens

$$\frac{F, F \Rightarrow G}{G}$$

If we rewrite it into the conjunctive normal form:

F	$\equiv C_1$	} parent clauses
$\neg F \vee G$	$\equiv C_2$	
G	$\equiv C_3$	resolvent of C_1 and C_2

resolved literals

Resolution principle (Robinson, 1965) is an extension of MP

$$\begin{array}{l}
 C_1 \equiv L_0 \vee L_1 \vee L_2 \\
 C_2 \equiv \neg L_0 \vee L_3 \vee L_4 \vee L_5 \\
 \hline
 C_3 \equiv L_1 \vee L_2 \vee L_3 \vee L_4 \vee L_5
 \end{array}$$

resolvent of C_1, C_2

Th. $\{C_1, C_2\} \vdash C_3$

Def. Let B be a set of clauses. Resolution deduction of C from B is a finite sequence C_1, C_2, \dots, C_n such that

(a) each C_i is either a clause in B or a resolvent of clauses preceding C_i ,

(b) $C_n \equiv C$

Th. Resolution principle is complete, i.e. a set of clauses B is inconsistent iff there is a resolution deduction of \square from B .

Consequently,

$B \vdash G$ iff $B \cup \{ \neg G \}$ is inconsistent

iff there is a res. deduction of \square from $B \cup \{ \neg G \}$

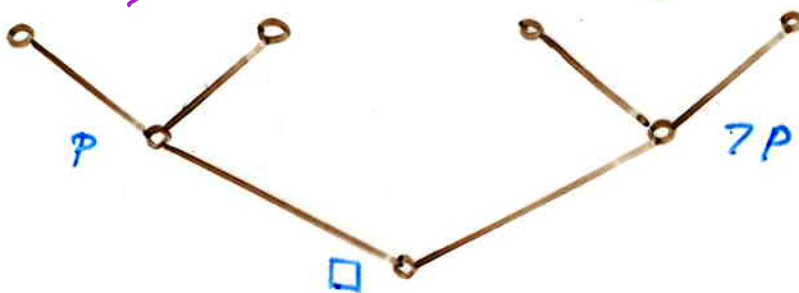
Example.

$B = \{ Q \Rightarrow P, P \Rightarrow Q, P \Rightarrow \neg Q \}$

$G \equiv \neg P \ \& \ \neg Q$

Prove $B \vdash G$

$B \cup \{ \neg G \} = \{ P \vee Q, P \vee \neg Q, \neg P \vee Q, \neg P \vee \neg Q \}$



Done.

4.4. UNIFICATION ALGORITHM

Now consider Predicate Calculus:

$$C_1 = P(f(y)) \vee Q(f(y))$$

$$C_2 = \neg P(f(y)) \vee R(y)$$

We can resolve $P(x)$ and $\neg P(f(y))$ only if

$$x \equiv f(y)$$

Resolvent of C_1, C_2

$$C_3 = Q(f(y)) \vee R(y)$$

The process of matching two literals is called unification algorithm.

Example. Unify $P(a, x, f(g(y)))$
and $P(z, f(z), f(u))$

1. step $z \equiv a$: $P(a, x, f(g(y)))$
 $P(a, f(a), f(u))$

2. step $x \equiv f(a)$: $P(a, f(a), f(g(y)))$
 $P(a, f(a), f(u))$

3. step $u \equiv g(y)$: $P(a, f(a), f(g(y)))$

The above literals match if the above substitutions are applied.

Resolution principle for Pre C:

Before resolving literals, try to unify literals of the two clauses.

Example. Let $B = \{B_1, B_2\}$ where

$$B_1 \equiv (\exists x) [P(x) \ \& \ (\forall y) (D(y) \Rightarrow L(x, y))]$$

$$B_2 \equiv (\forall x) [P(x) \Rightarrow (\forall y) (Q(y) \Rightarrow \neg L(x, y))]$$

and let $G \equiv (\forall x) (D(x) \Rightarrow \neg Q(x))$

Prove $B \vdash G$

We will prove that $B \cup \{ \neg G \}$ is inconsistent.

Set of clauses corresponding to $B \cup \{ \neg G \}$:

$$B_1: \quad P(c_1) \tag{1}$$

$$\neg D(y_2) \vee L(c_1, y_2) \tag{2}$$

$$B_2: \quad (\forall x) (\forall y) [\neg P(x) \vee \neg Q(y) \vee \neg L(x, y)]$$

$$\neg P(x_3) \vee \neg Q(y_3) \vee \neg L(x_3, y_3) \tag{3}$$

$$\neg G: \quad \neg (\forall x) (\neg D(x) \vee \neg Q(x))$$

$$(\exists x) (D(x) \ \& \ Q(x))$$

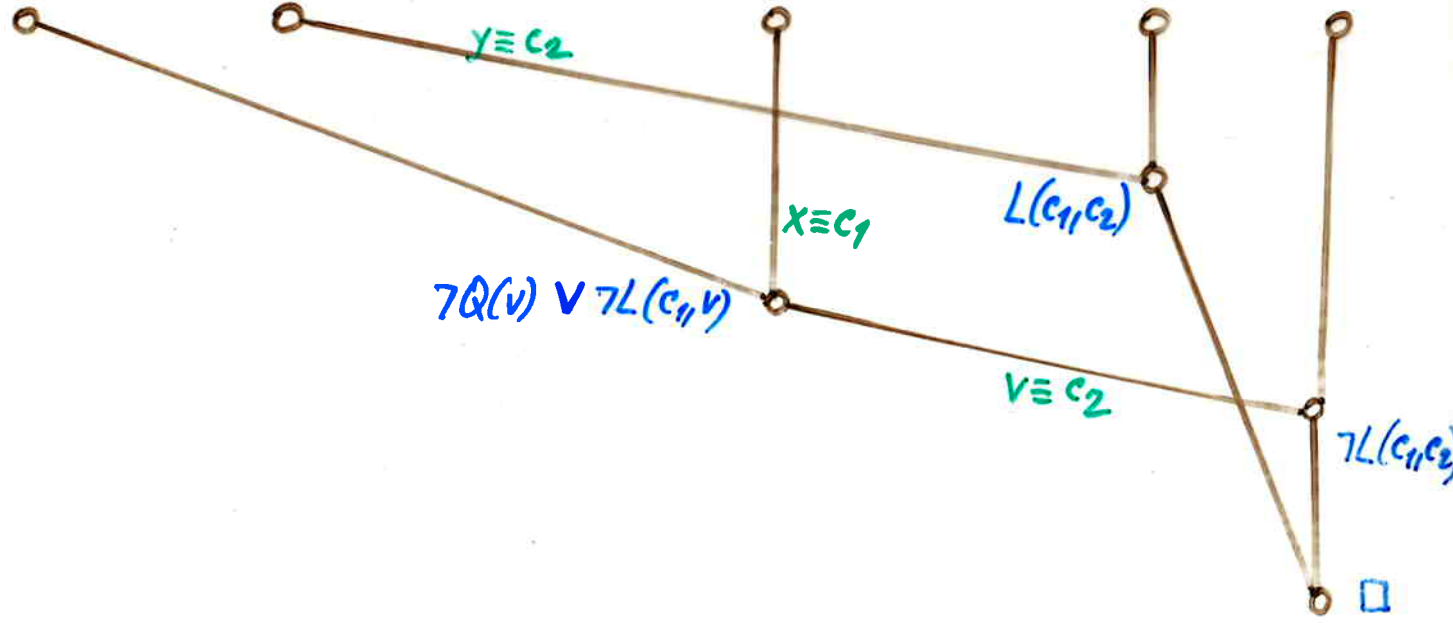
$$D(c_2) \ \& \ Q(c_2)$$

$$D(c_2) \tag{4}$$

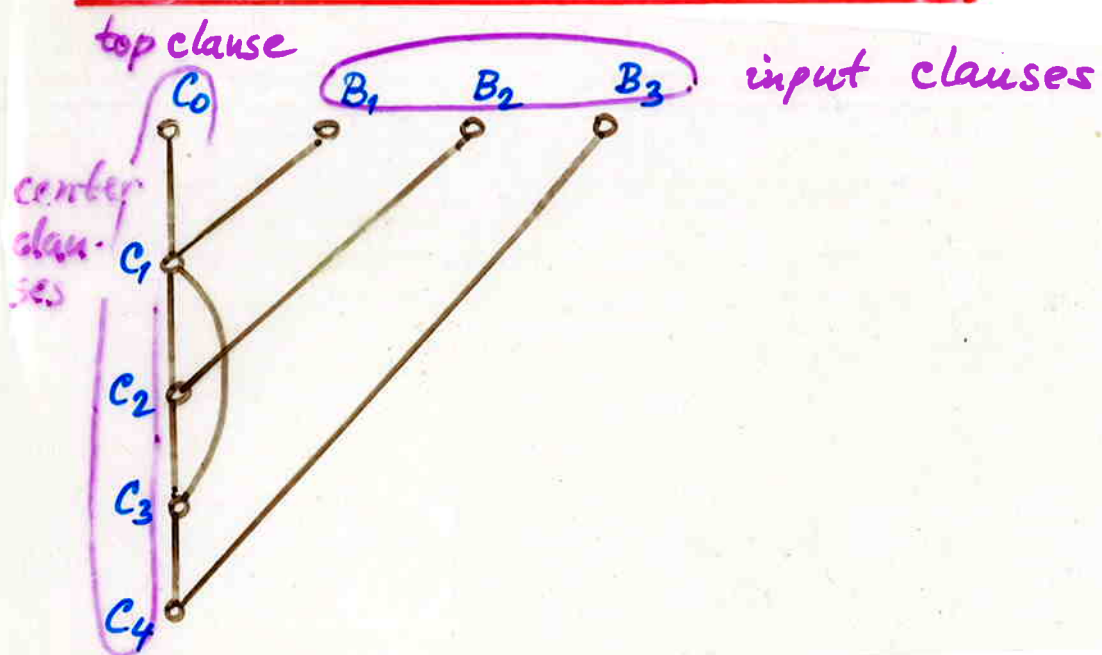
$$Q(c_2) \tag{5}$$

Resolution deduction of \square from the above set:

$P(c_1) \quad \neg D(y) \vee L(c_1, y) \quad \neg P(x) \vee \neg Q(v) \vee \neg L(x, v) \quad D(c_2) \quad Q(c_2)$



4.5. LINEAR STRATEGY



Linear resolution:

C_{i+1} is a resolvent of the center clause C_i and
 { an input clause
 { a center clause $C_j, j < i$

Note. The top clause C_0 is usually the negation of the formula which we are to prove to be a theorem from some premises.

Example.

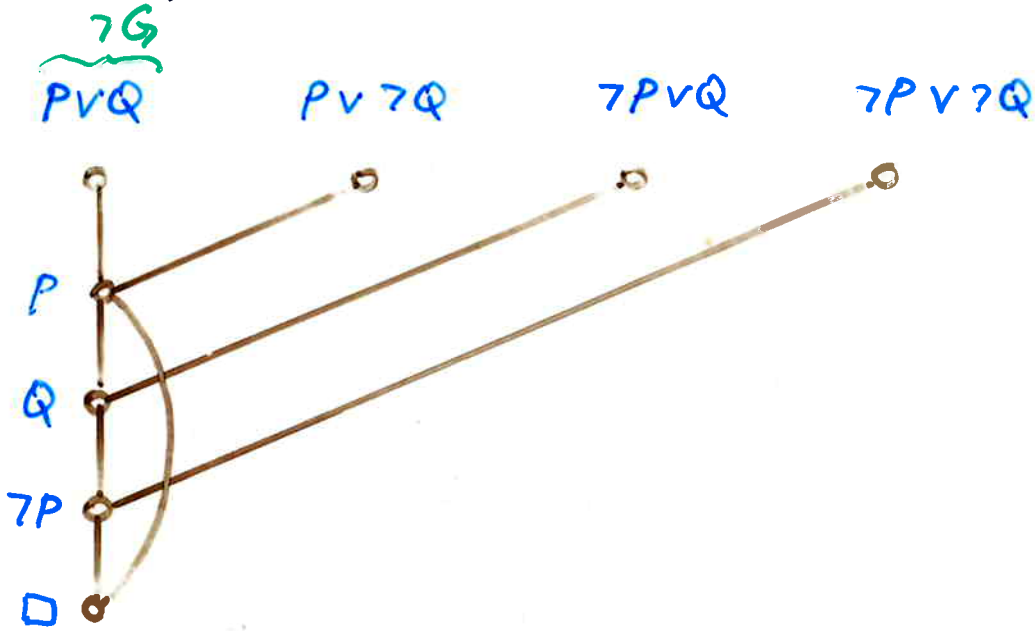
$$B = \{ Q \Rightarrow P, P \Rightarrow Q, P \Rightarrow \neg Q \}$$

$$G = \neg P \ \& \ \neg Q$$

Prove $B \vdash G$

a) Linear resolution

We will prove that $B \cup \{\neg G\}$ is inconsistent.



b) Syntactically

$$\vdash (Q \Rightarrow P) \ \& \ (P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow \neg Q)$$

known theorem

$$\vdash (F \Rightarrow G) \ \& \ (G \Rightarrow H) \Rightarrow (F \Rightarrow H)$$

$$B \vdash A \Rightarrow \neg Q$$

2x MP

$$\vdash (Q \Rightarrow \neg Q) \Rightarrow \neg Q$$

known theorem

$$\vdash (\neg F \Rightarrow F) \Rightarrow F$$

$$\rightarrow B \vdash \neg Q$$

MP

$$\vdash (P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$$

MP

$$\rightarrow B \vdash \neg Q \Rightarrow \neg P$$

MP

$$B \vdash \neg P$$

$$\vdash \neg P \Rightarrow (\neg Q \Rightarrow \neg P \ \& \ \neg Q)$$

known theorem

$$\vdash F \Rightarrow (G \Rightarrow F \ \& \ G)$$

$$B \vdash \neg P \ \& \ \neg Q$$

2x MP

G

g) Semantically

		B			G
P	Q	$Q \Rightarrow P$	$P \Rightarrow Q$	$P \Rightarrow \neg Q$	$\neg P \wedge \neg Q$
0	0	1	1	1	1
0	1	0	1	1	0
1	0	1	0	1	0
1	1	1	1	0	0

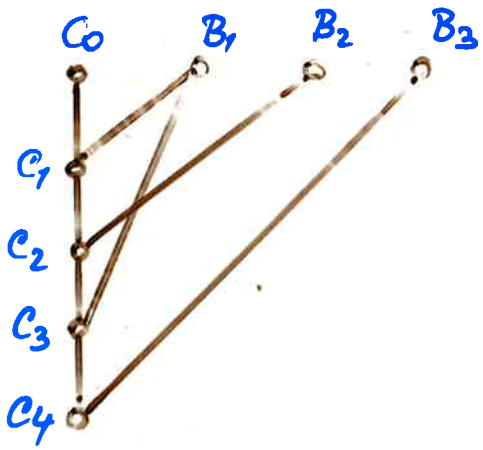
the model of B is a model of G as well.

- Linear resolution is complete.

4.6. INPUT RESOLUTION

Linear resolution is complete but we must store the entire sequence of center clauses (inefficient!)

Input resolution is linear resolution but a center clause is resolved with an input clause only.



Input resolution is incomplete, but we need not store the center clauses (completeness \times efficiency).

Def. Horn clause is a clause which has at most one literal without \neg

Th. Input resolution with Horn clauses and the top clause containing only literals with \neg is complete.

- PROLOG is equivalent to the Input resolution for Horn clauses.