
or, equivalently, as

$$\begin{aligned} & \text{minimize} && \|y\| \\ & \text{subject to} && I_C(x) \leq 0 \\ & && x_0 - x = y \end{aligned}$$

where the variables are x and y . The dual function of this problem is

$$\begin{aligned} g(z, \lambda) &= \inf_{x,y} (\|y\| + \lambda I_C(x) + z^T(x_0 - x - y)) \\ &= \begin{cases} z^T x_0 + \inf_x (-z^T x + I_C(x)) & \|z\|_* \leq 1, \quad \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} z^T x_0 - S_C(z) & \|z\|_* \leq 1, \quad \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

so we obtain the dual problem

$$\begin{aligned} & \text{maximize} && z^T x_0 - S_C(z) \\ & \text{subject to} && \|z\|_* \leq 1. \end{aligned}$$

If z is dual optimal with a positive objective value, then $z^T x_0 > z^T x$ for all $x \in C$, *i.e.*, z defines a separating hyperplane.

8.2 Distance between sets

The distance between two sets C and D , in a norm $\|\cdot\|$, is defined as

$$\mathbf{dist}(C, D) = \inf\{\|x - y\| \mid x \in C, y \in D\}.$$

The two sets C and D do not intersect if $\mathbf{dist}(C, D) > 0$. They intersect if $\mathbf{dist}(C, D) = 0$ and the infimum in the definition is attained (which is the case, for example, if the sets are closed and one of the sets is bounded).

The distance between sets can be expressed in terms of the distance between a point and a set,

$$\mathbf{dist}(C, D) = \mathbf{dist}(0, D - C),$$

so the results of the previous section can be applied. In this section, however, we derive results specifically for problems involving distance between sets. This allows us to exploit the structure of the set $C - D$, and makes the interpretation easier.

8.2.1 Computing the distance between convex sets

Suppose C and D are described by two sets of convex inequalities

$$C = \{x \mid f_i(x) \leq 0, i = 1, \dots, m\}, \quad D = \{x \mid g_i(x) \leq 0, i = 1, \dots, p\}.$$

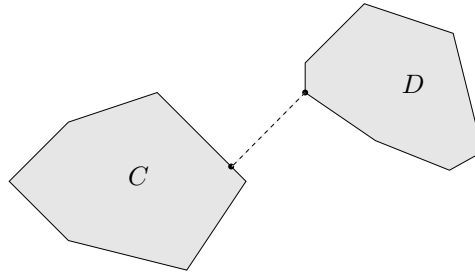


Figure 8.2 Euclidean distance between polyhedra C and D . The dashed line connects the two points in C and D , respectively, that are closest to each other in Euclidean norm. These points can be found by solving a QP.

(We can include linear equalities, but exclude them here for simplicity.) We can find $\mathbf{dist}(C, D)$ by solving the convex optimization problem

$$\begin{aligned} & \text{minimize} && \|x - y\| \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && g_i(y) \leq 0, \quad i = 1, \dots, p. \end{aligned} \tag{8.3}$$

Euclidean distance between polyhedra

Let C and D be two polyhedra described by the sets of linear inequalities $A_1x \preceq b_1$ and $A_2x \preceq b_2$, respectively. The distance between C and D is the distance between the closest pair of points, one in C and the other in D , as illustrated in figure 8.2. The distance between them is the optimal value of the problem

$$\begin{aligned} & \text{minimize} && \|x - y\|_2 \\ & \text{subject to} && A_1x \preceq b_1 \\ & && A_2y \preceq b_2. \end{aligned} \tag{8.4}$$

We can square the objective to obtain an equivalent QP.